Second Edition

MECHANICS

☑ Concepts illustrated with examples

☑ Large number of practice problems and review questions

■ H S HANS ■ S P PURI

Mechanics

Second Edition

H S HANS
and
S P PURI
Former Professors
Panjab University
Chandigarh

Tata McGraw-Hill Publishing Company Limited
NEW DELHI

McGraw-Hill Offices
New Delhi New York St Louis San Francisco Auckland Bogotá Caracas
Kuala Lumpur Lisbon London Madrid Mexico City Milan Montreal
San Juan Santiago Singapore Sydney Tokyo Toronto

Tata McGraw-Hill

Third reprint 2006
RLLCRRQKRAZZZ

ISBN 0-07-047360-9

Published by Tata McGraw-Hill Publishing Company Limited
7 West Patel Nagar, New Delhi 110 008 and typeset in Times at The Composers, 20/5 Old Market, West Patel Nagar, New Delhi 110 008 and printed at Sai Printo-Pack, Y-56, Okhla Industrial Area, Phase II, New Delhi 110 020

Cover: Meenakshi Printers

RZLLCRQDRAZXL

The McGraw-Hill Companies

Contents

List of Symbols

Vector quantities are represented by boldface letters and the same symbol in normal type represents the magnitude of the vector quantity only. A hat over a vector quantity represents a unit vector. A prime above a symbol denotes the quantity in the primed coordinate system, whereas a dot above a symbol denotes differentiation with respect to time. This list gives only the important symbols and is not intended to be comprehensive.

$\mathbf{A}(A_x, A_y, A_z)$	Vector
A	Undetermined constant, atomic weight, area
$\mathbf{a}(a_x, a_y, a_z)$	Acceleration in the unprimed system
a	Semi-major axis of the ellipse
a_{ij}	Direction cosine of the ith component with the jth axis, the undetermined constants
$\mathbf{B}$	Vector
B	Undetermined constant; bulk modulus
b	Amplitude of forced oscillator; semi-minor axis of the ellipse; impact parameter
$\mathbf{C}$	Vector
C, C_1, C_2	Undetermined constants
C_0	Undetermined constant, amplitude
c	Velocity of light in vacuum
$D \equiv \frac{d}{dt}$	Differential operator
D_1, D_2	Unknown constants
$\mathbf{E}$	Field intensity
E	Total energy
E_p	Potential energy
E_μ	Energy of the μ – meson
$\hat{\mathbf{e}}$	Unit vector
e	Electronic charge
$\mathbf{F}(F_x, F_y, F_z)$	Force
f_o	Frequency
G	Universal gravitational constant; a function in Virial theorem
g	Acceleration due to gravity

$\overset{*}{g}$	Apparent acceleration due to gravity
g_β	Coupling constant in $\beta-$decay
g_n	Coupling constant in nuclear interaction
$H(p,q)$	Hamiltonian
$\hbar$	Planck's constant
I	Moment of inertia, flux
$\mathbf{i},\mathbf{j},\mathbf{k}$	Unit vectors along x-,y- and z-axes
$\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$	Unit base vectors in S
i,j	Variable indices
$\mathbf{J}$	Current density, total angular momentum
K	Kinetic energy
k	Dielectric constant, force constant
$\mathbf{L}$	Angular momentum
$L(q,q)$	Lagrangian
l_o	Length in S, proper length
M	Mass
M_E	Mass of the earth
M_g	Gravitational mass
M_i	Inertial mass
M_s	Mass of sun or star
m	Mass of a particle, relativistic mass of a particle
m_o	Proper mass of a particle
m_s	Mass of the sun
m_c	Mass of a comet
$\hat{\mathbf{n}}$	Unit vector
n	Number of scatterers; refractive index, gyro-frequency
N	bending moment of the beam
$\mathbf{p}(p_x, p_y, p_z)$	Momentum of a particle in S
P_{av}	Average power supplied in one cycle
p	Generalized momenta
Q	Energy balance, quality factor (also figure of merit)
q	Charge, frequency; generalized coordinates
R_m	Mechanical resistance
R_s	Radius of a star
$\overset{*}{\mathbf{R}}$	Radial vector of centre of mass
$\mathbf{r}$	Position vector
r	Gyro-radius
r_o	Classical radius of the electron
r_{12}	Distance between two points 1 and 2
S	Stiffness constant, spin vector, displacement
S_{12}	Tensor function in molecular physics
$\mathbf{S}$	Spin
t	Time, kinetic energy
T	Periodic time, kinetic energy
T_μ	Kinetic energy of $\mu-$meson

U	Work, potential energy
U_s	Self energy
$\mathbf{u}(u_x, u_y, u_z)$	Velocity in S.
V	Volume, potential
$\mathbf{v}$	Uniform velocity of relative motion
v_e	Escape velocity
v_o	Orbital velocity
$\bar{v}$	Mean molecular speed
W	Work
X_m	Mechanical resistance
$\hat{\mathbf{x}}$	Unit vector along x-axis
$\mathbf{y}$	Unit vector along y-axis
y	Displacement along y-axis
Y	Young's modulus
y	
Z	Charge number
Z_m	Mechanical impedance
$\hat{\mathbf{z}}$	Unit vector along z-axis
z	Displacement along z-axis

Greek Symbols

α	Eigen-value, angular acceleration, angle Longitudinal strain per unit stress
β	Eigen-value, transverse strain per unit stress
γ	Contraction factor
δ	Phase constant
δ_s	Differential interval
ΔE	Energy change
Δm	Mass defect
$\Delta\omega$	Resonance width
$\in$	Eccentricity
ε_o	Permittivity of vacuum
ς	Real part of amplitude
η	Coefficient of viscosity, modulus of rigidity
ς, η, ξ	Eular angles
Θ	Angle
θ	Angular displacement
θ_o	Angular amplitude, zeroth angle
λ	Wavelength, imaginary component of the amplitude
μ	Reduced mass, variable index
μ_o	Permeability of vacuum
ν	Frequency, variable index
ρ	Component of radius vector in sphere coordinate system, mass density
σ	Area, cross-section, mass per unit length , Poisson ratio
τ	Damping time, couple for twisting the cylinder

φ, φ_o	Azimuthal angle, angle, scattering angle
φ_o	Phase constant at $t = 0$
ψ	Phase constant
Ω	Solid angle, rotational operator, constant in angular motion, precession velocity
ω	Angular frequency, angular velocity
ω_c	Cyclotron frequency

1

Scope and Historical Development

1.1 PHYSICS—THE FUNDAMENTAL SCIENCE

Physics is a very fundamental and vast science, of which the subject of mechanics forms one of the most important foundation stones. Physics is the most basic of the physical sciences, as its principles constitute the basis of other sciences such as chemistry, geology, electronics, meteorology; various branches of engineering, and the biological subjects like biophysics, molecular biology, microbiology and biochemistry. While other sciences are derived sciences of different orders of complexity, depending for their development and understanding on relatively less complex subjects; physics is the basic physical science of the first order and hence is self-contained in its logic. The concepts and the laws of physics are based on human intuition and are derived in a self-consistent manner from the basic experimental facts, and systematic mathematical reasoning.

Newtonian mechanics or the classical non-relativistic mechanics (hereafter, referred to only as mechanics)—the subject matter of the first part of this book—deals with the motion (or the lack of it) of material macroscopic bodies with velocities much less than the velocity of light.. Various laws of mechanics like Newton's three laws, or the concepts like force, momentum, energy, work, angular momentum, torque, etc. were suggested by experimental or observational facts as found by Galileo, Kepler and many others. These have been defined in a manner, so that they follow the dual criterion of intuitive understanding and mathematical self-consistency. For example, the concept of force not only corresponds to what is understood commonly by the layman but has also been precisely defined and related to mass and acceleration and used consistently in the subsequent development of the subject. Similar comments are true for the concepts of work, energy, torque etc.

The concepts developed in mechanics form the basis of the development of other branches of physics. For example, the concepts of force and energy are extensively used in the kinetic theory of matter, electrostatics, magnetism and acoustics. Of course, various new concepts have been introduced in different branches to explain the experimental observations.

For instance, in electricity and magnetism concepts such as charge, scalar and vector potentials, moving electric fields etc. were added on the basis of observations

made by Coulomb, Ampère, Faraday and others. Similarly the subject of optics, which is a case of the application of electromagnetic theory, also contains new concepts of fields and wave motion. The concepts of mechanics, however, are consistently used in the development of both electrostatics and electromagnetic theory.

Statistical mechanics and kinetic theory of matter in heat are conceptually a case of an application of the laws of probability to randomly moving particles, each of which obeys the laws of mechanics. It is interesting to note how seemingly very diverse subjects such as heat and mechanics have been correlated in a self-consistent manner in physics. It is this self-consistency and systematic logical development which is the hallmark of physics.

The subject of special theory of relativity, which constitutes the contents of the chapters 10, 11 and 12 of this book, has some extremely new principles, while dealing with the motion of particles with high velocities comparable to that of light. The postulates of the special theory of relativity, which apparently look so different from the basic principles of Newtonian mechanics, are, however, in confirmity with the experimental facts indicated in the experiments of Michelson and Morley and more importantly electromagnetic theory of Maxwell, and lead to the laws of Newtonian mechanics for low energies. For example, the Galilean transformation of frames of reference does not hold good in relativity. According to the special theory of relativity, mass, time and length are not invariant in different inertial frames of reference, as are assumed in mechanics. Similarly, the expressions of force, momentum, energy etc. are different in different frames. However, all the laws and concepts of Newtonian mechanics can be obtained from the special theory of relativity for small velocities.

The modern subjects of quantum mechanics, quantum electrodynamics and their applications to the microscopic systems such as atoms, molecules, nuclei, etc., form very fascinating and exciting chapters in the developments in physics in this century. They represent further modifications of the concepts developed in mechanics, electromagnetic theory and special theory of relativity as applicable to microscopic objects. These modifications were required by the experimental facts of blackbody radiation, photo-electric effect, Compton effect, diffraction of particles, etc. The explanation of these observations required the introduction of new concepts of quantisation of energy and angular momenta, which were not governed by Newton's laws of motion, but by the Schrödinger's wave equation. Though the modifications are highly profound, and fundamental, the concepts of position, momentum, force, energy, etc. as understood in classical mechanics are still basic.

The purpose of the above discussion was to bring out the point that the subject of physics is not only firmly grounded in experimental facts and observations, but is also a subject of fundamental and well-connected logic. The logic of physics—its postulates and theories—are not borrowed from other sciences, but are developed self-consistently relying on experimental facts, and under no conditions contradictory to them. It is always the new experimental facts that give rise to the new concepts and logic. And the self-consistency and continuity of logic are assured by the mathematical tools which the subject of physics uses for the development of various relationships in the quantities.

It may be emphasised that though the subject of mathematics forms the rock on which the foundations of physics are laid, physics (even theoretical physics) is not mathematics. While mathematics is a subject of pure logic with and without its application to real situations, the subject of physics is concerned with actual physical situations. Mathematics only serves as a logical tool for physics, whose foundations, however, are rooted in experimental or observational phenomena. Hence physics represents a very beautiful marriage between experimental facts and pure mathematical logic. It is essential, therefore, that for the healthy development of physics, both aspects—the experimental and theoretical—are understood and grasped properly.

1.2 HISTORICAL DEVELOPMENT OF MECHANICS

Though mechanics, in the modern sense, was first formulated by Newton in 1687 when he stated his famous three laws of motion, the development of the various concepts in mechanics has been going on for about two millennia before that. The motion of material bodies is an everyday experience; hence, it was natural for the human mind to start formulating theories about the motion of the bodies on the earth or those falling on it. Socrates and later his successor Aristotle ~ 400 BC were perhaps the first western philosophers who recorded their ideas about the motion of bodies. For example, according to them 'everything finds its natural position' or that 'planets move in circles, because the circle is a perfect figure' or that 'heavier things take less time in falling to the earth than light bodies'. These were erroneous, vague and untested laws of motion which were stated by those hoary philosophers and were believed for centuries by the layman and clergy. Another wrongly held concept that prevailed for many centuries was that the earth is stationary, and the sun and all the planets and stars move around it. As a matter of fact, the Egyptian astronomer Claudius Ptolemy (early second century AD) carefully calculated the movement of each planet around the earth, and explained these motions by an epicycloidal path, with the earth displaced from the centre of the epicycloid. Of course, much earlier, Babylonians and Indians had carried out empirical calculations of the apparent motion of the planets by which they could predict the future events of the planetary motion and eclipses. All these efforts to understand the motion of the planets, the moon or sun on the one hand, and the motion of bodies on earth, on the other, were either only empirical or sometimes completely erroneous.

It seems that the first breakthrough in mechanics was brought about by Galileo's (1564-1643) famous experiments wherein he showed that two weights of different magnitudes, when allowed to fall from the tower of Pisa, took the same time to reach its bottom. In other words, the rate of fall of an object is independent of the weight, or in modern language, all of them fall with the same acceleration. This was, in fact, a case of triumph of the experimental facts over the fanciful words of authority. It was also Galileo who stated that the earth is not stationary but it is the sun which is stationary and all the planets and earth move around it. According to him, all stars were also stationary. For this belief, which were against the prevailing Christian belief in Europe, he was persecuted. Of course, many other philosophers

such as Copornicius and many Indian astronomers also believed that the earth rotates around the sun and that the sun is stationary. These ideas were, however, based on intuitive conjectures and not on experimental facts. It was the combination of three efforts coming successively that finally laid the foundation of mechanics and also solved the mystery of planetary motion. These were the efforts of Tycho Brahe (1546-1601), Kepler (1571-1630) and Newton (1642-1727) which gave birth to the present concepts of mechanics. Tycho collected a large amount of data on the exact location of various planets, which were then systematised by Kepler, who gave his famous three laws of planetary motion around the sun. This laid the groundwork for Newton's great work. The recognition by Newton that the same laws of mechanics that operate in planetary motion are also valid for bodies on the earth was a great step forward in understanding the laws of mechanics. Whether the story of the revelation of this concept from the observation of the fall of an apple is true or not, it was certainly a remarkable insight. Newton then enunciated his law of gravitation in 1683 and explained Kepler's three empirical laws in a natural and elegant manner. The three laws of motion were put forward as axioms in the treatise *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy) published in 1687. The three volumes of this treatise contain Newton's important contribution to the subject of mechanics.

Classical mechanics has been further developed by great stalwarts like Lagrange (1736-1813) and Hamilton (1805-1865) who gave the famous Lagrange's equation of motion and Hamilton's canonical equations, respectively. As very elegent result of their formulation of classical mechanics was the principle of least action. Others who have contributed deeply to the subject are: Jacobi (1804-1851), who is known for his work on the famous Hamilton-Jacobi equation; Poincaré (1854-1912), who gave very profound arguments about the relationship of the inertial mass of bodies and the universal gravitational field, and introduced the concept of integral invariants of Poincaré; and Poisson (1781-1840), who developed the Poisson equation and Poisson brackets. The developments of classical mechanics by these intellectual giants ultimately laid the foundation of the logic of modern quantum mechanics. The topics mentioned above are, however, beyond the purview of this book. Readers desirous of acquainting themselves with these topics can consult some of the books listed at the end of this chapter.

1.3 SOME IMPLICATIONS OF THE PRINCIPLES OF MECHANICS

Both historically and logically, mechanics was the first subject of physics to be developed; therefore its postulates and the underlying assumptions were based on the intuitive idealisation of experimental facts and observations. The principles of mechanics seem to possess an obviousness which is not shared by other branches of physics. But this did not make the job for formulating the laws easy for the earlier physicists. The obvious had to be idealised before being given the form of a law or relationship.

As it stands now, Newton's three laws of motion form the basis of mechanics. These laws are introduced to the students of science even at schools with the tacit assumption that these are very easy to comprehend and apply. But it took about two

thousand years—between the views expressed by Aristotle in the fourth century BC and the 17th century when Newton's laws were put forward—to reach the right conclusions.

A number of difficulties were experienced in developing various concepts. For example, from the intuitive concept of force, the first law of Newton is somewhat obvious, but the second law is not so evident. According to the writings of Newton as given in *Principia Mathematica*, the second law of motion reads: 'the change of motion is proportional to the force and takes place in the direction of the straight line in which the force acts'. This statement passed through a lot of controversy and argumentation before it was decided that force is proportional to acceleration. As a matter of fact, the second law of motion is basically a definition of force in terms of the inertial mass and acceleration. In the form of an equation, it reads as:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) = m\,(d^2\mathbf{x}/dt^2) \qquad (1.1)$$

which is a fundamental equation of mechanics. Quantities such as force ($\mathbf{F}$), mass (m), displacement ($\mathbf{x}$), and time (t) used in this equation are indeed defined only intuitively. It, however, required a genius like Newton to correlate these physical ideas in a convenient and correct manner. Some other definitions of force could also have been tried, which would have given rise to different relationships in mechanics; making it perhaps, more cumbersome than it is today.

It may be realised that a good amount of idealisation is assumed in enunciating these laws. The assertion of the first law that a body goes on moving in a straight line unless acted on by a force can never be proved experimentally. Still the definition presumes the ideal condition that, in a hypothetical case, if the surface had zero friction, the motion would have continued forever.

The third law of motion was, according to Mach, Newton's most important contribution to the principles of mechanics. It is interesting to note that this law basically requires forces and accelerations to occur in pairs. This is a view according to which a single force cannot be generated. In terms of observation, this seems to be based on facts but it has far-reaching consequences on the properties of space. This implies that free space, in itself, cannot create a force. While it is true that the third law is closely related to the second law, it was, however, still stated explicitly to introduce the concept of reaction, which is not obvious, unless clearly stated. It is because of this concept of reaction that Mach praised Newton for giving this law.

The basic quantities involved in mechanics are: mass, length and time. These three quantities were initially defined intuitively, based on the idealisation of daily experiences. These experiences have been further refined to make them more objective. The initial concept of mass as an amount of matter is too vague, wrong and also unmeasurable. Consequently, it has been replaced by mass as defined in Newton's second law of motion, according to which it is the amount of inertia associated with a body. The definition of length requires idealised rigid rods to measure other lengths with. It also assumes that space is homogeneous and has the same properties in all directions. The concept of time requires a uniformly flowing time so that an interval of time at present has the same meaning as in the infinite past or infinite

future. These are all idealisations whose validity can be proved only through experiments.

The second law of motion, which is used as a fundamental equation in mechanics uses a differential equation of the second order. The validity of such an equation demands that both space and time are continuous. This is only an assumption and it is hoped that it is correct.

All other concepts are derived from these basic concepts of mass, length and time. We have already discussed the concept of force. Other concepts such as momentum, work, kinetic energy, potential energy, angular momentum, torque, etc. are not just arbitrary mathematical functions of mass, length and time, but are defined appropriately to convey definite physical experiences. These quantities represent functions which are very convenient in perceiving conceptually the behaviour of the motion of the body. It is interesting to note that because of these correct concepts—related very closely to the physical situations—we could obtain the three laws of conservation of linear momentum, angular momentum and energy.

1.4 THE SCOPE OF MECHANICS

As stated earlier, mechanics deals with the motion of material macroscopic bodies moving at comparatively small velocities—much smaller than the velocity of light. As a matter of fact, the historical and conceptual development of the subject started with an attempt to understand the motion of the bodies in everyday life on one side and heavenly bodies such as planets on the other. What are the limits of sizes, masses, distances, velocities and times for which the laws of mechanics hold good? And what are the assumptions made in deriving these laws of motion or developing other concepts of mechanics, that require us to set these limits? Let us discuss the basic assumptions in mechanics to arrive at these limits.

Newton's first law of motion requires that a body will continue moving in a straight line, with a constant velocity, if no external force acts on it. Does the law hold good for any length of distance? Similarly, in the second law of motion, the acceleration is generally calculated using the properties of flat space, i.e. Euclidean geometry. Does Eulidean geometry hold good over any length? If so, does it mean, that the space is 'flat' even if we go to very large lengths—say astronomical lengths?

From the astronomical observations it has been inferred that if the space has any curvature at all, the radius of curvature of space is at least 5×10^{17} cm, if not more. This was specifically inferred from the measurements on the positions of Neptune and Pluto and comparing them, with the expected positions.

Similarly, if the space is 'flat' then it would obey Euclidean geometry according to which the sum of the three angles in a triangle is 180°. On the other hand, if the space is curved, say a convex surface like the surface of a sphere, the sum of the angles of the triangle will be more than 180°. Such an experiment was performed on the suggestion of Schwarzschild by measuring the angles formed by the two positions of the earth six months apart with the position of the sun. The measurements showed that if space has any curvature, its radius of curvature is larger than 6×10^{19} cm. It is assumed that a characteristic length in astronomical measurements is the radius of the universe with its value of 10^{28} cm. Hence the radius of curvature of the

space—if it exists—is approximately eight orders of magnitude less than the radius of the universe.

Under these conditions, one can say that the application of Euclidean geometry in mechanics, is justified up to very large distances—definitely within our solar system; remembering that the distance of the earth from the sun is only 1.5×10^{13} cm and that of Pluto, the farthest planet in the solar system is 5.9×10^{14} cm from the sun. Also, these distances are certainly much smaller than the possible radius of curvature of the space. One is not sure of such 'flat' Euclidean geometry for very far-off stars in the universe.

The second law of motion, i.e. $\mathbf{F} = m\,(d^2\mathbf{x}/dt^2)$, may be interpreted so that the law remains same whether t is positive or negative (this can be seen by replacing positive t with the negative t and still the equation remains the same). Thus for a given time, the equation of motion does not distinguish between future and past. However, we know, in practice, that time can only move from present to future and never from present to past. It is now known that the Schrödinger equation in quantum mechanics handles this aspect properly.

We further assume in mechanics that the mass of a body remains constant during the motion. As we shall see in the special theory of relativity, this is strictly not true. According to this theory, the mass of a body depends on its speed $|\mathbf{v}|$ with respect to the frame of reference of the observer. The effective mass m is related to the rest mass m_0 by

$$m = \frac{m_0}{\sqrt{(1 - v^2/c^2)}}$$

Therefore if $|\mathbf{v}|$ is even 1% of c, the mass of the body will change by about 0.005%, which is quite substantial for large bodies. Also, under these conditions of fast-moving bodies, the time is dilated and length contracted. These problems are dealt with in the special theory of relativity.

On the other extreme, classical mechanics also does not hold good for very short distances or short times for bodies that are microscopic, such as atoms, molecules or electrons. Without going into details, we shall state that for these microscopic bodies, distances and times, these are the laws of quantum mechanics which are applicable. These microscopic limits are governed by the uncertainty principle according to which

$$\Delta x\, \Delta p_x \geq \hbar \tag{1.2a}$$

$$\Delta t\, \Delta E \geq \hbar \tag{1.2b}$$

where $\hbar = h/2\pi$; h being Plank's constant given by 6.626×10^{-27} erg s, and Δx is the uncertainty in displacement, Δp_x the uncertainty in linear momentum along the x-axis; Δt the uncertainty in time and ΔE, the uncertainty in energy. Such relations do hold good for macroscopic objects also but the uncertainties involved are too small to be measured. For example, if we consider a sphere of mass of 1 g and radius 1 cm moving with velocity 10 cm/s, then an uncertainty of 10^{-9} cm in displacement and that of 10^{-18} cm/s in velocity will satisfy the uncertainty principle. These uncertainties will be much smaller than the size and the velocity of the sphere and hence will not be observed. On the other hand, since the mass of particles of atomic size is

10^{-24} g, an uncertainty of 10^{-9} cm in displacement will be compatible with an uncertainty of 10^{6} cm/s in velocity and these uncertainties are of the order of the sizes and velocities of the atoms. Consequently, these can be measured and for these the laws of classical mechanics are not valid.

Because of these limitations of length, size, time and velocity, the laws of classical mechanics are applied only to objects of relatively large size, but moving slowly. In practice, this means that the problems of engineering and astronomy are amenable to the treatment of mechanics. Mechanical engineering, which is concerned with static and dynamic equilibriums and dynamic motions uses the laws of mechanics. The same is true of astronomy and planetary motion.

Some of the formulations of classical mechanics as given by Lagrange, Hamilton, Poisson and Jacobi were, however, taken over in quantum mechanics and paved the way for the development of this subject.

SUGGESTED READINGS

1. Lindsay R. B. and Margenau, H., *Physics* (Dover Pub., New York), 1957.
2. Gaillispie, C. C. (Ed. in-chief), *Dictionary of Scientific Biography*, Vols. I-XIV, (Charles Scribner's Sons, New York), 1976.
3. Capek, M., *The Philosophical Impact of Contemporary Physics* (Van. Nostrand, New York), 1961.
4. Mach, E., *The Science of Mechanics* (La Salle, Chicago), 1960.
5. Goldstein, H., *Classical Mechanics* (Addison-Wesley, Reading, Mass), 1980.

QUESTIONS

1.1 "Mathematics is the science of zeroth order and physics is the science of first order." Justify this statement.

1.2 Give proper arguments and one example to support the statement that physics is based on mathematical reasoning and experimental facts in a self-consistent manner.

1.3 Bring out the importance of studying mechanics by arguing that the concepts introduced here are of great utility in other branches of physics also.

1.4 Many authors call Galileo, the Father of Mechanics. Give your opinion in this regard.

1.5 Comment on Mach's remarks that the third law of motion was the most important work of Newton in mechanics.

1.6 Discuss briefly the limits on length, time and mass over which the laws of classical mechanics are valid.

1.7 "The laws of mechanics can be applied only to objects of large size and moving with velocities less than that of light." Discuss.

2

Vector Analysis

We know that in classical mechanics (as also in the whole of physics), many quantities such as displacement, momentum and force have associated with them not only magnitudes but also directions in space. Such quantities are called '*vectors*' and are symbolically represented by a line with an arrow, so that the length of the line represents the magnitude of the quantity and the arrow represents the direction. There are other quantities, such as mass, energy, etc. which can be represented by a magnitude in appropriate units only, but do not have any direction associated with them. Such quantities are called *scalars*. While in the case of scalars, laws of multiplication etc. are only arithmetical, in the case of vectors, they also involve the direction. Hence a specific algebra—called vector algebra–has been developed for this purpose. In the case of scalars, say masses m_1 and m_2, the addition is only arithmetical, e.g. $m = m_1 + m_2$; on the other hand, if there are two displacements, one following the other but in different directions, the resultant cannot be obtained by simply adding them arithmetically. Geometrical laws have to be applied. Hence masses are scalar and displacements are vectors.

We now carefully develop these laws for vectors in a precise and quantitative manner. It may, however, be pointed out at the very start that a very interesting feature of vector algebra is that it is independent of the coordinate system used for their description. This makes these laws attain a universality which is very helpful in physics. In fact, the statement of the laws of physics in the language of vector algebra renders them compact and simple in appearance. It is pertinent to know that the vector algebra as developed here assumes 'flat' space and hence Euclidean geometry. For spherical surfaces, the laws of vector algebra will be different than given here, and do not have the universality of the vector algebra developed for Euclidian space. The assumption of 'flat' space, in general, is used in physics. As pointed in Chapter 1, such an assumption is reasonably justified.

2.1 VECTOR—DEFINITIONS, CONCEPTS AND NOTATIONS

A physical quantity which has both magnitude and direction in space is called a vector quantity. Examples in classical mechanics are: linear displacement, linear velocity, linear momentum, linear acceleration, force, angular displacement, angular velocity, angular momentum, angular acceleration and torque. In classical electro-

dynamics, one comes across vector quantities such as electric and magnetic fields and many quantities connected with these basic vectors.

Vector Representation

A vector quantity is represented by a straight line with an arrow; the length of the line denoting the magnitude of the quantity. The direction of the line denoted by the arrow gives the direction of the vector quantity. For example, if a body moves 3 m in a given direction, then its displacement may be represented by a vector $\overrightarrow{ab}$ of length 3 cm with an arrow as shown in Fig. 2.1. The representation of displacement of 3 m by 3 cm is a question of selecting the proper scale. In this representation a represents the starting point and b the end point.

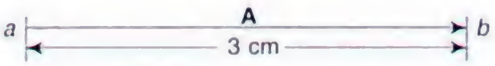


Fig. 2.1 Displacement vector

Notation

Symbolically, a vector is represented by a letter say A, on which one puts an arrow →; and denotes it as $\vec{A}$. It is also represented only by a bold letter say **A**. The letter demoting the vector may be capital as stated above or small. Hence a displacement along the x-axis may be written as **X** or **x**. Similarly, a force may be represented by **F** or **f**. Capital letters are many times preferred, but there is no set rule. The convention in this regard varies from quantity to quantity.

The symbol **A** written in this manner, contains both the magnitude and the direction. The magnitude of the vector can be written as $|\mathbf{A}|$ and sometimes only as A. The complete vector **A** can then be written as

$$\mathbf{A} = \hat{\mathbf{A}}\,A = A\,\hat{\mathbf{A}} \tag{2.1a}$$

where $\hat{\mathbf{A}}$ is called a unit vector, and denotes the direction of **A** and has magnitude of one unit. Sometimes **A** is also represented by

$$\mathbf{A} = \hat{\mathbf{e}}_A\,A \tag{2.1b}$$

where $\hat{\mathbf{e}}_A$ is unit vector, which has the same meaning as $\hat{\mathbf{A}}$ and is only an alternative way of writing a unit vector. A displacement of 3 m in any direction as given in Fig. 2.1 can then be represented as

$$\mathbf{A} = 3\hat{\mathbf{A}}$$

where unit vector $\hat{\mathbf{A}}$ has the same direction as **A** (Fig. 2.2). A displacement **A′** of, say 2 m, in the same direction as **A** can be written as

$$\mathbf{A}' = 2\hat{\mathbf{A}}$$

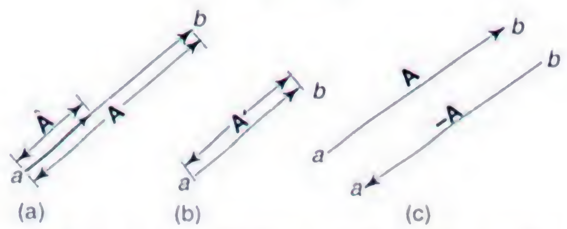


Fig. 2.2 Displacements in terms of base vectors

(c) Law of Association

It may be realised by drawing the vectors diagrammatically, that addition of vectors is commutative, i.e.

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{2.6}$$

Similarly, one can see that the addition of vectors is associative, i.e.

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \tag{2.7}$$

(d) Null Vector

If two vectors **A** and **B** have exactly the same magnitude but are opposite in direction, it is apparent that their resultant will be zero, i.e. if $\mathbf{B} = -\mathbf{A}$, then

$$\mathbf{A} + \mathbf{B} = \mathbf{A} - \mathbf{A} = \mathbf{O} \tag{2.8}$$

The resultant vector represented by **O**, has zero magnitude and is called the null or zero vector.

(e) Components of a Vector

An arbitrary vector **A** can have, in principle, any direction. It can, however, be decomposed into its components along, *x, y, z* direction. In other words, vector **A** may be written as

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z \tag{2.9}$$

where $\mathbf{A}_x$, $\mathbf{A}_y$ and $\mathbf{A}_z$ are the vector components of **A** along the three axes *x, y* and *z*. Defining **x, y** and **z** as unit vectors along *x, y* and *z*-axes respectively, we can write Eq. (2.9) as

$$\mathbf{A} = \mathbf{x}A_x + \mathbf{y}A_y + \mathbf{z}A_z \tag{2.10}$$

Let us assume for simplicity that the vector **A** is in the *xy* plane; then

$$\mathbf{A}_z = \mathbf{0} \quad \text{and} \quad \mathbf{A} = \mathbf{A}_x + \mathbf{A}_y \tag{2.10a}$$

This is shown diagrammatically in Fig. 2.6.

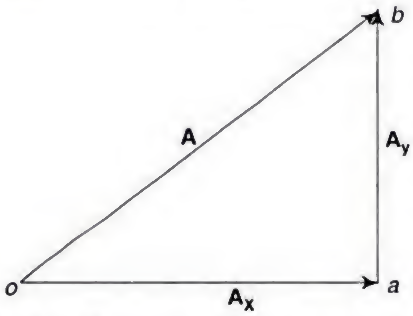


Fig. 2.6 Resolution of vector **A** along *x*- and *y*-axes

We can further represent Eq. (2.10a) in terms of unit vectors along the direction of the coordinates. Using the notation of Eq. (2.10), we can write Eq. (2.10a) as

$$\mathbf{A}A = \mathbf{x}A_x + \mathbf{y}A_y \tag{2.10b}$$

Alternatively **x** is represented by **i**, **y**, by **j**, and **z** by **k** so that an arbitrary vector **A**, in three dimension, may be written as

$$\mathbf{A} = \mathbf{A}A = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z$$
$$\equiv A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$
$$\equiv (A_x, A_y, A_z) \tag{2.11}$$

(f) Vector Derivatives

We will introduce the concept of vector derivative using the idea of vector addition, because of its usefulness in classical mechanics.

A vector derivative is basically different from the derivative of a scalar quantity, because a change in the vector involves not only a change in magnitude of the vector but also a change in its direction. This can be seen as follows : Let $\mathbf{r}(t_1)$ and $\mathbf{r}(t_2)$ represent the displacements of a body at two instants of time t_1 and t_2, as shown in Fig. 2.7.

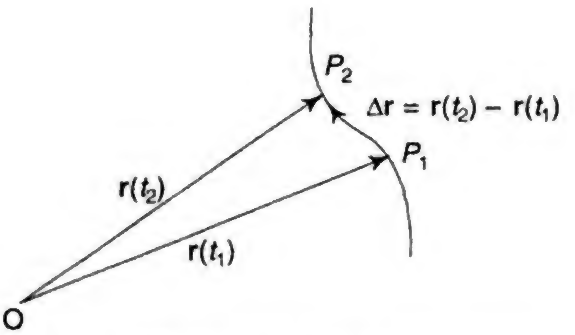


Fig. 2.7 Infinitesimal change in a vector

If the interval between t_2 and t_1 i.e. Δt is small so that $\Delta \mathbf{r}$ is quite small, then evidently, by using the law of vector addition

$$\Delta \mathbf{r} = \mathbf{r}(t_2) - \mathbf{r}(t_1)$$

The ratio $\Delta \mathbf{r}/\Delta t$ gives the rate of change of the vector $\mathbf{r}$. Also the direction of this ratio will be along P_1P_2 i.e. along the direction $\Delta \mathbf{r}$. When $\Delta t \rightarrow 0$, we represent it by dt; then $\Delta \mathbf{r} \rightarrow d\mathbf{r}$ and the ratio becomes $\Delta \mathbf{r}/\Delta t \rightarrow d\mathbf{r}/dt$. This gives us the vector definition of velocity $\mathbf{v}$, which can be written as,

$$\mathbf{v} = \frac{d(\mathbf{r})}{dt} \tag{2.12}$$

It is important to remember that $d\mathbf{r}$ is a vector increment, and involves not only a change in the magnitude of $|\mathbf{r}|$ but also a change in its direction.

EXAMPLE 2.1

A car travels due east on a level road for 10 min at 60 kmph and then due north at 50 kmph for 12 min before stopping. Find the resultant displacement from the starting point.

Solution

We define the x- and y-axes along the east and north directions respectively. Now displacement along east is given by

$$|\mathbf{A}| = \frac{60 \times 10}{60} = 10 \text{ km}$$

multiplying two vector quantities, (force **F** and displacement **r**) in such a manner that the component of one quantity **F** along the other vector quantity **r** is multiplied. The final result is the arithmetical product of two vector quantities in the same direction, i.e. **F** cos θ and **r**, hence the name 'scalar' for this product.

Symbolically, the scalar product of two vectors is written by putting a centre dot ($\cdot$) between the two vectors and is also called their dot product. As for example, in the above example

$$W = \mathbf{F} \cdot \mathbf{r} \tag{2.14}$$

In general, the dot product of any two vectors, say **A** and **B**, is written as **A** · **B**, and is read as **A** dot **B** (Fig. 2.11). Obviously, the scalar product depends only on the relative directions of **A** and **B** and hence on the angle between the two vectors. It is independent of the absolute directions of **A** and **B**, and therefore, will have the same value irrespective of the coordinate system.

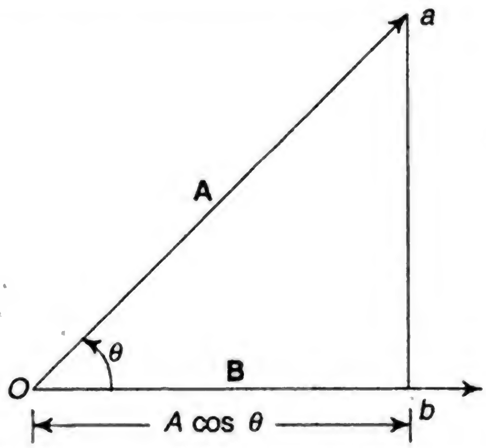


Fig. 2.11 Scalar (or dot product) of vectors

Apart from the example of work as a scalar product of two vectors, there are many other examples of scalar product in physics, such as

1. Power = **F** · **v**
   were **v** is the velocity vector and **F** is the force vector.
2. Potential energy $U(r)$, given by

$$-U(r) = \int_r^{\infty} \mathbf{F} \cdot d\mathbf{r} = -\int \frac{\partial U}{\partial r} \cdot d\mathbf{r}$$

Some of the properties of the scalar product or dot product of two vectors are given below.

(a) The scalar product is commutative, i.e. it is independent of the order of multiplication

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{2.15}$$

This can be seen by realising that cos θ = cos $(-\theta)$ so that whether the angle in Fig. 2.11 is measured from **B** to **A**, i.e. θ or from **A** to **B**, i.e. $-\theta$, the value of the cos θ of the angle between the two vectors is the same.

(b) The scalar product of vectors obeys the distributive law, i.e.

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \tag{2.16}$$

which is the same as that of the rotation caused by it. We shall see in Sec. 2.5 that this direction is perpendicular to both **r** and **F**. Such a situation corresponds to the vector product of the two vectors **r** and **F** and is written as

$$\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F} \tag{2.24}$$

Other examples of vector products are: angular momentum, couple, force due to a magnetic field on a moving charge, etc. as we will show subsequently.

In general, the vector product **C** of two vectors **A** and **B** is symbolically represented as

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AB \sin \theta)\, \hat{\mathbf{C}} \tag{2.25}$$

Here $\hat{\mathbf{C}}$ is the unit vector along the direction of **C**. Because of the convention of putting the cross (×) between the multiplying vectors, the vector product is also sometimes called cross product and is read as **A** cross **B**.

The physical meaning of the vector product is that given in the example of torque, as stated above. The magnitude of the vector product of two vectors, is the product of the magnitude of vector **A** and the component of **B** in the direction perpendicular to that of **A**, i.e. $|\mathbf{C}| = C = AB \sin \theta$, where θ is the angle between the first vector **A** and the second vector **B**. The direction of **C** is taken perpendicular to both **A** and **B** and it is taken to be positive when **C** is in the direction represented by the rotation of **A** towards **B**. This convention of defining the direction of the vector product is called the right-hand rule, according to which if we imaginarily grasp the vector **C** with the right-hand so that the grasping fingers represent the rotation of **A** towards **B**, then the direction of the thumb represents the direction of **C**. This is shown in Fig. 2.14. Sometimes a right-hand screw is used to describe the direction of the vector **A** × **B**. The direction of vector **A** × **B** is taken in such a manner that if the rotation of **A** towards **B** represents the rotation of the right-hand screw through θ, then the motion of the screw is along the vector **A** × **B**, as shown in Fig. 2.15. After this introduction, the different properties of the vector product can now be discussed.

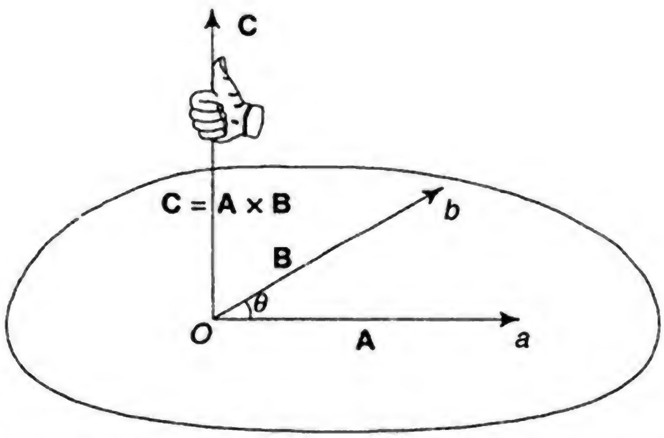


Fig. 2.14 Right-hand rule and the vector product of two vectors

(i) The vector product of the two vectors is not commutative: This means that if we change the order of the vectors, the resultant product is not the same. In fact, it can be shown that

$$\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A}) \tag{2.26}$$

This can be seen by comparing Fig. 2.15 representing vector $\mathbf{A} \times \mathbf{B}$ and Fig. 2.16 depicting the vector product $\mathbf{B} \times \mathbf{A}$. It is obvious that while the magnitude of both the vector products is the same, i.e. $|\mathbf{A} \times \mathbf{B}| = |\mathbf{B} \times \mathbf{A}| = AB \sin \theta$, their directions are opposite to each other. Hence the negative sign on the right side in Eq. (2.26).

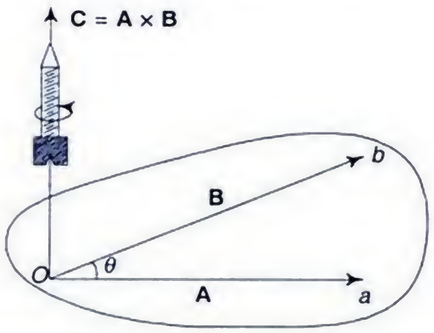


Fig. 2.15 The right-hand screw and the vector product $\mathbf{A} \times \mathbf{B}$

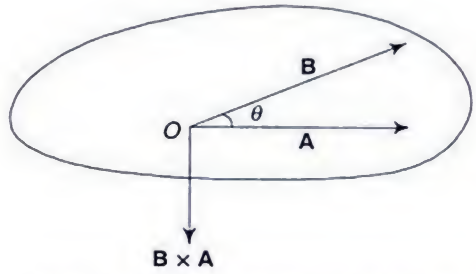


Fig. 2.16 The representation of the vector product $\mathbf{B} \times \mathbf{A}$

(ii) *Surface area as a vector product:* Suppose two vectors $\mathbf{A}$ and $\mathbf{B}$ inclined at angle θ constitute two adjacent sides of a parallelogram whose area is given by $AB \sin \theta$ (Fig. 2.17). Comparing this with Eq. (2.25), one can see that the cross product of $\mathbf{A}$ and $\mathbf{B}$ represents the area of the parallelogram with sides A and B. The magnitude of $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is equal to the area of the parallelogram, and as stated earlier, its direction represents the order in which $\mathbf{A}$ and $\mathbf{B}$ occur. Because of the information which one gets in this manner, not only about the magnitude of the area but also about the order in which $\mathbf{A}$ and $\mathbf{B}$ occur, it is a convention to express the area of a surface by the vector product.

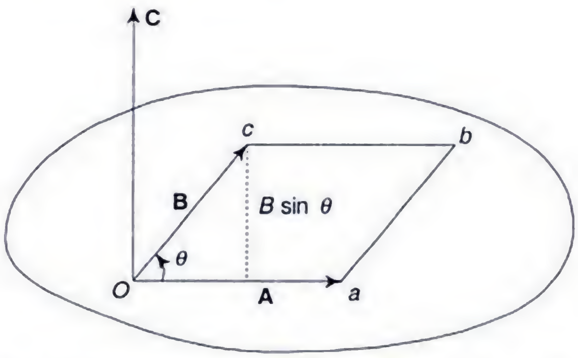


Fig. 2.17 Area of a parallelogram as a vector product of the vectors representing its adjacent sides

Even if the area is irregular, then too, the vector representing the area can be expressed in such a manner that the direction of the vector denotes the sense in which the boundary curve is traversed in accordance with the right-hand rule (Fig. 2.18). The vector representing the area is, of course, perpendicular to the area. In this diagram, $d\sigma$ denotes the vector, representing the area and $\mathbf{n}$ is the unit vector

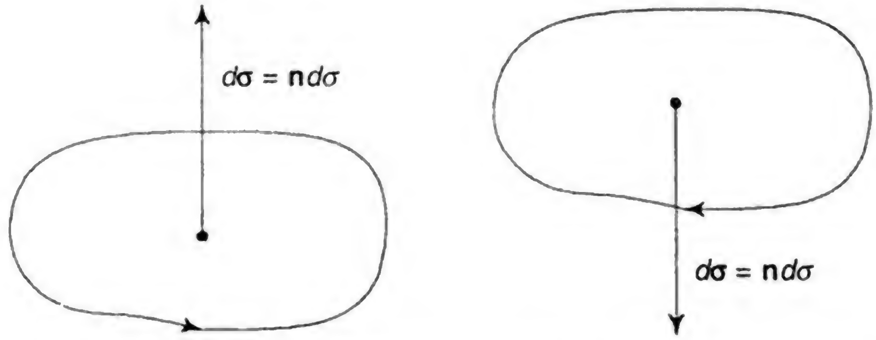


Fig. 2.18 Vector representation of area of an irregular surface

along $d\boldsymbol{\sigma}$ and $|d\boldsymbol{\sigma}|$ is its magnitude. (The change in direction of $d\boldsymbol{\sigma}$ with reversal in the sense of the boundary curve should be noted.)

In the case of closed surfaces, such as those of a sphere, parallelepiped or tetrahedron, the direction of the sides is selected in such a manner that the vector representing the area of the surface always points outward.

The total vector sum of the area of a closed surface can easily be seen to be zero, because for every vector area in one direction there is another vector area of the same magnitude but in the opposite direction. It can be easily verified for a sphere or cube, holds good for all closed surfaces.

(iii) *Distributive law:* The vector product obeys the distributive law with respect to addition, i.e. for given three vectors **A**, **B** and **C**

$$\mathbf{C} \times (\mathbf{A} + \mathbf{B}) = \mathbf{C} \times \mathbf{A} + \mathbf{C} \times \mathbf{B} \tag{2.27}$$

This can be seen, by referring to Fig. 2.19. In this figure the prism *MNOPQR* has sides **A**, **B**, **A** + **B** and **C**. The areas of the five faces of the prism are:

$$\text{Area } MRQN = \mathbf{C} \times \mathbf{A}$$

$$\text{Area } NOPQ = \mathbf{C} \times \mathbf{B}$$

$$\text{Area } MOPR = (\mathbf{A} + \mathbf{B}) \times \mathbf{C}$$

$$\text{Area } MNO = \frac{1}{2}\mathbf{A} \times \mathbf{B},$$

and

$$\text{Area } PRQ = \frac{1}{2}(-\mathbf{B}) \times (-\mathbf{A}) = \frac{1}{2}\mathbf{B} \times \mathbf{A}$$

In view of the comments made in the above discussion of area, the vector sum of the entire surface area of the prism is zero.

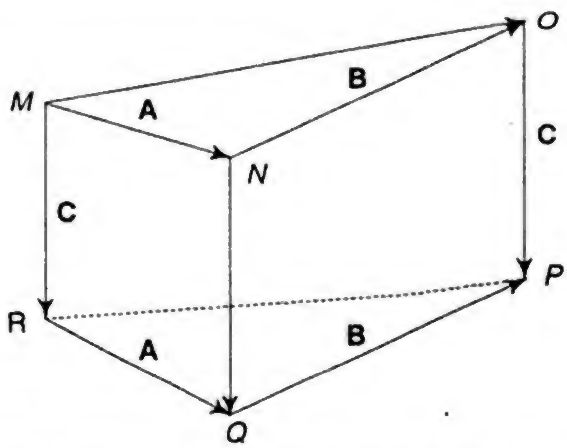


Fig. 2.19 Prism MNOPQR with the sides **A**, **B**, **A** + **B** and **C**, used in establishing the distribution law for vector product

Hence $$\mathbf{C} \times \mathbf{A} + \mathbf{C} \times \mathbf{B} + (\mathbf{A} + \mathbf{B}) \times \mathbf{C} + \frac{1}{2}\,\mathbf{A} \times \mathbf{B} + \frac{1}{2}\,\mathbf{B} \times \mathbf{A} = 0$$

Using the relation,

$$-(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \times \mathbf{A}$$

we get

$$\mathbf{C} \times \mathbf{A} + \mathbf{C} \times \mathbf{B} = -(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{C} \times (\mathbf{A} + \mathbf{B})$$

which proves Eq. (2.27).

(iv) *Vectors product of unit vectors:* It can be easily verified from the basic definition of vector product that the following relationships hold good for vector products of unit vectors along right-handed orthogonal coordinates of Fig. 2.20 (a).

1. $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$
2. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$; $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
3. $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$; $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ (2.28)

For the left-handed coordinate system, as given in Fig. 2.20(b), we get the following relations:

$$\mathbf{i} \times \mathbf{j} = -\,\mathbf{k};\ \mathbf{j} \times \mathbf{k} = -\mathbf{i};\ \mathbf{k} \times \mathbf{i} = -\mathbf{j}$$

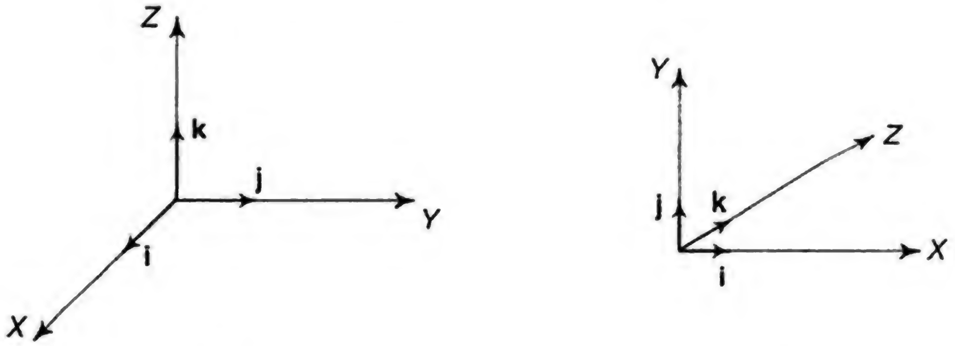


Fig. 2.20 Unit vectors in the right-handed and left-handed coordinate systems

(v) *Vector product in terms of components:* One can represent the vector product $\mathbf{A} \times \mathbf{B}$ in terms of three unit vector. For this purpose, it may be recalled that

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_x\mathbf{k}$$
$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

and using the various relationships of vector products of unit vectors as given above in Eq. (2.28), we get

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= A_xB_y\,(\mathbf{i} \times \mathbf{j}) + A_xB_z\,(\mathbf{i} \times \mathbf{k}) + A_yB_x\,(\mathbf{j} \times \mathbf{i}) + A_yB_z\,(\mathbf{j} \times \mathbf{k})\\ &\quad + A_zB_x\,(\mathbf{k} \times \mathbf{i}) + A_zB_y\,(\mathbf{k} \times \mathbf{j})\\ &= (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j}\\ &\quad + (A_xB_y - A_yB_x)\,\mathbf{k} \qquad (2.29)\end{aligned}$$

From the structure of this equation, it is easily seen that the three components of the vector $\mathbf{A} \times \mathbf{B}$ are given by

$$\begin{aligned}(\mathbf{A} \times \mathbf{B})_x &= (A_yB_z - A_zB_y)\\ (\mathbf{A} \times \mathbf{B})_y &= (A_zB_x - A_xB_z)\\ (\mathbf{A} \times \mathbf{B})_z &= (A_xB_y - A_yB_x) \qquad (2.30)\end{aligned}$$

The vector $\mathbf{A} \times \mathbf{B}$ can also be represented in terms of a determinant as follows:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{2.31}$$

EXAMPLE 2.5

With respect to a particular coordinate system, a force $\mathbf{F} = (4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$ N is applied at the point B (5, –1, 3) of a body which rotates about an axis through the point A (1, 2, –1). Find the torque acting on the body if the position coordinates are expressed in metres.

Solution

In this case the force is applied at B (5, –1, 3) and the body rotates about an axis through the point A (1, 2, –1), therefore,

$$\mathbf{r} = \overrightarrow{AB} = [(5-1)\mathbf{i} + (-1-2)\mathbf{j} + (3+1)\,\mathbf{k}]\text{ m}$$
$$= (4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})\text{ m}$$

Applied force

$$\mathbf{F} = (4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})\text{N}$$

Therefore, torque $\boldsymbol{\Gamma}$ is given by

$$\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix}$$
$$= [(-9+8)\,\mathbf{i} - (12-16)\,\mathbf{j} + (-8+12)\,\mathbf{k}]\text{ N m}$$
$$= (-\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})\text{ N m}$$

Magnitude of torque,

$$|\boldsymbol{\Gamma}| = [(-1)^2 + (4)^2 + (4)^2]^{1/2}$$
$$= \sqrt{33}\text{ N m}$$

Further,

$$\mathbf{r} \cdot \boldsymbol{\Gamma} = (4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \cdot (-\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$$
$$= -4 - 12 + 16 = 0$$
$$\mathbf{F} \cdot \boldsymbol{\Gamma} = (4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot (-\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$$
$$= -4 - 8 + 12 = 0$$

These results show that $\boldsymbol{\Gamma}$ is perpendicular to both $\mathbf{r}$ and $\mathbf{F}$ as anticipated on the basis of the statement made in the text.

EXAMPLE 2.6

Find the area of the parallelogram whose adjacent sides are given, in metres, by

$$\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{B} = 3\mathbf{i} + 2\mathbf{k}$$

Solution

If $\mathbf{A}$ and $\mathbf{B}$ are the adjacent sides of a parallelogram, then its area is given by

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 3 & 0 & 2 \end{vmatrix}$$
$$= (2-0)\,\mathbf{i} - (2-3)\,\mathbf{j} + (0-3)\,\mathbf{k}$$
$$= (2\mathbf{i} + \mathbf{j} - 3\mathbf{k})\text{ m}^2$$

Magnitude of the area,

$$|\mathbf{C}| = [(2)^2 + (1)^2 + (-3)^2]^{1/2}\ \mathrm{m}^2$$
$$= \sqrt{14}\ \mathrm{m}^2$$

The vector **C** is perpendicular to the plane of **A** and **B**.

2.4 PRODUCT OF THREE VECTORS

Many times one comes across repeated products of vectors involving three or more vectors. Among these, the following two products of three vectors are of special physical interest:

(i) Scalar triple product, which is written as $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$, and
(ii) vector triple product, which is written as $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

We shall see later that the volume of a parallelepiped can be represented by a scalar product, also called mixed triple product.

2.4.1 Scalar Triple Product

(i) For any three vectors **A**, **B** and **C** expressed in terms of their components, one can write scalar triple product

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= A_x\,(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z\,(B_xC_y - B_yC_x)$$

This can be written in the determinant form as

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \qquad (2.32)$$

(ii) Since interchanging of two rows in a determinant changes its sign, the determinant for $\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C})$ should be negative of that for $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

Interchanging the rows once more, we have

$$\mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = -\,\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C})$$
$$= \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$

Repeating these steps, it can be shown that

(a) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$ (2.33)

i.e. the scalar product of the three vectors **A**, **B** and **C** is the same if these are written in the cyclic or clockwise order as shown in Fig. 2.21 (a).

(b)
$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = -\,\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C})$$
$$= -\mathbf{C} \cdot (\mathbf{B} \times \mathbf{A})$$
$$= -\mathbf{A} \cdot (\mathbf{C} \times \mathbf{B}) \qquad (2.34)$$

i.e. the sign of scalar triple product is changed if the vectors are considered in anticlockwise or anticyclic order, as in Fig. 2.21 (b).

(iii) From Eq. (2.33), we have

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \qquad (2.35)$$

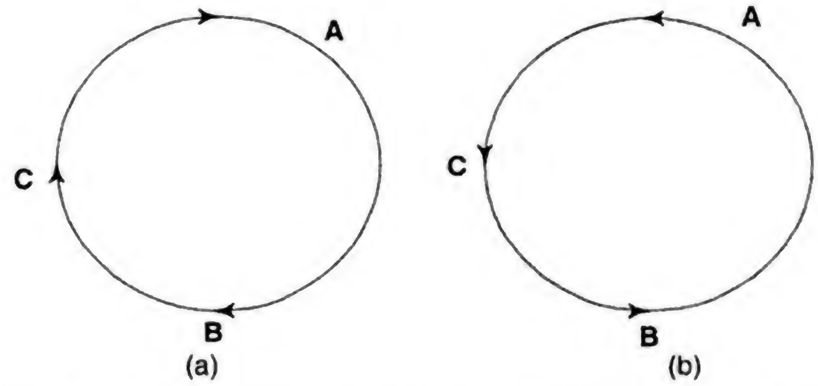


Fig. 2.21 Diagram illustrating the cyclic and anti-cyclic sequence as used in scalar triple product

because the scalar product of two vectors ($\mathbf{A} \times \mathbf{B}$) and $\mathbf{C}$ is commutative. Equation (2.35) shows that the dot and cross in the scalar triple product may be interchanged without altering the product. In view of this, the scalar triple product is also written as (**ABC**).

(iv) In order to obtain geometrical meaning of scalar triple product, we suppose that the vectors **A**, **B** and **C** define the sides of a parallelepiped, as shown in Fig. 2.22.

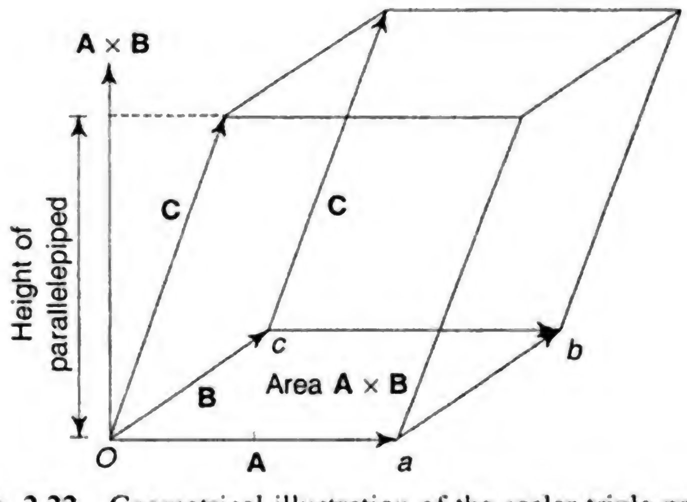


Fig. 2.22 Geometrical illustration of the scalar triple product

As discussed in the previous section, the area of the face *Oabc* is given by $\mathbf{A} \times \mathbf{B}$. This is represented by means of a vector perpendicular to the plane *Oabc*. Now the scalar product of $\mathbf{C}$ with $\mathbf{A} \times \mathbf{B}$, is given by

$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ = (Magnitude of the vector $\mathbf{A} \times \mathbf{B}$) × (projection of vector **C** on the vector $\mathbf{A} \times \mathbf{B}$)
= (Area of the base parallelogram defined by vectors **A** and **B**) × (height of the parallelepiped)
= volume of the parallelepiped

In view of Eqs (2.33) and (2.35), we can say that the volume of the parallelepiped is given by

$$V = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$
$$= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$$

$$= \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$
$$= (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \tag{2.36}$$

Geometrically, Eq. (2.36) means that the volume of a parallelepiped can be found by using any of its faces as a base. Hence the scalar triple product of the three sides of a parallelepiped taken in cyclic order defines the volume of the parallelepiped. In this case the volume will have a positive sign. However, if the order of the vectors is taken as anticyclic, then the volume comes out to be negative.

It is interesting to note that if $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = 0$, then either the magnitude of one of the vectors is zero, or $\mathbf{C}$ is orthogonal to $\mathbf{A} \times \mathbf{B}$, which implies that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are coplanar.

2.4.2. Vector Triple Product

The vector triple product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ satisfies the following relationship

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B}) \tag{2.37}$$

This can be seen as follows:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x \end{vmatrix}$$

$$= [A_y\,(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)]\mathbf{i} + [A_x(B_yC_z - B_zC_y) - A_x(B_xC_y - B_yC_x)]\mathbf{j} + [A_x\,(B_zC_x - B_xC_z) - A_y\,(B_yC_z - B_zC_y)]\mathbf{k}$$

$$= [B_xA_xC_x + B_xA_yC_y + B_xA_zC_z - C_xA_xB_x - C_xA_yB_y - C_xA_zB_z]\mathbf{i} + [B_yA_xC_x + B_yA_yC_y + B_yA_zC_z - C_yA_xB_x - C_yA_yB_y - C_xA_zB_z]\mathbf{j} + [B_zA_xC_x + B_zA_yC_y + B_zA_zC_z - C_zA_xB_x - C_xA_yB_y - C_zA_zB_z]\mathbf{k}$$

$$= B_x(A_xC_x + A_yC_y + A_zC_z)\mathbf{i} + B_y(A_xC_x + A_yC_y + A_zC_z)\mathbf{j} + B_z(A_xC_x + A_yC_y + A_zC_z)\mathbf{k} - C_x(A_xB_x + A_yB_y + A_zB_z)\mathbf{i} - C_y(A_xB_x + A_yB_y + A_zB_z)\mathbf{j} - C_z(A_xB_x + A_yB_y + A_zB_z)\mathbf{k}$$

$$= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

It may be realised that the resultant vector $\mathbf{E} \equiv \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is perpendicular to both vectors $\mathbf{A}$ and $(\mathbf{B} \times \mathbf{C})$. If $(\mathbf{B} \times \mathbf{C}) \equiv \mathbf{D}$, then $\mathbf{E}$ is perpendicular to $\mathbf{D}$. But $\mathbf{D}$ is perpendicular to the plane containing $\mathbf{B}$ and $\mathbf{C}$. It is apparent, therefore, that $\mathbf{E}$ is in the same plane as $\mathbf{B}$ and $\mathbf{C}$. Furthermore, $\mathbf{E}$ is perpendicular to the plane defined by the vectors $\mathbf{A}$ and $\mathbf{D}$. Since $\mathbf{E}$ lies in plane of $\mathbf{B}$ and $\mathbf{C}$, the plane defined by these two vectors, $\mathbf{B}$ and $\mathbf{C}$ should be perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{D}$. We leave it to the reader to draw such planes.

EXAMPLE 2.7

The vectors defining the three edges of a parallelepiped are given to be

$$\mathbf{A} = -4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$$
$$\mathbf{B} = -5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{C} = 4\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$$

Find the volume of the parallelepiped if the coordinates are expressed in metres.

Solution

Vectors **A**, **B** and **C** representing the edges of the given parallelepiped are shown in Fig. 2.23. The volume of the parallelepiped is given by

$$V = \mathbf{A}\quad(\mathbf{B}\times\mathbf{C})\ \text{m}^3$$

$$= \begin{vmatrix} -4 & 3 & 5 \\ -5 & 4 & 2 \\ 4 & 5 & -3 \end{vmatrix}\ \text{m}^3$$

$$= [-4(-12-10) + 3(8-15) + 5(-25-16)]\ \text{m}^3$$

$$= (88 - 21 - 205)\ \text{m}^3$$

$$= -138\ \text{m}^3$$

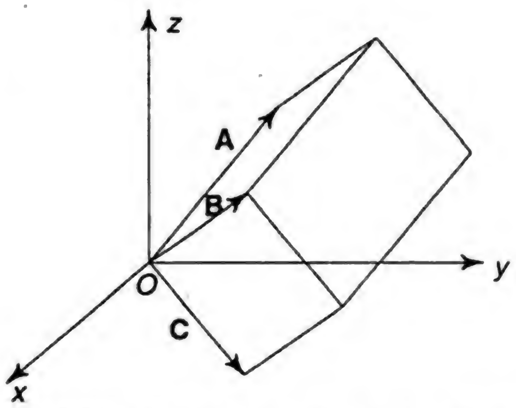


Fig. 2.23 Geometrical representation of the vectors **A**, **B** and **C** as edges of a parallelepiped

EXAMPLE 2.8

With reference to a particular coordinate system, the three vectors **A**, **B** and **C** are given to be

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} - \mathbf{k},\ \mathbf{B} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k},\ \mathbf{C} = 3\mathbf{i} - 4\mathbf{k}$$

Determine $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ and show that it is equal to $\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$.

Solution

Here

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$
$$\mathbf{B} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$
$$\mathbf{C} = 3\mathbf{i} + 0\mathbf{j} - 4\mathbf{k}$$

Therefore

$$\mathbf{B}\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 3 & 0 & -4 \end{vmatrix}$$
$$= (4-0)\,\mathbf{i} - (-8-9)\,\mathbf{j} + (0+3)\,\mathbf{k}$$
$$= 4\mathbf{i} + 17\mathbf{j} + 3\mathbf{k}$$

and

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 4 & 17 & 3 \end{vmatrix}$$
$$= (6+17)\mathbf{i} - (3+4)\mathbf{j} + (17-8)\mathbf{k}$$
$$= 23\mathbf{i} - 7\mathbf{j} + 9\mathbf{k}$$

Now

$$\mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}\,(\mathbf{A}\cdot\mathbf{B})$$
$$= (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})\,[(\mathbf{i} + 2\mathbf{j} - \mathbf{k})\cdot(3\mathbf{i} + 0\mathbf{j} - 4\mathbf{k})] - (3\mathbf{i} + 0\mathbf{j} - 4\mathbf{k})[(\mathbf{i} + 2\mathbf{j} - \mathbf{k})\cdot(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})]$$
$$= (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})\,[(3+0+4)] - (3\mathbf{i} + 0\mathbf{j} - 4\mathbf{k})]\,[(2-2-3)]$$
$$= 7(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) + 3(3\mathbf{i} - 4\mathbf{k})$$
$$= 23\mathbf{i} - 7\mathbf{j} + 9\mathbf{k}$$

Comparing this result with that obtained for $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$, we see that

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}\,(\mathbf{A}\cdot\mathbf{B})$$

2.5 ROTATION AS A VECTOR

Till now we have dealt with vector quantities that have linear direction associated with them. Such vectors are called polar or radial vectors and are exemplified by displacement, velocity, force, etc.

There are, however, certain quantities, such as angular displacement, angular velocity, torque, etc. which are associated with the rotational motion. These quantities can also be represented by vectors, the sense of the rotation being connected with the direction of the vector. For example, in Fig. 2.24, a very small angular displacement $\Delta\theta$ is represented by a vector along OZ. The convention adopted in this regard is that if the rotation of a right-handed screw follows the rotational direction (i.e. the rotation of the position vector $\overrightarrow{OP}$ to $\overrightarrow{OQ}$), then the forward motion of the screw, represents the direction of the vector representing $\Delta\boldsymbol{\theta}$. Evidently, if the rotation is in the opposite direction, the vector representing it will also be reversed. Thus, the physical quantity associated with rotational motion and the sense of rotation can be represented by vectors and these are helpful in analysing rotational motion through techniques of vector algebra.

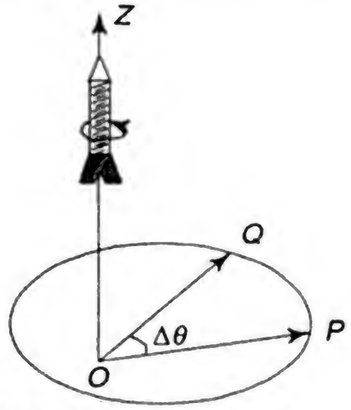


Fig. 2.24 Representation of angular displacement

2.5.1 Vector Algebra and Vector Calculus

Rotational Quantities

Some of the rotational quantities that can be represented by vectors are small angular displacement $\quad \Delta\boldsymbol{\theta}$

$$\text{Angular velocity} \qquad \boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt}$$

$$\text{Angular momentum} \qquad \mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{2.38}$$

$$\text{Torque} \qquad \boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}$$

Suppose a vector $\mathbf{r} = \overrightarrow{OP_1}$ initially at angular position θ is rotated through a small angle $\Delta\boldsymbol{\theta}$ (Fig. 2.25).

How should we represent the small angular displacement $\Delta\boldsymbol{\theta}$? It is easy to see that the magnitude $|\Delta\boldsymbol{\theta}|$ is given by

$$|\Delta\boldsymbol{\theta}| = \frac{|\Delta\mathbf{s}|}{r} \tag{2.39}$$

It can be seen that if $\Delta\mathbf{s}$ goes from P_1 to P_2, $\Delta\boldsymbol{\theta}$ has one sense, if $\Delta\mathbf{s}$ goes from P_2 to P_1, then $\Delta\boldsymbol{\theta}$ will have the opposite sense. For the case when $\Delta\mathbf{s}$ goes from P_1 to P_2, as shown in Fig. 2.25, $\Delta\boldsymbol{\theta}$ is conventionally represented by a vector along OZ, so that it is perpendicular to the plane containing $\mathbf{r}$ and $\Delta\mathbf{s}$ and points along the direction of the forward motion of the right handed screw, the rotational motion of the screw being in the same sense in which OP_1 goes to OP_2.

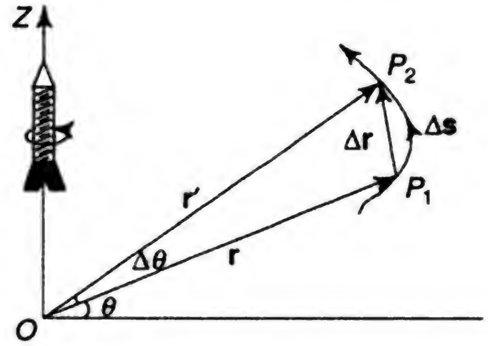


Fig. 2.25 Illustration of a convention for rotational vectors

As a matter of fact, for the case when P_1P_2 is very small, and is the arc of a circle, it can be seen that P_1P_2, OP_1 and OZ are perpendicular to each other. Hence, one can rewrite Eq. (2.39) vectorially for very small value of $\Delta\theta$ as

$$\Delta s = \Delta\boldsymbol{\theta} \times \mathbf{r}$$

For very small values of $\Delta\boldsymbol{\theta}$, one can write $\Delta s \equiv \Delta\mathbf{r}$, so that the above equation becomes,

$$\Delta\mathbf{r} = \Delta\boldsymbol{\theta} \times \mathbf{r} \tag{2.40}$$

Obviously $\Delta\boldsymbol{\theta}$ is perpendicular to both $\mathbf{r}$ and $\Delta\mathbf{r}$.

Because of the smallness of $\Delta\boldsymbol{\theta}$ and $\Delta\mathbf{r}$, we can write Eq. (2.40) as

$$d\mathbf{r} = d\boldsymbol{\theta} \times \mathbf{r} \tag{2.41}$$

Equations (2.40) and (2.41) fix the direction of $\Delta\boldsymbol{\theta}$ or $d\boldsymbol{\theta}$. Comparing these equations with the relationship for cross product, we see that the direction of $d\boldsymbol{\theta}$ is such that if it is rotated towards $\mathbf{r}$, this corresponds to the rotation of a right-handed screw whose forward motion is represented by $d\mathbf{r}$. Obviously, $d\boldsymbol{\theta}$ is perpendicular to both $\mathbf{r}$ and $d\mathbf{r}$. We have thus established that $\Delta\boldsymbol{\theta}$ or $d\boldsymbol{\theta}$ behaves like a vector.

We can now prove that the addition of angular displacement vectors obeys the commutative relationship, i.e.

$$d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2 = d\boldsymbol{\theta}_2 + d\boldsymbol{\theta}_1 \tag{2.42}$$

Let us consider the case of two successive rotations, $d\boldsymbol{\theta}_1$ and $d\boldsymbol{\theta}_2$ such that initial position vectors for the two cases are $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Then

$$d\mathbf{r}_1 = d\boldsymbol{\theta}_1 \times \mathbf{r}_1$$

and

$$d\mathbf{r}_2 = d\boldsymbol{\theta}_2 \times \mathbf{r}_2 \tag{2.43}$$

But $d\mathbf{r}_1$ is obtained from the rotation of $\mathbf{r}_1$ to $\mathbf{r}_2$ through the angle $d\boldsymbol{\theta}_1$ and

$$\mathbf{r}_2 = \mathbf{r}_1 + d\mathbf{r}_1$$

as shown in Fig. 2.26. In view of this, the expression for $d\mathbf{r}_2$ becomes

$$d\mathbf{r}_2 = d\boldsymbol{\theta}_2 \times [\mathbf{r}_1 + d\mathbf{r}_1]$$

The resultant vector is given by

$$\begin{aligned}\overrightarrow{OP_3} &= \mathbf{r}_1 + d\mathbf{r}_{12} \\ &= \mathbf{r}_1 + d\mathbf{r}_1 + d\mathbf{r}_2 \\ &= \mathbf{r}_1 + d\boldsymbol{\theta}_1 \times \mathbf{r}_1 + d\boldsymbol{\theta}_2 \times (\mathbf{r}_1 + d\mathbf{r}_1) \\ &= \mathbf{r}_1 + (d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2) \times \mathbf{r}_1\end{aligned}$$

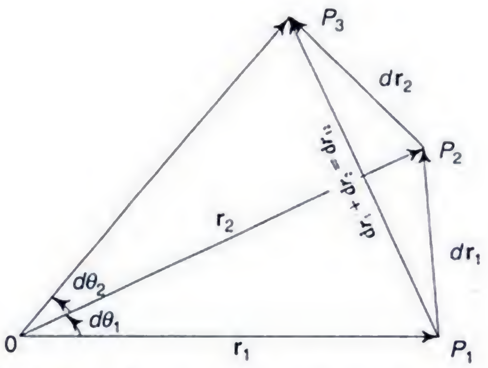


Fig. 2.26 Addition of two small angular displacements

Here we have neglected $d\theta_2 \times d\mathbf{r}_1$, for which the magnitude is very small.

In a similar manner, it can be shown that if we first rotate vector $\mathbf{r}_1$ through $d\theta_2$ and then through $d\theta_1$, the final vector $\overrightarrow{OP_3}$ is given by

$$\overrightarrow{OP_3} = \mathbf{r}_1 + d\mathbf{r}_{21} = \mathbf{r}_1 + d\mathbf{r}_2 + d\mathbf{r}_1$$
$$= \mathbf{r}_1 + (d\theta_2 + d\theta_1) \times \mathbf{r}_1$$

Equating the right-hand sides of the above equations, we see that

$$d\theta_1 + d\theta_2 = d\theta_2 + d\theta_1$$

i.e. the sum of infinitesimal angular displacements is commutative in vector notation.

It may further be noted that while infinitesimal displacement $d\theta$ behaves like a vector whose addition is commutative, the large angular rotations are not vectors. In Fig. 2.27, we have shown the case when rotation is 90°, and the rotations are sequentially around y and x-axes [Fig. 2.27 (a), (b) (c)] or x and y-axis [Fig. 2.27, (a′), (b′), (c′)]. The resultant is different in two cases. Hence it does not obey the commutative rule. Therefore finite rotations cannot be regarded as vector quantities.

The angular velocity ω, is, of course, connected with $d\theta$ through the following equation

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

The vector properties of ω are contained in the quantity $d\theta$. All the arguments for $d\theta$ will apply to the vector representing angular velocity ω. Thus the angular velocity vector is in the same direction as $d\theta$.

Further, dividing both sides of Eq. (2.40) by Δt and finding the limit when $\Delta t \to 0$, we get

$$\lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} \times \mathbf{r}$$

or
$$\frac{d\mathbf{r}}{dt} = \frac{d\theta}{dt} \times \mathbf{r}$$

or
$$\mathbf{v} = \omega \times \mathbf{r} \qquad (2.44)$$

This equation gives the relationship of linear velocity $\mathbf{v}$ associated with the radial increment $d\mathbf{r}$ and the angular velocity ω and the radial vector $\mathbf{r}$.

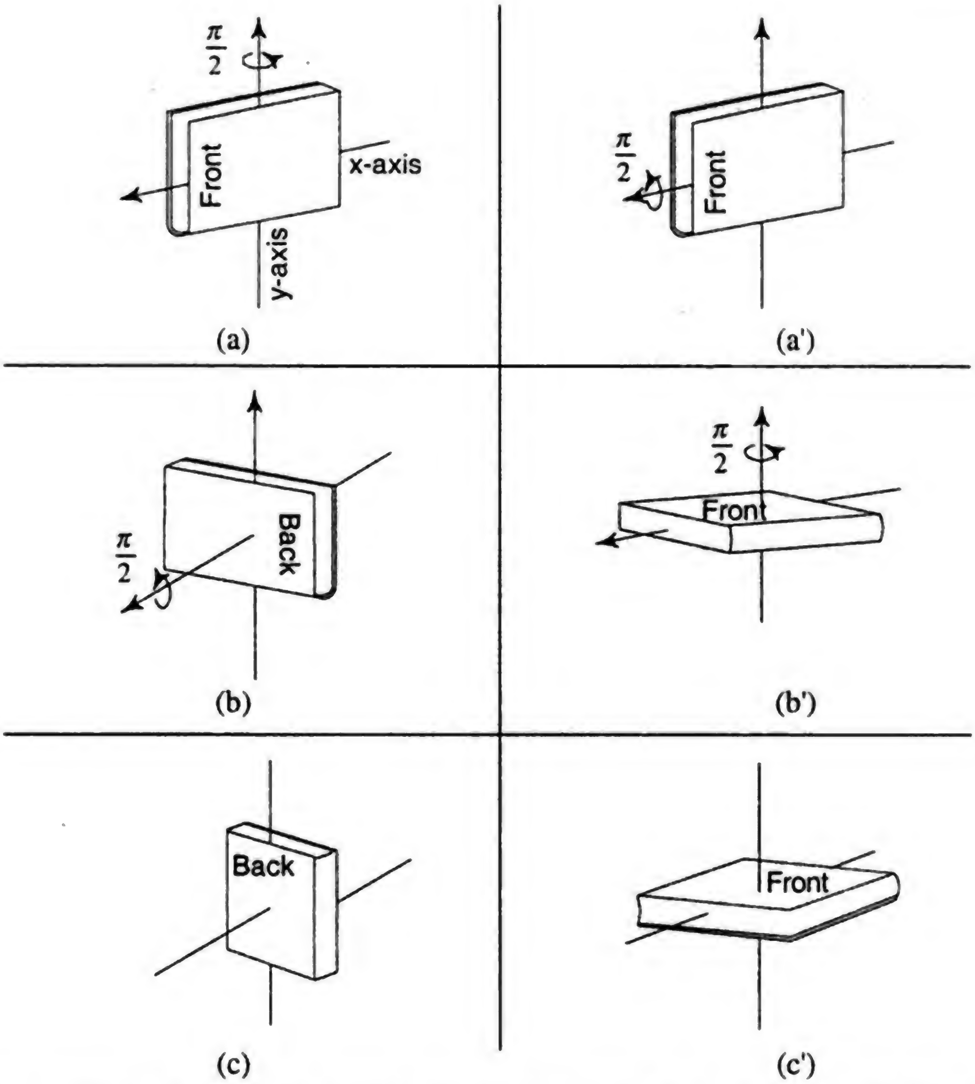


Fig. 2.27 Illustration of the finite rotation, showing it to be not obeying the commutative rule, *a, b, c* represent rotations about *y*- and *x*-axes, and a', b', c' represent rotations about *x*- and *y*-axes.

From the definition of torque $\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F}$ and angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, it is easy to see that $\mathbf{\Gamma}$ and $\mathbf{L}$ are also vectors. The direction of these vectors is, of course, such that: (a) in the case of torque $\mathbf{\Gamma}$, its direction is given by the forward motion of a right-handed screw, whose rotation is the same as that of $\mathbf{r}$ towards $\mathbf{F}$; and (b) in the case of angular momentum, again, the direction of $\mathbf{L}$ is given by the forward motion of the right-hand screw rotating in the direction of $\mathbf{r}$ towards $\mathbf{p}$.

2.5.2 Rotation of Coordinate Axes

The solution of many physical problems becomes easier by rotating the whole coordinate system. For example, in the study of rigid bodies, one normally considers two coordinate systems: one fixed in space and the other in the rigid body. The second or body coordinate system shares the motion of the rigid body including rotation with respect to the space coordinate system. One, therefore, is required to understand such a relative rotation of the two coordinate systems.

Though we will discuss different types of coordinate systems and their use in different situations in the next chapter, we use here the three-dimensional cartesian coordinate system *xyz*. Consider a rotation around the *z*-axis, as shown in Fig. 2.28. Consider the point $P(x, y, z)$ in the coordinate system x, y, z. The displacement vector $\mathbf{r} = \overrightarrow{OP}$, is then given, in the x, y, z coordinate system by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

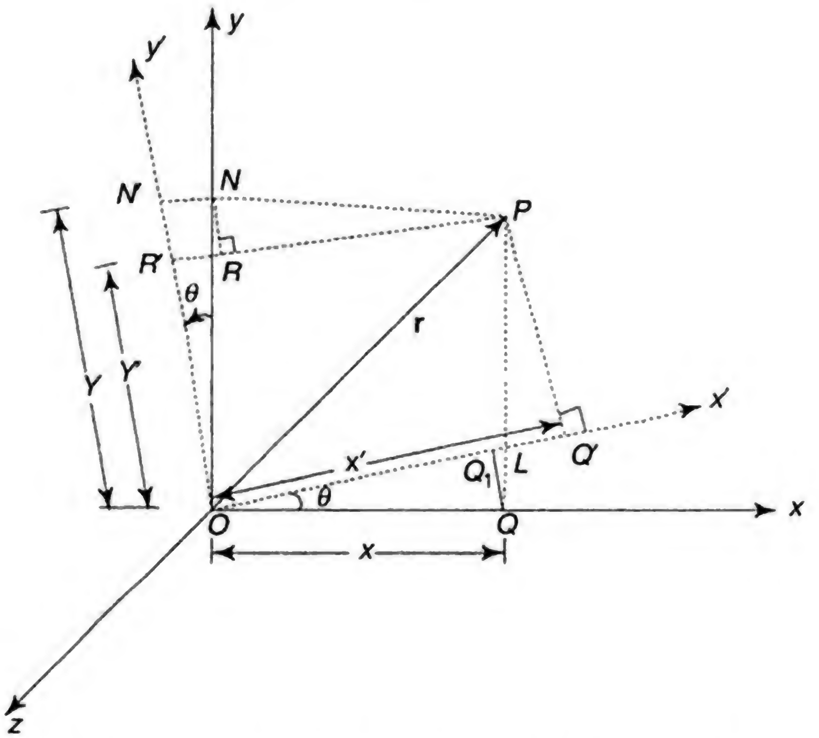


Fig. 2.28 Rotation of the coordinate system *xyz* into $x'\ y'\ z'$ around the *z*-axis

On the other hand, in the new coordinate system, x', y', z', the vector $\mathbf{r}'$ can be written as

$$\mathbf{r}' = x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}$$

Evidently the values of x', y', z' are different from x, y and z.

Considering the rotation around the *z*-axis and taking the point *P* in the *xy* plane, it will be seen that new coordinates of *P* in the $x'y'$ plane after rotation are given by (Fig. 2.28)

$$\begin{aligned} x' &= OQ' = OQ_1 + Q_1Q' \\ &= OQ \cos\theta + Q_1L + LQ' \\ &= OQ \cos\theta + QL \sin\theta + LP \sin\theta \\ &= x \cos\theta + y_1 \sin\theta + y_2 \sin\theta \\ &= x \cos\theta + y \sin\theta \end{aligned}$$

Similarly

$$\begin{aligned} y' &= OR' = ON' - R'N' \\ &= y \cos\theta - x \sin\theta \\ &= -\sin\theta + y \cos\theta \end{aligned}$$

and $\quad z' = z$

We can, therefore, write them in a composite manner as:

$$x' = (\cos\theta)\, x + (\sin\theta)\, y + (0)\, z$$
$$y' = (-\sin\theta)\, x + (\cos\theta)\, y + (0)\, z \qquad (2.45)$$
$$z' = (0)\, x + (0)\, y + (1)\, z$$

Equation (2.45) may also be expressed as

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (2.46)$$

Writing

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv \mathbf{r} \quad \text{and} \quad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \equiv \mathbf{r}'$$

we can write Eq. (2.46) as

$$\mathbf{r}' = \mathbf{R}\,(\theta)\mathbf{r} \qquad (2.47)$$

Here $\mathbf{R}(\theta)$ represents the first matrix on the right side of Eq. (2.46). Equation (2.47) represents symbolically that: (i) if the coordinate system is rotated in the anticlockwise direction around the z-axis, the new components of the vector are obtained by operating the matrix $\mathbf{R}(\theta)$ on the old coordinates. (ii) instead of rotating the coordinate system x, y, z in the anticlockwise direction, one can obtain the same results by rotating $\overrightarrow{OP}$ in the clockwise direction to new position $\overrightarrow{OP'}$, then the new coordinates of $\mathbf{r}' = \overrightarrow{OP'}$ will be similar for the x, y, z coordinate system; as the coordinates of the unrotated $\mathbf{r}$-vector in the x', y', z' coordinate system. Equation (2.47) represents this second possibility also. This is shown in Fig. 2.29.

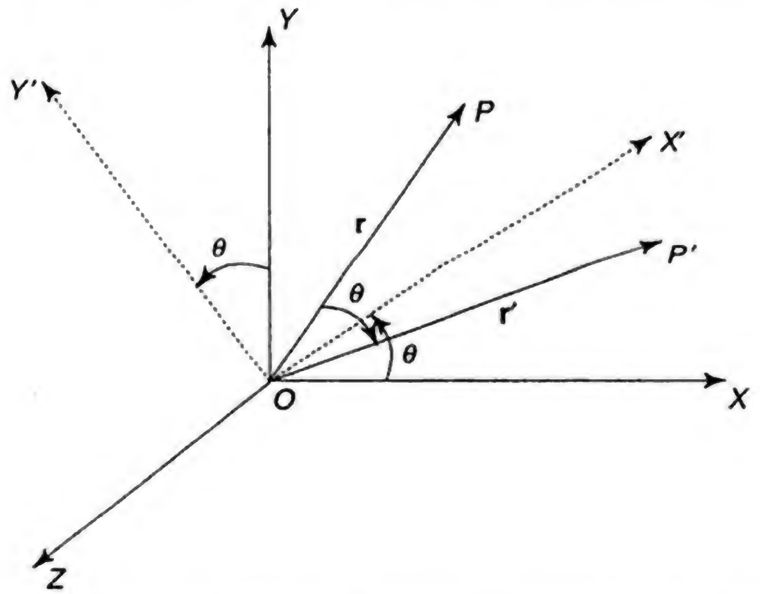


Fig. 2.29 Equivalence of clockwise rotation of the position vector **r**, in xy-plane to the anticlockwise rotation of the coordinate system around z-axis

The matrix $\mathbf{R}(\theta)$ is called the transformation matrix and is such that when it operates on the position vector $\mathbf{r}$, it gives a new vector $\mathbf{r}'$. It is evident that matrix $\mathbf{R}(\theta)$, given in Eq. (2.46) is true for rotation around the z-axis only.

For no rotation, $\theta = 0$ and the transformation matrix $\mathbf{R}(0)$ becomes

$$\mathbf{R}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = +1 \tag{2.48}$$

It corresponds to no rotation and is called a unit matrix. Next consider the matrix,

$$\mathbf{R}_i = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{2.49}$$

Equation (2.49) does not represent any physical rotation, but the inversion of the coordinates as $\mathbf{r}' = \mathbf{R}_i\mathbf{r} = -\mathbf{r}$, i.e. its operation leads to $x \to -x, y \to -y$, and $z \to -z$. The operation $\mathbf{R}_i\mathbf{r}$ represents what is called an improper rotation or inversion as shown in Fig. 2.30. $\mathbf{R}_i$ is called the inversion matrix. It converts a right-handed coordinate system to a left-handed co-ordinate system. Matrices $\mathbf{R}_i$ and $\mathbf{R}(\theta)$ will be used in Chapter 8 and also in understanding the inversion properties of some physical quantities in classical mechanics.

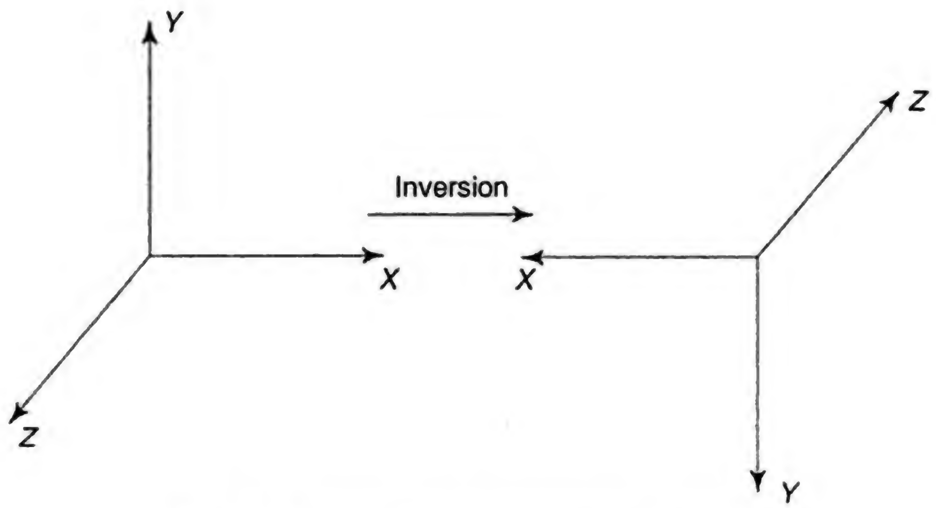


Fig. 2.30 Inversion of the coordinate axes

2.5.3 Pseudovectors and Pseudoscalars

Pseudovectors

The preceding discussion shows that the vectors can be classified into two categories.

(i) Vectors such as displacement $d\mathbf{r}$, velocity $\mathbf{v}$, force $\mathbf{F}$, momentum $\mathbf{p}$, etc., which directly point along the direction of the vector quantity. These are called the polar or radial vectors.

(ii) Vectors representing rotational quantities, e.g. angular velocity $\boldsymbol{\omega}$, torque $\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}$, and angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ are called axial vectors because they point along the axis of the rotational quantity, obeying the principle of the right-handed screw.

Apart from these obvious differences between these two types of vectors there are formal differences based on their different behaviour under reflection and inversion. By reflection, we mean the reflection through a mirror parallel to a plane of the coordinate system. As an example, consider the reflection in the mirror parallel to the yz-plane (Fig. 2.31). In this case, the direction of the x-coordinate is changed whereas the directions of y- and z-axes remaines unchanged. Thus, a right-handed system is transformed into a left-handed system on reflection. Furthermore, the coordinates x, y, z of point p are changed to $-x$, y, z. Thus the direction of the position vector $\mathbf{r}$ is changed on reflection. In fact, this is true for all polar vectors that their directions are changed on reflection. Of course, their magnitude is not affected. Similarly, on inversion, the polar vectors change their sign as $\mathbf{R}_i\mathbf{r} = -\mathbf{r}$, etc. In other words, both reflection and inversion change a right-handed coordinate system to a left-handed one and also the polar vectors change their direction.

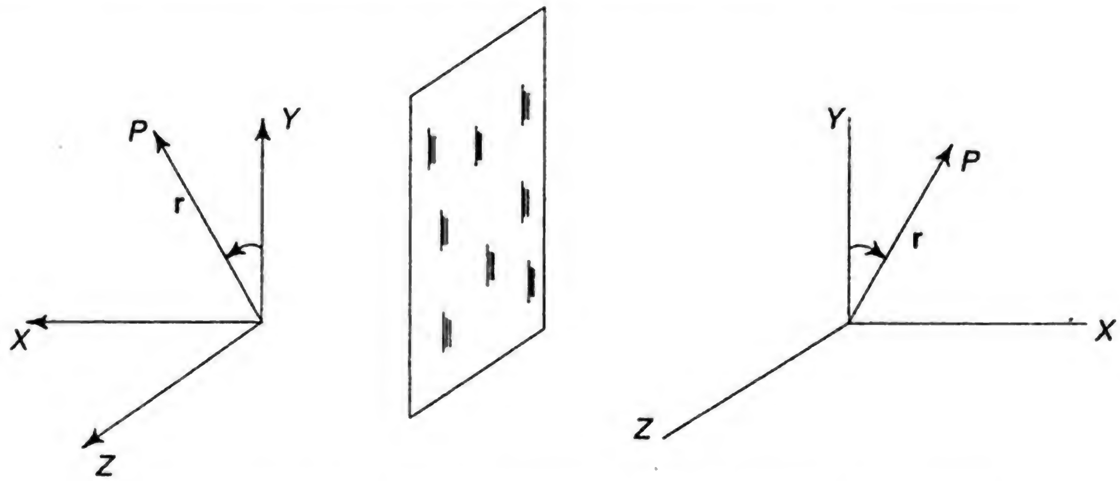


Fig. 2.31 Reflection of xyz coordinates in a mirror parallel to yz-plane

On the other hand, an axial vector does not change its sign on reflection. We take the example of torque given by

$$\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}$$

On inversion, $\mathbf{r} \to -\mathbf{r}$, $\mathbf{F} \to -\mathbf{F}$; hence $\boldsymbol{\Gamma} \to \boldsymbol{\Gamma}$ i.e. while $\boldsymbol{\Gamma}$ is a vector in the sense that it has direction, it does not obey the important property of vectors, i.e. of change of direction on reflection or inversion. Such vector quantities are called pseudovectors (pseudo means false). All axial vectors are pseudovectors.

Pseudoscalars

We have already mentioned, that scalars have no direction associated with them. These are specified by their magnitudes only, hence they should not change their sign on reflection or inversion. One such quantity which does not behave that way is the scalar triple product exemplified by the volume of a parallelepiped

$$\text{Volume } V = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \tag{2.51}$$

On inversion $\mathbf{A} \to -\mathbf{A}$, $\mathbf{B} \to -\mathbf{B}$ and $\mathbf{C} \to -\mathbf{C}$, hence $V \to -V$. Such quantities are called pseudoscalars.

EXAMPLE 2.9

A sphere of radius 0.1 m with centre at the origin is rotating about its axis in such a way that its angular velocity is given by $\boldsymbol{\omega} = 3\mathbf{j} + 4\mathbf{k}$ rad/s. Find the angle subtended by the axis of rotation with the z-axis. Also determine the velocity vector $\mathbf{v}$ of a point P on the surface of the sphere if its position at the given instant is $\left(\sqrt{0.005}\,\mathbf{i} + \sqrt{0.003}\,\mathbf{j} - \sqrt{0.002}\,\mathbf{k}\right)$.

Solution

By definition a rotational vector is taken along the axis of rotation. Therefore, the vector for the axis will be given by the same expression as that for $\boldsymbol{\omega}$. If it subtends angle θ with the z-axis, then

$$\begin{aligned}\cos\theta &= \frac{\mathbf{k}\cdot\boldsymbol{\omega}}{|\mathbf{k}|\,|\boldsymbol{\omega}|}\\ &= \frac{\mathbf{k}\cdot(3\mathbf{j}+4\mathbf{k})}{1\cdot(3^2+4^2)^{1/2}}\\ &= 4/5 = 0.8\\ \therefore\qquad \theta &= 36°52'\end{aligned}$$

The axis of rotation is inclined at θ with the z-axis.

Now the linear velocity $\mathbf{v}$ of a particle with radius vector $\mathbf{r}$ and rotating with angular velocity $\boldsymbol{\omega}$ is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

In the present example

$$\boldsymbol{\omega} = 3\mathbf{j} + 4\mathbf{k} \text{ rad/s.}$$

$$\mathbf{r} = \left(\sqrt{0.005}\,\mathbf{i} + \sqrt{0.003}\,\mathbf{j} - \sqrt{0.002}\,\mathbf{k}\right) \text{ m}$$

Therefore

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 4 \\ \sqrt{0.005} & \sqrt{0.003} & -\sqrt{0.002} \end{vmatrix}$$

$$\begin{aligned} &= (-3\sqrt{0.002} - 4\sqrt{0.003})\,\mathbf{i} + 4\sqrt{0.005}\,\mathbf{j}\\ &\quad + (0 - 3\sqrt{0.005})\mathbf{k} \text{ m/s}\\ &= -0.353\mathbf{i} + 0.283\mathbf{j} - 0.212\,\mathbf{k} \text{ m/s}\\ |\mathbf{v}| &= [(-0.353)^2 + (0.283)^2 + (-0.212)^2]^{1/2} \text{ m/s}\\ &= 0.5 \text{ m/s}\end{aligned}$$

EXAMPLE 2.10

A constant vector $\mathbf{A}$ is given by

$$\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

with respect to a particular coordinate system. Find the form of this vector with respect to a coordinate system which is obtained from the first by rotating it about

the z-axis through angle α in the anticlockwise direction. Determine its magnitude for $\alpha = 30°$ and compare with that of $|\mathbf{A}|$.

Solution

Since the new coordinate system is obtained from the old one by rotating it through angle α around the z-axis, the transformation matrix will be

$$\mathbf{R}(a) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Accordingly, the vector $\mathbf{A}$ will be transformed to $\mathbf{A}'$ such that

$$\mathbf{A}' = \mathbf{R}(\alpha)\mathbf{A}$$

Here $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ and, therefore, in matrix form it can be written as:

$$\mathbf{A} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

Hence

$$\mathbf{A}' = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cos\alpha + 2\sin\alpha \\ -3\sin\alpha + 2\cos\alpha \\ -4 \end{bmatrix}$$

or $\quad \mathbf{A}' = (3\cos\alpha + 2\sin\alpha)\,\mathbf{i} + (-3\sin\alpha + 2\cos\alpha)\,\mathbf{j} - 4\mathbf{k}$

For $\quad \alpha = 30°$, the vector becomes

$$\mathbf{A}' = (3\cos 30° + 2\sin 30°)\,\mathbf{i} + (-3\sin 30° + 2\cos 30°)\,\mathbf{j} - 4\mathbf{k}$$

$$= (3 \times \sqrt{3}/2 + 2.1/2)\mathbf{i} + (-3.1/2 + 2.\ \sqrt{3}/2)\mathbf{j} - 4\mathbf{k}$$

$$= \left(\frac{3\sqrt{3}}{2} + 1\right)\mathbf{i} + \left(\sqrt{3} - \frac{3}{2}\right)\mathbf{j} - 4\mathbf{k}$$

Therefore

$$|\mathbf{A}'| = \left[\left(\frac{3\sqrt{3}}{2} + 1\right)^2 + \left(\sqrt{3} - \frac{3}{2}\right)^2 + (-4)^2\right]^{1/2}$$

$$= \left(\frac{27}{4} + 1 + 3\sqrt{3} + 3 + \frac{9}{4} - 3\sqrt{3} + 16\right)^{1/2}$$

$$= (29)^{1/2}$$

Also

$$|\mathbf{A}| = [(3)^2 + (2)^2 + (-4)^2]^{1/2}$$

$$= (9 + 4 + 16)^{1/2}$$

$$= (29)^{1/2}$$

Hence

$$|\mathbf{A}'| = |\mathbf{A}|$$

as expected, because $|\mathbf{A}|$ is invariant under rotation.

2.6 VECTOR CALCULUS

2.6.1 Differentiation of Radial Vector

A vector can be differentiated only with respect to a scalar. The differentiation of a radial vector, say $\mathbf{r}$, with respect to a scalar, say time 't', can be expressed as

$$\frac{d\mathbf{r}}{dt} = \underset{\Delta t \to 0}{\mathrm{Lt}} \frac{\mathbf{r}_2 - \mathbf{r}_1}{\Delta t} = \underset{\Delta t \to 0}{\mathrm{Lt}} \frac{\Delta \mathbf{r}}{\Delta t} \tag{2.52}$$

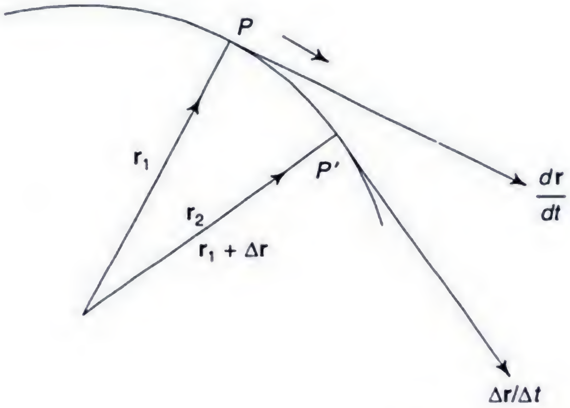


Fig. 2.32 The relationship of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\frac{d\mathbf{r}}{dt}$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are vectors, at time t_1 and t_2 respectively. Evidently, the change in r involves not only the magnitude but also the direction.

Physically, $\frac{d\mathbf{r}}{dt}$ represents the velocity, represented by the vector $\mathbf{v}$, given by

$$v = \frac{d\mathbf{r}}{dt} \tag{2.53}$$

One can mathematically describe successive derivatives like $\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d^2\mathbf{r}}{dt^2}$; or $\frac{d}{dt}\left(\frac{d^2\mathbf{r}}{dt^2}\right) = \frac{d^3\mathbf{r}}{dt^3}$. However, only $\frac{d^2\mathbf{r}}{dt^3}$ is physically significant. It is called acceleration $\mathbf{a}$, so that

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \tag{2.54}$$

As $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, we can write Eqs (2.52) and (2.54) as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \tag{2.55a}$$

and

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} \tag{2.55b}$$

2.6.1.1 Radial and Transverse Components of Velocity

Putting $\mathbf{r} = r\hat{\mathbf{r}}$ where r is the magnitude of $\mathbf{r}$; and $\hat{\mathbf{r}}$ is the unit vector, along its direction, we write Eqs (2.55a) as

$$\mathbf{v} = \frac{d}{dt}(r\hat{\mathbf{r}}) = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \tag{2.56}$$

In Eq. (2.56), $\frac{dr}{dt}\hat{\mathbf{r}}$ is called the radial component of velocity $\mathbf{v}$ and $r\frac{d\hat{\mathbf{r}}}{dt}$ is called the transverse component of $\mathbf{v}$ because it is perpendicular to $\mathbf{r}$. This can be easily seen by differentiating $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$, from which we obtain

$$\frac{d\hat{\mathbf{r}}}{dt}\cdot\hat{\mathbf{r}} + \hat{\mathbf{r}}\cdot\frac{d\hat{\mathbf{r}}}{dt} = 2\hat{\mathbf{r}}\cdot\frac{d\hat{\mathbf{r}}}{dt} = 0$$

or

$$\hat{\mathbf{r}}\cdot\frac{d\hat{\mathbf{r}}}{dt} = 0 \tag{2.57}$$

This shows that $\hat{\mathbf{r}}$ and $\frac{d\hat{\mathbf{r}}}{dt}$ are perpendicular to each other.

EXAMPLE 2.11

Show that differential coefficients of the sum of two vectors, $\mathbf{A} + \mathbf{B}$, is equal to the sum of the individual differential coefficients.

Solution

Assuming that differential is taken with respect to time t, for the increment δt in the value of t, we have increments $\delta\mathbf{A}$ in $\mathbf{A}$, and $\delta\mathbf{B}$ in $\mathbf{B}$, so that

$$\delta(\mathbf{A} + \mathbf{B}) = (\mathbf{A} + \delta\mathbf{A} + \mathbf{B} + \delta\mathbf{B}) - (\mathbf{A} + \mathbf{B})$$

$$= \delta\mathbf{A} + \delta\mathbf{B}$$

and hence, when $\delta t \to 0$, we can write

$$\underset{\delta t \to 0}{\frac{\delta}{\delta t}}(\mathbf{A} + \mathbf{B}) = \frac{d}{dt}(\mathbf{A} + \mathbf{B}) = \underset{\delta t \to 0}{\frac{\delta\mathbf{A}}{\delta t}} + \underset{\delta t \to 0}{\frac{\delta\mathbf{B}}{\delta t}}$$

$$= \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}$$

EXAMPLE 2.12

Write the differential of the product of a scalar and a vector.

Solution

Let the scalar be s; and let vector be $\mathbf{r}$. Let both be differentiable functions of variable t. Then one can write

$$\delta(s\mathbf{r}) = (s + \delta s)(\mathbf{r} + \delta\mathbf{r}) - (s\mathbf{r})$$
$$= \delta s\mathbf{r} + s\delta\mathbf{r} + \delta s\delta\mathbf{r}$$

Dividing by δt; and for limit $\delta t \to 0$, we get

$$\underset{\delta t \to 0}{\frac{\delta(s\mathbf{r})}{\delta t}} = \underset{\delta t \to 0}{\frac{\delta s}{\delta t}}\,\mathbf{r} + s\,\underset{\delta t \to 0}{\frac{\delta\mathbf{r}}{\delta t}} + \underset{\delta t \to 0}{\frac{\delta s}{\delta t}}\,\delta\mathbf{r}$$

or
$$\frac{d(s\mathbf{r})}{dt} = \frac{ds}{dt}\,\mathbf{r} + s\,\frac{d\mathbf{r}}{dt}$$

The last term becomes zero, as $\delta\mathbf{r} \to 0$.

2.6.2 Scalar and Vector Fields

If a physical quantity—a scalar or a vector—is expressed as a continuous function of the position of a point in the region of space, it is referred to as a point function. Then this region is known as scalar field for scalar quantity, and vector field for vector quantity. Each point may then be expressed as a single-valued function, say $\phi\,(x, y, z)$, for scalar field, and $\mathbf{F}\,(x, y, z)$ for vector field.

Examples of scalar fields are temperature, potential (electric or magnetic). Examples of vector fields are electric or magnetic intensities **E** and **H** and velocity distributions and so on. Starting from any desired point in the vector field; and proceeding through infinitesimal distances, from point to point, we can draw a line of flow or flux line – which will in general, be curvilinear. The tangent at any point to this curve gives the direction of the vector at that point. Generally, the measure or magnitude of the vector is given by the number of flux lines passing through the surface, perpendicular to the direction of the lines.

2.6.2.1 Operator ∇

We now discuss a case where not only the differentiation of vectors is involved; but also where differentiation operator itself is a vector quantity called 'del' (∇), defined as

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{2.58}$$

It may be reiterated that by definition ∇ is a vector, and x, y, z are scalars representing the displacements along x, y, and z directions. The operator ∇ can operate on a scalar ϕ or on a vector **V** either through dot product or cross product. Therefore, one has three possibilities of the end products for the operation by ∇. They are respectively called gradient ϕ, ($\nabla\phi$); divergence **V**, ($\nabla \cdot \mathbf{V}$); and curl **V**, ($\nabla \times \mathbf{V}$). The physical significance and the mathematical expressions for these three cases are given below.

2.6.2.2 The Gradient

The operator ∇ operating on a scalar ϕ, that is, $\nabla\phi$ is called the gradient of ϕ or grad ϕ. Evidently, grad ϕ is a vector and is expressed as

$$\nabla\phi = \text{grad } \phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \tag{2.59}$$

To understand the physical significance of grad ϕ, let us consider a family of surfaces for which a scalar quantity, say temperature, represented by $\phi(x, y, z)$ is constant, which means

$$d\phi = \left(\frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial \phi}{\partial z}\right) dz = 0 \qquad (2.60)$$

Then a vector-increment, $d\mathbf{R}$, starting from point P on such a surface can be represented by

$$d\mathbf{R} = \mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz \qquad (2.61)$$

Hence,
$$\nabla\phi \cdot d\mathbf{R} = d\phi = 0 \qquad (2.62)$$

Equation (2.62) gives the physical significance of $\nabla\phi$. It can be inferred from this equation that it is perpendicular to $d\mathbf{R}$, and perpendicular to the surface of the family which passes through P (Fig. 2.33). The direction of $\nabla\phi$ is generally taken to be that in which ϕ is increasing. Hence, grad ϕ represents the change for scalar quantity ϕ, per unit displacement, perpendicular to the vectors on a surface of a constant ϕ.

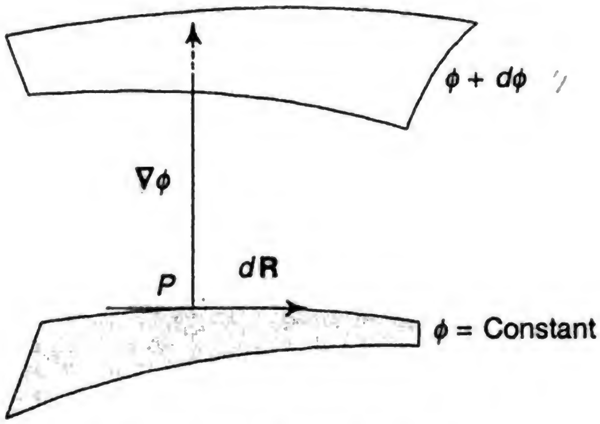


Fig. 2.33 The gradiant of a scalar function

2.6.3 The Divergence

By the very definition, we state

$$\nabla \cdot \mathbf{V} = \text{div } \mathbf{V} \qquad (2.63a)$$

where
$$\nabla \cdot \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\mathbf{i}V_x + \mathbf{j}V_y + \mathbf{k}V_z)$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \qquad (2.63b)$$

Evidently, div $\mathbf{V}$ is a scalar because it is the dot product of two vectors ∇ and $\mathbf{V}$.

The physical significance of div $\mathbf{V}$ can be understood by considering the flow of a fluid (gas, liquid, or magnetic flux) through a parallelepiped of volume $dx\,dy\,dz = d\tau$ through different faces.

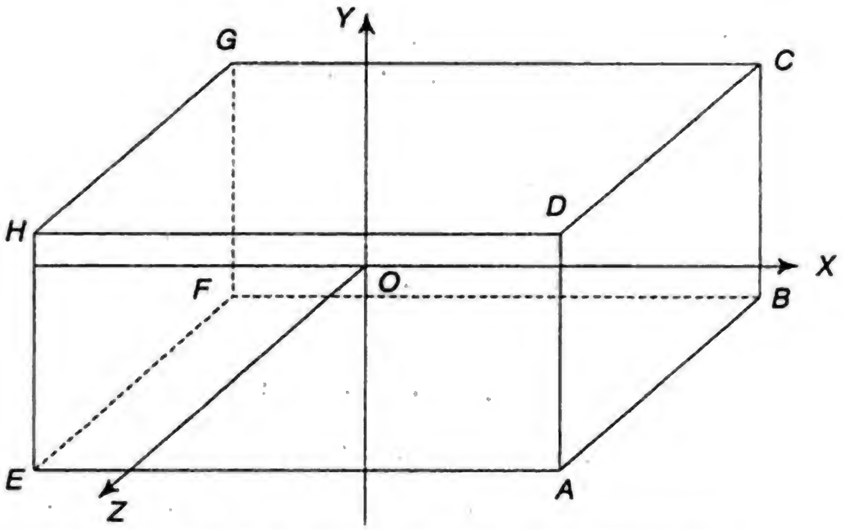


Fig. 2.34 The flow of fluid through a parallelepiped

One can, for example, write the loss of fluid mass through face *ABCD* per unit time as

$$\mathbf{i}\cdot\left[\mathbf{V}(x,y,z)+\frac{\partial \mathbf{V}}{\partial x}\frac{dx}{2}\right]dydz \tag{2.64}$$

where $\mathbf{V}$ is a vector field representing at each point in space, the direction and magnitude of the flow (density, time, velocity), and depicting the total flow per unit cross section per unit time. Then $\partial\mathbf{V}/\partial x$ is gradient of $\mathbf{V}$, along the x-axis. Equation (2.64) represents the value of flow at face *ABCD*, if $\mathbf{V}$ is taken at the centre (x, y, z) of parallelopiped. Similarly at face *EFGH*, one can write the loss of fluid mass as

$$\mathbf{i}\cdot\left[\mathbf{V}(x,y,z)-\frac{\partial \mathbf{V}}{\partial x}\frac{dx}{2}\right]dydz \tag{2.65}$$

Therefore, the net loss through these faces is obtained by subtracting Eq. (2.65) from Eq. (2.64) as

$$\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x}dxdydz \tag{2.66a}$$

Similarly, the losses through the other two pairs of faces are

$$\mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y}dxdydz \tag{2.66b}$$

and

$$\mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z}dxdydz \tag{2.67}$$

Hence the total loss of fluid from the parallelopiped is

$$\left[\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x}+\mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y}+\mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z}\right]dxdydz$$

$$=\nabla\cdot\mathbf{V}\,d\tau \tag{2.68}$$

Equation (2.68) illustrates the physical meaning of $\nabla \cdot \mathbf{V}$ or div $\mathbf{V}$. It is the total loss of mass per unit volume per unit time from an enclosed volume. The term divergence denotes this physical situation.

If $\mathbf{v}$ is the velocity of the fluid, then $\mathbf{V} = \rho\mathbf{v}$, where ρ is the density (mass per unit volume) of the fluid. Then Eq. (2.63a) can be written as

$$\nabla \cdot \mathbf{V} = (\nabla \cdot \mathbf{v})\rho$$

$$= -\frac{\partial \rho}{\partial t} \tag{2.69}$$

The minus sign indicates that an outward flow decreases the liquid left in the volume enclosed, since whatever flows out through the surface must come out at the expense of the liquid remaining inside the volume element. This is called the equation of continuity. In an incompressible liquid , $\partial\rho/\partial t = 0$. Hence,

$$\nabla \cdot \mathbf{V} = 0 \tag{2.70}$$

2.6.4 The Curl

The vector product of ∇ and $\mathbf{V}$, that is, $\nabla \times \mathbf{V}$ is called the curl $\mathbf{V}$. Evidently, it is a vector because it represents the vector product of two vectors. We can expand curl $\mathbf{V}$ as

$$\text{curl } \mathbf{V} = \nabla \times \mathbf{V}$$

$$= \mathbf{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \tag{2.71}$$

The physical significance of curl $\mathbf{V}$ may be understood by considering the rotation of a rigid body around an axis. We have already seen in Eq. (2.44) that

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{R} \tag{2.44}$$

where $\mathbf{v}$ is the linear velocity of a point on the rigid body, $\boldsymbol{\omega}$ is the angular velocity, and $\mathbf{R}$ the radial vector of the point.

Therefore $$\text{curl } \mathbf{v} = \nabla \times (\boldsymbol{\omega} \times \mathbf{R}) \tag{2.72}$$

From Eqs (2.44) and (2.72)

$$\text{curl } \mathbf{v} = \boldsymbol{\omega}(\nabla \cdot \mathbf{R}) - (\nabla \cdot \boldsymbol{\omega})\mathbf{R} \tag{2.73}$$

Knowing that $$\mathbf{R} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \tag{2.74}$$

we can easily see that $$\nabla \cdot \mathbf{R} = 3 \tag{2.75}$$

If ω is a constant vector (for the motion with constant angular velocity), then

$$\nabla \cdot \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \tag{2.76}$$

Hence, $$(\nabla \cdot \boldsymbol{\omega})\mathbf{R} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{R} \tag{2.77}$$

$$= \left[\omega_x \frac{\partial}{\partial x} + \omega_y \frac{\partial}{\partial y} + \omega_z \frac{\partial}{\partial z}\right]\mathbf{R}$$

$$= \mathbf{i}\omega_x + \mathbf{j}\omega_y + \mathbf{k}\omega_z$$

$$= \boldsymbol{\omega} \tag{2.78}$$

Hence, Eq. (2.73) reduces to

$$\text{curl } \mathbf{v} - = 3\boldsymbol{\omega} - \boldsymbol{\omega}$$

$$= 2\boldsymbol{\omega} \tag{2.79}$$

In other words, the curl of a linear vector like velocity converts it into the angular velocity (twice of it, of course). The directions of $\mathbf{v}$ and $\boldsymbol{\omega}$ are, of course, perpendicular to each other. This gives the justification of the term 'curl' to this operation.

Solenoidal Vector

For a vector point function $\mathbf{F}$, if $\nabla \cdot \mathbf{F} = 0$, then flux across any closed surface around the point of $\mathbf{F}$ is zero, as has been explained in the previous section. Then function $\mathbf{F}$ is said to solenoidal, for which either the lines of flow of its flux should form closed curves (like the lines of force in the magnetic field or of an electric current), or extend to infinity. Such a solenoidal function is a curl of some function. Since the div. curl = 0, as shown below, it follows that the curl of every function is necessarily solenoidal.

EXAMPLE 2.13

Show that $r^n\mathbf{r}$ is an irrotational vector for any value of n but is solenoidal for $n = -3$, where $\mathbf{r}$ is the position vector and r is its magnitude.

Solution

A vector $\mathbf{A}$ is solenoidal if $\nabla \cdot \mathbf{A} = 0$, and is irrotational if $\nabla \times \mathbf{A} = 0$. Now, applying the divergence and curl operations to the given vector $r^n\mathbf{r}$ we see immediately that

$$\nabla \times (r^n\mathbf{r}) = 0$$

since $r^n\mathbf{r}$ is in the direction of $\mathbf{r}$ and its curl will give zero for any value of n. To see for what value of n it is solenoidal, let us calculate its divergence.

Now $\quad \nabla \cdot (r^n\mathbf{r}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot [(x^2 + y^2 + z^2)^{\frac{n}{2}} (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)]$

$$= \Sigma\left[(x^2 + y^2 + z^2)^{\frac{n}{2}} + nx^2(x^2 + y^2 + z^2)^{\frac{n}{2}-1}\right]$$

$$= \Sigma\,(r^n + nx^2r^{n-2})$$

$$= 3\,r^n + nr^n$$

$$= (3 + n)r^n$$

For $\quad n = -3,$

$$\nabla \cdot (r^n\mathbf{r}) = 0$$

implying thereby, that it is solenoidal for $n = -3$ only.

EXAMPLE 2.14

Show that

(i) $\nabla \cdot \mathbf{r} = 3$
(ii) $\nabla \cdot (\nabla\phi) = \nabla^2\phi$

Solution

(i) $$\nabla \cdot \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

(ii) $$\nabla \cdot \nabla\,\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \nabla^2\phi$$

where $$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is called the Laplacian operator.

EXAMPLE 2.15

Prove the following:

(a) $\nabla \times (\nabla\,\phi) = 0$
(b) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

where ϕ is a scalar function and $\mathbf{A}$ is a vector function.

Solution

(a) $$\nabla \times \nabla\,\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right] + \mathbf{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right] + \mathbf{k}\left[\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right]$$

$$= 0$$

(b) $$\nabla \cdot \nabla \times \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left[\mathbf{i}\left(\frac{\partial A_Z}{\partial y} - \frac{\partial A_y}{\partial x}\right) + \right.$$

$$\left. \mathbf{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= 0$$

EXAMPLE 2.16

Show that

$$\nabla \cdot \hat{\mathbf{r}} = \frac{2}{r}$$

where $\hat{\mathbf{r}}$ is a unit vector along $\mathbf{r}$ and r is the magnitude of $\mathbf{r}$.

Solution

$$\nabla \cdot \hat{\mathbf{r}} = \nabla \cdot \frac{\mathbf{r}}{r}$$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}z}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

Let us evaluate the term

$$\frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{\partial}{\partial x}\left[\frac{x}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}\right]$$

$$= \frac{(x^2 + y^2 + z^2)^{\frac{1}{2}} - x^2 (x^2 + y^2 + z^2)^{-\frac{1}{2}}}{(x^2 + y^2 + z^2)}$$

$$= \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \tag{1}$$

Analogously,

$$\frac{\partial}{\partial y}\left(\frac{y}{r}\right) = \frac{z^2 + x^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \tag{2}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{r}\right) = \frac{x^2 + y^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \tag{3}$$

Adding (1), (2) and (3), we get

$$\nabla \cdot \hat{\mathbf{r}} = \frac{2(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$= \frac{2}{r}$$

EXAMPLE 2.17

Show that the curl of the velocity of any particle of a rigid body is equal to twice the angular velocity of the body.

Solution

The velocity $\mathbf{v}$ is given by the relation

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

where $\mathbf{r}$ is the position vector relative to a fixed point and ω is the angular velocity of the body.

Now,
$$\begin{aligned}\nabla \times \mathbf{v} &= \nabla \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= \boldsymbol{\omega}\,(\nabla \cdot \mathbf{r}) - (\boldsymbol{\omega} \cdot \nabla)\mathbf{r} \\ &= 3\boldsymbol{\omega} - \boldsymbol{\omega}\end{aligned}$$

Eq. (2.37) was used in deriving the above result, since $\nabla \cdot \mathbf{r} = 3$, and

$$\begin{aligned}(\boldsymbol{\omega} \cdot \nabla)\mathbf{r} &= \left(\omega_x \frac{\partial}{\partial x} + \omega_y \frac{\partial}{\partial y} + \omega_z \frac{\partial}{\partial z}\right)(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) \\ &= \mathbf{i}\omega_x + \mathbf{j}\omega_y + \mathbf{k}\omega_z = \boldsymbol{\omega}\end{aligned}$$

Hence,
$$\nabla \times \mathbf{v} = 2\boldsymbol{\omega}$$

or
$$\boldsymbol{\omega} = \frac{1}{2}\, \nabla \times \mathbf{v}$$

EXAMPLE 2.18

Calculate the following:

(a) $\nabla f(r)$

(b) $\nabla \cdot [\mathbf{r} f(r)]$

(c) $\nabla^2 f(r)$

(d) $\nabla^2 \left[\nabla \cdot \dfrac{\mathbf{r}}{r^2}\right]$

Solution

(a)
$$\nabla f(r) = f'(r)\hat{\mathbf{r}} = f(r)\, \frac{\mathbf{r}}{r}$$

(b)
$$\nabla \cdot [\hat{\mathbf{r}} f(r)] = f(r)\nabla \cdot \hat{\mathbf{r}} + \hat{\mathbf{r}} \cdot \nabla f(r)$$

Now,
$$\begin{aligned}\nabla \cdot \hat{\mathbf{r}} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left(\mathbf{i}\frac{x}{r} + \mathbf{j}\frac{y}{r} + \mathbf{k}\frac{z}{r}\right) \\ &= \sum \mathbf{i}\frac{\partial}{\partial x}\left(\frac{x}{r}\right) \\ &= \sum \mathbf{i}\frac{\partial}{\partial x}\left[\frac{x}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}\right] \\ &= \sum \mathbf{i}\frac{(x^2 + y^2 + z^2)^{\frac{1}{2}} - x^2(x^2 + y^2 + z^2)^{-\frac{1}{2}}}{(x^2 + y^2 + z^2)}\end{aligned}$$

$$= \sum \mathbf{i} \frac{(x^2+y^2+z^2)-x^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{2}{r}$$

$$\hat{\mathbf{r}} \cdot \nabla f(r) = \hat{\mathbf{r}} \cdot f'(r)\hat{\mathbf{r}} = f'(r)$$

Thus, $$\nabla \cdot [\hat{\mathbf{r}} f(r)] = \frac{2f(r)}{r} + f'(r)$$

(c) $$\nabla^2 f(r) = \nabla \cdot \nabla f(r)$$

$$= \nabla \cdot (f'(r)\hat{\mathbf{r}})$$

$$= f'(r)\,(\nabla \cdot \hat{\mathbf{r}}) + \hat{\mathbf{r}} \cdot \nabla f'(r)$$

Now, $$\nabla \cdot \hat{\mathbf{r}} = \frac{2}{r}$$

Therefore, $$\nabla^2 f(r) = \frac{2}{r} f'(r) + \hat{\mathbf{r}} \cdot f''(r)\hat{\mathbf{r}}$$

$$= f''(r) + \frac{2f'(r)}{r}$$

(d) $$\nabla^2 \left[\nabla \cdot \frac{\mathbf{r}}{r^2}\right]$$

Now, $$\nabla \cdot \frac{\mathbf{r}}{r^2} = \frac{\nabla \cdot \mathbf{r}}{r^2} + \mathbf{r} \cdot \nabla\left(\frac{1}{r^2}\right)$$

$$= \frac{3}{r^2} + \mathbf{r} \cdot \left(-\frac{2}{r^3}\hat{\mathbf{r}}\right)$$

$$= \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2}$$

$$\nabla \cdot \nabla\left(\frac{1}{r^2}\right) = \nabla \cdot \left[-\frac{2}{r^3}\hat{\mathbf{r}}\right]$$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{-2(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)}{r^4}\right)$$

$$= \sum \mathbf{i} \frac{\partial}{\partial x}\left(\frac{-2x}{r^4}\right)$$

Now, $$\frac{\partial}{\partial x}\left(\frac{-2x}{r^4}\right)$$

$$= \frac{-2r^5 + 8x^2 r^3}{r^9}$$

$$\text{Therefore, } \nabla \cdot \nabla \left(\frac{1}{r^2}\right) = \frac{-6r^5 + 8r^5}{r^9}$$

$$= \frac{2}{r^4}$$

EXAMPLE 2.19

Calculate the following:

(i) ∇r^n
(ii) $\nabla \cdot (\nabla r^n)$
(iii) $\nabla \times (\nabla r^n)$
(iv) $\nabla^2 r^n$

where r is the distance of any point from the origin.

Solution

(i) Let $\phi(x, y, z) = r^n = (x^2 + y^2 + z^2)^{n/2}$

$$\frac{\partial \phi}{\partial x} = nx\,(x^2 + y^2 + z^2)^{\frac{n}{2}-1} = nxr^{n-2}$$

Analogously, $\dfrac{\partial \phi}{\partial y} = nyr^{n-2}$; $\quad \dfrac{\partial \phi}{\partial z} = nzr^{n-2}$

Combining these results, we get

$$\nabla r^n = \Sigma \mathbf{i} \frac{\partial \phi}{\partial x} = \Sigma \mathbf{i} x\,(nr^{n-2})$$

$$= nr^{n-2}\mathbf{r} = nr^{n-1}\hat{\mathbf{r}}$$

(ii) $\nabla \cdot \nabla r^n$

$$\nabla r^n = nr^{n-2}\mathbf{r}$$
$$= n(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)r^{n-2}$$

Therefore,

$$\nabla \cdot \nabla r^n = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (n(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z))r^{n-2}$$

$$= \Sigma \mathbf{i} \frac{\partial}{\partial x}\,(nr^{n-2}\,x)$$

$$\frac{\partial}{\partial x}\,(nr^{n-2}x) = n(n-2)r^{n-3}\,\frac{\partial r}{\partial x}\,x + nr^{n-2}$$

Now, $\quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

$$\text{Thus,}\quad \Sigma \mathbf{i} \frac{\partial}{\partial x}\,(nr^{n-2}\,x) = \Sigma \mathbf{i} \left[n(n-2)r^{n-3}\frac{x^2}{r} + nr^{n-2}\right]$$

$$= n(n-2)r^{n-3}\,\frac{r^2}{r} + 3nr^{n-2}$$
$$= n(n-2)r^{n-2} + 3nr^{n-2}$$

$$= (n^2 - 2n + 3n)r^{n-3}$$
$$= n(n + 1)r^{n-2}$$

or $$\nabla \cdot \nabla r^n = n(n + 1)r^{n-2}$$

(iii)

$$\nabla \times \nabla r^n = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ nr^{n-2}\,x & nr^{n-2}y & nr^{n-2}\,z \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}\left(nr^{n-2}\,z\right) - \frac{\partial}{\partial z}\left(nr^{n-2}\,y\right)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}\left(nr^{n-2}\,y\right) -\right.$$

$$\left.\frac{\partial}{\partial x}\left(nr^{n-2}\,x\right)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}\left(nr^{n-2}\,y\right) - \frac{\partial}{\partial y}\left(nr^{n-2}\,x\right)\right]$$

Let us consider the term which is the coefficient of **i**

$$\frac{\partial}{\partial y}\,(nr^{n-2}z) - \frac{\partial}{\partial z}\,(nr^{n-2}y)$$

$$= n(n-2)r^{n-3}\,\frac{\partial r}{\partial y}\,z - n(n-2)\,\frac{\partial r}{\partial z}\,y$$

$$= n(n-2)r^{n-3}\,\frac{yz}{r} - n(n-2)r^{n-3}\,\frac{zy}{r}$$

$$= 0$$

Analogously, the other two terms are zero.

Therefore, $$\nabla \times \nabla r^n = 0$$

Geometrically one can anticipate the result since ∇r^n will be in the direction of **r**, and thus, $\nabla \times \nabla r^n = 0$.

(iv) Laplacian is given by

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Thus, $$\nabla^2 r^n = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}}$$

$$= \Sigma\frac{\partial^2}{\partial x^2}\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}}$$

$$= \Sigma\frac{\partial}{\partial x}\left[\frac{n}{2}\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}-1}(2x)\right]$$

$$= \Sigma\frac{\partial}{\partial x}\left(nx\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}}\right)$$

$$= \Sigma\left(n\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}+n(n-2)\,x^2\left(x^2+y^2+z^2\right)^{\frac{n}{2}-2}\right)$$

$$= 3nr^{n-2} + n(n-2)r^{n-2}$$

$$= n(n+1)r^{n-2}$$

EXAMPLE 2.20

Prove that

(i) $\operatorname{div}(\phi\mathbf{A}) = \phi\operatorname{div}\mathbf{A} + \mathbf{A} \times \operatorname{grad}\phi$

(ii) $\operatorname{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \operatorname{curl}\mathbf{A} - \mathbf{A} \cdot \operatorname{curl}\mathbf{B}$

Solution

(i)
$$\operatorname{div}(\phi\mathbf{A}) = \sum \mathbf{i} \cdot \frac{\partial(\phi\mathbf{A})}{\partial x}$$

$$= \sum \mathbf{i}\frac{\partial}{\partial x} \cdot (\phi\mathbf{A})$$

$$= \sum \mathbf{i} \cdot \phi\frac{\partial\mathbf{A}}{\partial x} + \sum \mathbf{i} \cdot \frac{\partial\phi}{\partial x}\mathbf{A}$$

$$= \phi\sum \mathbf{i} \cdot \frac{\partial\mathbf{A}}{\partial x} + \sum \mathbf{i}\frac{\partial\phi}{\partial x} \cdot \mathbf{A}$$

$$= \phi\operatorname{div}\mathbf{A} + \operatorname{grad}\phi \cdot \mathbf{A}$$

(ii)
$$\operatorname{div}(\mathbf{A} \times \mathbf{B}) = \sum \mathbf{i} \cdot \frac{\partial(\mathbf{A} \times \mathbf{B})}{\partial x}$$

$$= \sum \mathbf{i} \cdot \left(\frac{\partial\mathbf{A}}{\partial x} \times \mathbf{B} + \mathbf{A} \times \frac{\partial\mathbf{B}}{\partial x}\right)$$

$$= \sum \mathbf{i} \cdot \frac{\partial\mathbf{A}}{\partial x} \times \mathbf{B} + \sum \mathbf{i} \cdot \mathbf{A} \times \frac{\partial\mathbf{B}}{\partial x}$$

$$= \sum \mathbf{i} \cdot \frac{\partial\mathbf{A}}{\partial x} \cdot \mathbf{B} - \sum \mathbf{i} \times \frac{\partial\mathbf{B}}{\partial x} \cdot \mathbf{A}$$

$$= \mathbf{B} \,.\, \operatorname{curl}\mathbf{A} - \mathbf{A} .\, \operatorname{curl}\mathbf{B}$$

2.7 VECTOR INTEGRATION

Vector integration can be carried out either over a scalar quantity or over a vector quantity. This gives rise to the following situations:

1. Integration of a vector over a scalar quantity
2. Integration of a scalar over a vector quantity
3. Integration of a vector over a vector quantity

Typical scalar quantities are time, temperature, displacement in a specific direction or a volume. A typical vector quantity is a $d\mathbf{s}$ (a general vector displacement) or a surface element $d\mathbf{S}$.

1. A simple example of vector integration over a scalar is the case of integration of the radial vector in the equation corresponding to the motion of a particle under constant acceleration due to gravity. For example,

$$\frac{d^2\mathbf{s}}{dt^2} = \mathbf{g} \tag{2.80}$$

Integration twice leads to

$$\mathbf{s} = \mathbf{g}\frac{t^2}{2} + \mathbf{v}_0 t + \mathbf{d}_0 \tag{2.81}$$

where $\mathbf{v}_0$ and $\mathbf{d}_0$ are constants of integration and in the above physical example are the initial velocity $\mathbf{v}_0$ and displacement $\mathbf{d}_0$. This example is straightforward and corresponds to the first case.

2. *Line Integral*: This integral is over a displacement $d\mathbf{s}$ and includes (i) integral over $d\mathbf{s}$, (ii) of a vector dotted to $d\mathbf{s}$, and (iii) of a vector crossed to $d\mathbf{s}$.

(i) The case of an integral of a scalar over a displacement $d\mathbf{s}$ can be represented by $\int_c \phi \; d\mathbf{s}$, where c represents the path of integration. This is a line integral, and can be evaluated as follows:

Since, $\quad d\mathbf{s} = \mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz \quad$ (2.82)

we can write

$$\int_c \phi \; d\mathbf{s} = \int_A^B \phi \;(x, y, z)\,(\mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz)$$

$$= \int_{x_1}^{x_2} \phi \;(x, y, z)\mathbf{i}dx + \int_{y_1}^{y_2} \phi \;(x, y, z)\mathbf{j}dy + \int_{z_1}^{z_2} \phi \;(x, y, z)\mathbf{k}dz \tag{2.83}$$

where A and B are initial and final points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2). If coordinates y and z can be correlated to x, x and z to y, and x, y to z, then each of the above three equations can be easily solved. In practice, such an integration can be carried out numerically, if the curve between A and B is not expressible analytically. Also, the direction of the motion of the integration should be kept in mind.

(ii) The integration of a vector $\mathbf{V}$ dotted to a radial vector $d\mathbf{s}$, may be represented as

$$\int \mathbf{V} \cdot d\mathbf{s} \tag{2.84}$$

Using the expressions of $\mathbf{V}$ and $d\mathbf{s}$, it is straightforward to evaluate Eq. (2.84) because it is a scalar. An interesting case is when

$$\mathbf{V} = \mathrm{grad}\phi = \nabla\phi \tag{2.85}$$

Then, $\quad \int_A^B \mathbf{V} \cdot d\mathbf{s} = \int_A^B \nabla\phi \cdot d\mathbf{s}$

$$= \int_A^B \left[\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right]$$

$$= \int_A^B d\phi$$

$$= \phi_B - \phi_A \tag{2.86}$$

If we integrate around a closed curve, that is, A = B, then Eq. (2.86) becomes

$$\oint \mathbf{V} \cdot d\mathbf{s} = 0$$

Converse is also true, that is, if $\oint \mathbf{V} \cdot d\mathbf{s} = 0$ then **V** is the gradient of some scalar point function ϕ.

(iii) One can calculate

$$\oint \mathbf{V} \times d\mathbf{s} \tag{2.87}$$

in a similar manner by expanding the vectors in terms of x, y, and z.

3. *Surface and Volume Integrals:* A surface **S** can be represented by a vector (Fig. 2.18). Hence the problem of an integral over a surface also has three possibilities like the line integral. For example,

(a) $\iint_s \phi d\mathbf{S}$

(b) $\iint_s \mathbf{V} \cdot d\mathbf{S}$

(c) $\iint_s \mathbf{V} \times d\mathbf{S}$ (2.88)

where ϕ is a scalar and **V** is a vector.

The same methods of integrals are used as in the line integral except that now $\mathbf{S} = \mathbf{A} \times \mathbf{B}$ and $d\mathbf{S} = d\mathbf{A} \times d\mathbf{B}$. So, if we know the expressions of **A** and **B**, say, in terms of x, y, and z, then $d\mathbf{S}$ can be evaluated, and hence, the integral. It may be pointed out that the direction of $d\mathbf{S}$ is normal to the surface, and if the surface encloses a portion of space, dS is taken as the outward pointing normal.

The integral $\iint_s \mathbf{V} \cdot d\mathbf{S}$ is an important quantity and is called the flux of **V** through the surface. If **V** is the product of density ρ and velocity **v** of the fluid; then this integral is the amount of fluid flowing through a surface in unit time.

The volume, on the other hand, is a scalar quantity. Let $d\tau = dx\, dy\, dz$ be an element of volume. Then we have two types of integrals over volume, that is,

(a) $\iiint_\tau \phi \, d\tau$

(b) $\iiint_\tau \mathbf{V} \, d\tau$ (2.89)

The integration of these is straightforward but has to be carried out over three independent coordinates.

2.7.1 Gauss's Theorem of Divergence

According to this theorem, the normal surface integral of a function **F**, over the boundaries of a closed surface is equal to the volume integral of the divergence of a function over the volume V, enclosed by the surface. This can be expressed as

$$\text{Flux} = \iint_s \mathbf{F} \cdot d\mathbf{S} \equiv \iiint \text{div } \mathbf{F} dV$$

$$= \iiint (\nabla \cdot \mathbf{F})\, dV \tag{2.90}$$

This can be proved as follows:

If we consider a small segment of rectangular parallelopipeds of surface, so that the flux enters say perpendicular to the surface $dydz$, and leaves after travelling distance dx (from x, y, z to $(x + dx, y, z)$; and because of the smallness of surface $dx\, dy$, we consider only the $F_x\,(P_1)$ at the centre of the face to be applicable to the whole surface $dx\, dy$, we can write the expression (Fig. 2.35), for incoming and outgoing flux.

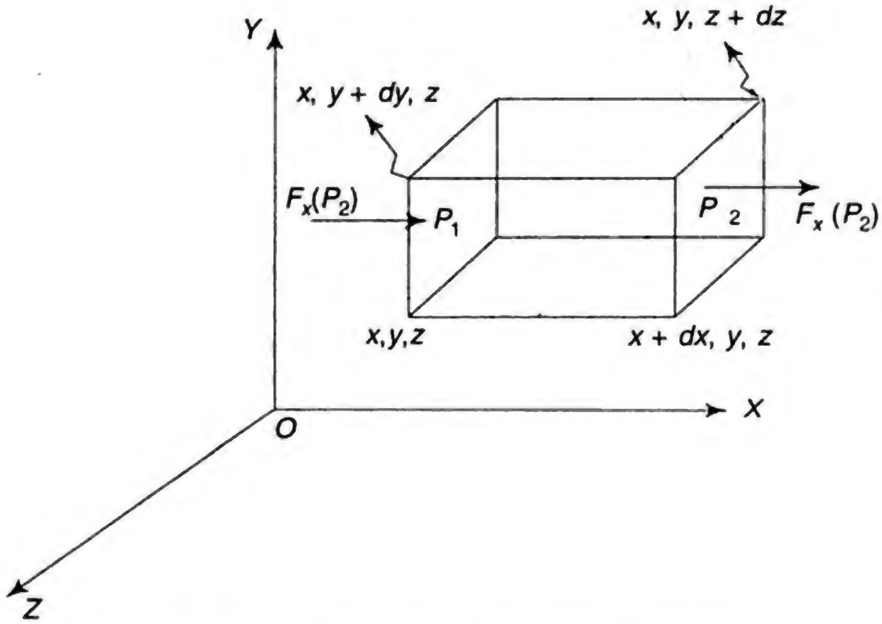


Fig. 2.35 The flow of fluid through a cube

Thus,

Flux through left side of the cube

$$= -F_x\,(P_1)\, dy\, dz \tag{2.91a}$$

Similarly, flux through the right side of the cube

$$= F_x\,(P_2)\, dy\, dz \tag{2.91b}$$

Taking
$$F_x\,(P_2) = F_x\,(P_1) + \frac{\partial F_x}{\partial x}\, dx \tag{2.91c}$$

the net flux through the left and right face is given by

$$\left[F_x(P_1) + \frac{\partial F_x}{\partial x}\, dx\right] dy\, dz - F_x(P_1)\, dy\, dz = \frac{\partial F_x}{\partial x}\, dx\, dy\, dz \tag{2.92}$$

Similarly, the net flux through the top and down face is given by

$$\frac{\partial F_y}{\partial y}\, dx\, dy\, dz \tag{2.93}$$

and net flux from the remaining one side to the other is given by

$$\frac{\partial F_z}{\partial z}\, dx\, dy\, dz \tag{2.94}$$

The flux through all the faces of the cube is given by:

$$\iint_s \mathbf{F} \cdot d\mathbf{S} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx\, dy\, dz = (\nabla \cdot \mathbf{F})\, dV \tag{2.95}$$

where $dV = dx\, dy\, dz$.

If we sum up the flux through all the elemental cubes constituting the surface we have

$$\iint_s \mathbf{F} \cdot d\mathbf{S} = \iiint (\nabla \cdot \mathbf{F})\, dV \tag{2.96}$$

More rigorously, this expression may be written as

$$\iint_s \mathbf{F} \cdot \hat{\mathbf{n}} d\mathbf{S} = \iiint (\nabla \cdot \mathbf{F})\, dV \tag{2.97}$$

where $\hat{\mathbf{n}}$ is the outward drawn normal unit vector.

EXAMPLE 2.21

If the charge distribution has a simple symmetry, it is a good candidate for the application of Gauss's law for evaluating the electric field. Applying Gauss's law, evaluate the electric field outside as well as inside an isolated charged sphere.

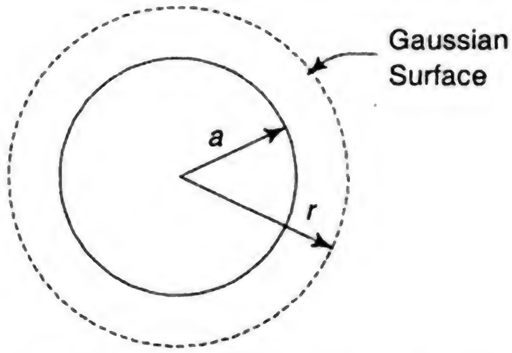


Fig. E2.21(a) A uniformly charged sphere

Solution

Imagine a spherical surface at radius r (called a Gaussian surface), as in Fig. E2.22(a) and according to Gauss's law, we get

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} q \tag{1}$$

where q is the total charge of the sphere. $\mathbf{E}$ points outwards as does $d\mathbf{a}$, so we can dispense with the dot product and employ only the magnitudes. Thus,

$$E 4\pi r^2 = \frac{q}{\varepsilon_0}$$

or

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \tag{2}$$

This is the same as the field produced by the point charge q at the centre of the sphere. If the charge is distributed uniformly through the volume given by the volume charge density ρ, then,

$$q = \frac{4}{3} \pi a^3 \rho$$

so that

$$E = \frac{1}{4\pi\varepsilon_0} \frac{4\pi a^3 \rho}{3r^2}$$

$$= \frac{a^3 \rho}{3\varepsilon_0 r^2} \tag{3}$$

For finding the field at a point inside the sphere, imagine a sphere passing through the point. The charge in the shell of thickness $(R - r)$ does not contribute to the field at the point of observation since it lies within the shell. Now, applying Gauss's law, we get

$$4\pi r^2 E = \frac{1}{\varepsilon_0}\frac{4\pi r^3 \rho}{3}$$

or
$$E = \frac{r\rho}{3\varepsilon_0} \tag{4}$$

One can visualise the fields better when plotted as a function of distance from the centre of the sphere as is shown in Fig. E2.21(b). The values of E as given by Eqs (3) and (4) match at the boundary $r = a$.

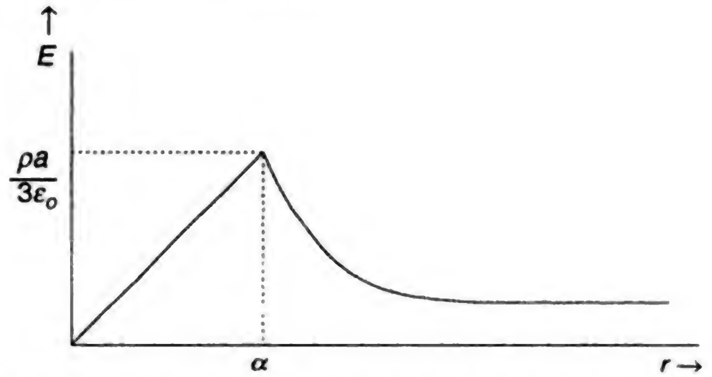


Fig. Ex 2.21(b) Electric field as a function of distance

It is easy to understand that the field outside a thin spherical shell of charge is the same as if the charge on the shell is located at the center of the shell. However, to evaluate the field at a point inside the shell, one has to discuss it out.

Let P be the point of observation inside the spherical shell. Imagine a cone with apex at P and extending on either side cutting surface elements da_1 and da_2 at a distance of r_1 and r_2 from P, respectively, Fig. E2.21 (c). Assuming that σ denotes the surface density of charge, the fields due to surface elements are $\frac{1}{4\pi\varepsilon_0}\frac{\sigma da_1}{r_1^2}$ and $\frac{1}{4\pi\varepsilon_0}\frac{\sigma da_2}{r_2^2}$ and directed in opposite directions.

Since,
$$\frac{da_1}{r_1^2} = \frac{da_2}{r_2^2} = d\Omega$$

where $d\Omega$ is the solid angle subtended at P by these surface elements. The contributions being equal and opposite cancel each other. One can divide the total surface by opposite differential areas. The total field at point P is zero, since each pair of differential areas gives no contribution.

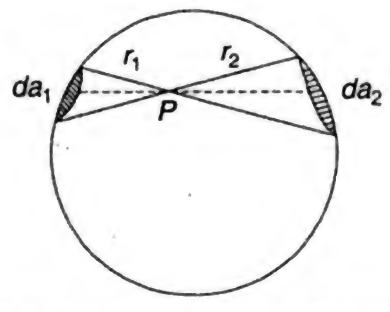


Fig. Ex 2.21(c) A spherical shell of charge

2.7.2 Stokes' Theorem

The theorem connects a line integral to a surface integral and is stated in the form

$$\oint \mathbf{V} \cdot ds = \iint \nabla \times \mathbf{V} \cdot d\mathbf{S} \tag{2.101}$$

It means that if a vector $\mathbf{W} \equiv \nabla \times V$ is integrated over a surface $d\mathbf{S}$; then it is equivalent to the integration of V over the boundary s of the surface, that is,

$$\iint_s \mathbf{W} \cdot d\mathbf{S} = \oint \mathbf{V} \cdot ds \tag{2.102}$$

Physically, the vector $\mathbf{V}$ may be taken as flux density of a fluid or as the field of a mechanical or electrical force. In the special case, when the work done is independent of the path, the line integral vanishes, and hence, one can state that the requirement of the path of integration is

$$\nabla \times \mathbf{V} = 0 \tag{2.103}$$

Physically, Stokes' theorem as stated in Eq. (2.101) may be represented by the integration over surface in such a manner that it results in the line integral over the contour of the surface as given in Fig. (2.36). It is easily seen that a sum of the surface integrals gives rise to the line integral at the outside contour. Within the surface these line integrals cancel.

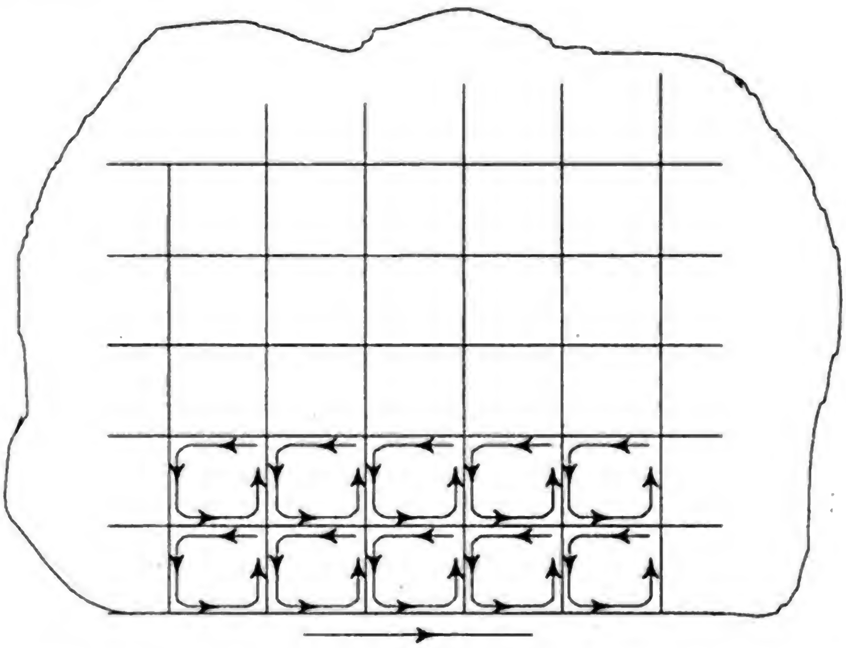


Fig. 2.36 The integration over surface resulting in the line integral over the contour of the surface

Proof of Stokes' Theorem

Consider a surface S bounded by the closed contour C. Let C' be the projection on the x-y plane. We then associate a point P (x, y) on the x-y plane with a point $P'(x, y, z)$ on the S-surface. So a function $u(x, y, z)$ on the S-surface reduces to another function $\phi(x, y)$ on the x-y plane, that is,

$$u(x, y, z) = \phi(x, y) \tag{2.104}$$

Similarly, if we project the surface S on (x-z) and (y-z) plane, we can state that

$$v(x,y,z) = \chi(x,z) \tag{2.105}$$

$$w(x,y,z) = \psi(y,z) \tag{2.106}$$

If we, now, define vector **V** at each point on the surface S as

$$\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \tag{2.107}$$

and take a unit vector **n**, perpendicular to the surface at any point, Eq. (2.101) becomes,

$$\iint_s \mathbf{n} \cdot \nabla \times \mathbf{V} dS = \iint_s \mathbf{n} \cdot (\nabla \times u\mathbf{i} + \nabla \times v\mathbf{j} + \nabla \times w\mathbf{k})dS \tag{2.108}$$

Among the three terms on the right-hand side of Eq. (2.108), one can calculate a typical term as follows:

$$\mathbf{n} \cdot \nabla \times u\mathbf{i} = \mathbf{n} \cdot \left(\mathbf{j}\frac{\partial u}{\partial z} - \mathbf{k}\frac{\partial u}{\partial y}\right) \tag{2.109}$$

Then, we realise that if

$$\mathbf{s} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{2.110}$$

then,

$$\frac{\partial \mathbf{s}}{\partial y} = \mathbf{j} + \mathbf{k}\,\frac{\partial z}{\partial y} \tag{2.111}$$

(x and y are independent-coordinates in $\phi(x, y)$ but z may depend on y). Equation (2.111) represents a vector, tangent to the curve cut from S by a plane y-z, perpendicular to the x-axis. This vector, $\frac{\partial \mathbf{s}}{\partial y}$, is, therefore, perpendicular to **n**.

Hence,

$$\mathbf{n} \cdot \left[\mathbf{j} + \mathbf{k}\frac{\partial z}{\partial y}\right] = 0 \tag{2.112}$$

Hence, keeping in mind from Eq. (2.104) that

$$\frac{\partial \phi}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} \tag{2.113}$$

we see from Eq. (2.109) that

$$\mathbf{n} \cdot \left[\mathbf{j}\frac{\partial u}{\partial z} - \mathbf{k}\frac{\partial u}{\partial y}\right] \tag{2.114}$$

$$= \mathbf{n} \cdot \left[\mathbf{j}\frac{\partial u}{\partial z} + \mathbf{k}\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right] - \mathbf{n} \cdot \mathbf{k}\frac{\partial \phi}{\partial y}$$

$$= -\,\mathbf{n} \cdot \mathbf{k}\frac{\partial \phi}{\partial y} \tag{2.115}$$

since the term in the rectangular brackets is zero in view of Eq. (2.112). Realising that $\mathbf{n} \cdot \mathbf{k} dS = dydx$, we may write

$$\iint_s \mathbf{n} \cdot \nabla \times u\mathbf{i} d\mathbf{S} = -\iint \frac{\partial \phi}{\partial y}\, dxdy$$

$$= \int (\phi_2 - \phi_1) dx \tag{2.116}$$

where ϕ_2 and ϕ_1 are values of ϕ at y_2 and y_1 (maxima and minima), respectively. If ds is a line element of the contour c′, we may write

$$dx = \pm \left(\frac{\partial x}{\partial s'} \right) ds \tag{2.117}$$

The sign of ds will be negative at y_2 (towards minima) and positive at y_1 (towards maximum), and hence, the integral (2.116) becomes

$$- \int (\phi_2 - \phi_1) \frac{\partial x}{\partial s'} ds' = \int_{C'} \phi dx = \int_C u dx \tag{2.118}$$

where c′ is the contour on the projected surface x-y and c on the real surface S.

Hence,
$$\iint_s \mathbf{n} \cdot \nabla \times u\mathbf{i}d\mathbf{S} = \int_C u dx \tag{2.119}$$

Similarly, it can be seen that

$$\iint_s \mathbf{n} \cdot \nabla \times v\mathbf{j}d\mathbf{S} = \int_C v dy \tag{2.120}$$

and
$$\iint_s \mathbf{n} \cdot \nabla \times w\mathbf{k} = \int_C w dz \tag{2.121}$$

Summing up the above three equations, and from Eq. (2.101), we get

$$\iint_s \mathbf{n} \cdot \nabla \times \mathbf{V} d\mathbf{S} = \int_C \mathbf{V} \cdot ds \tag{2.122}$$

which is the Stokes' theorem.

EXAMPLE 2.22

Verify Stokes' theorem for the function $\mathbf{F} = x(\mathbf{i}x + \mathbf{j}y)$, integrated around the square, in the plane, whose sides are along the line: $x = 0$; $y = 0$; $z = 0$.

Solution

Referring to the square shown in Fig. Ex 2.22, we can, evidently write

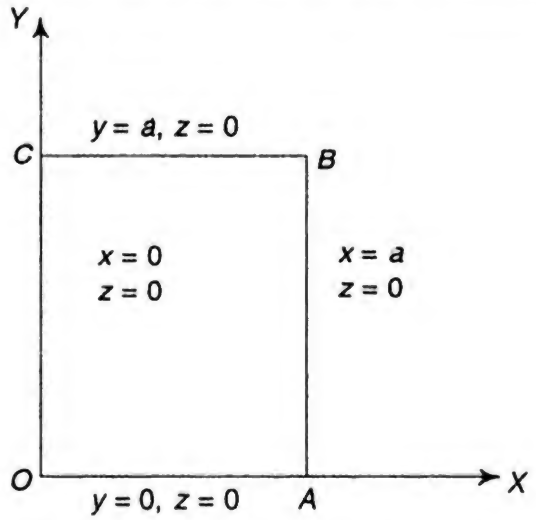


Fig. Ex 2.22 Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \int_{OA} \mathbf{F}\cdot d\mathbf{s} + \int_{AB} \mathbf{F}\cdot d\mathbf{s} + \int_{BC} \mathbf{F}\cdot d\mathbf{s} + \int_{CO} \mathbf{F}\cdot d\mathbf{s} \tag{1}$$

where
$$\int_{OA} \mathbf{F}\cdot d\mathbf{s} = \int_o^a x(\mathbf{i}x + \mathbf{j}y)\cdot \mathbf{i}dx = \int_o^a x^2\, dx = \frac{a^3}{3} \tag{2}$$

Similarly,
$$\int_{AB} \mathbf{F}\cdot d\mathbf{s} = \int_o^a x(\mathbf{i}x + \mathbf{j}y)\cdot \mathbf{j}dy = \int_o^a aydy = \frac{a^3}{3} \tag{3}$$

$$\int_{BC} \mathbf{F}\cdot d\mathbf{s} = \int_a^o x(\mathbf{i}x + \mathbf{j}y)\cdot \mathbf{i}dx = -\int_o^a x^2\, dx = -\frac{a^3}{3} \tag{4}$$

and
$$\int_{CO} \mathbf{F}\cdot d\mathbf{s} = \int_a^o x(\mathbf{i}x + \mathbf{j}y)\cdot \mathbf{j}dy = 0 \tag{5}$$

so
$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \frac{a^3}{3} + \frac{a^3}{3} - \frac{a^3}{3} + 0 = \frac{a^3}{3} \tag{6}$$

Now, according to Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \iint_s \text{curl } \mathbf{F}\cdot d\mathbf{S} \tag{7}$$

Since curl $x(\mathbf{i}x + \mathbf{j}y) = \mathbf{k}y$, we have

$$\iint_s \mathbf{F}\cdot d\mathbf{s} = \iint_s \text{curl}\, x(\mathbf{i}x + \mathbf{j}y)\cdot d\mathbf{S}$$

$$= \int_o^a\int_o^a \mathbf{k}y\cdot \mathbf{k}dxdy = \int_o^a\int_o^a ydxdy = \frac{a^3}{2} \tag{8}$$

which proves Stoke's theorem.

QUESTIONS

2.1 Define and differentiate the terms scalars and vectors. Classify the following quantities as scalars and vectors:
(i) Flight of an aeroplane from Delhi to Chandigarh; (ii) increase in the population of India by about 130 million in one decade; (iii) weight; (iv) power; (v) pressure; (vi) angular twist; (vii) density; and (viii) motion of hands of a clock.

2.2 Define the term 'unit vector' and comment on its importance.

2.3 State the principle of addition of vectors and illustrate it by considering the addition of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$.

2.4 Given two vectors **A** and **B**, when will these be called (i) equal, and (ii) negative vectors.

2.5 Addition of vectors is commutative as well as associative. Justify this statement.

2.6 A force acting in the north-east direction can never balance a force directed along north-west. Comment.

2.7 What are the base vectors **i, j, k**? Bring out their usefulness in vector algebra.

2.8 Bring out the meaning of derivative of a vector with respect to a scalar quantity. How is it different from the derivative of a scalar quantity?

2.9 In ordinary algebra, we talk of one type of multiplication, whereas in vector algebra two types of products are needed. Comment.
2.10 Define the scalar product of two vectors **A** and **B** and cite three examples where such a concept is used.
2.11 Show that the scalar product of two vectors is commutative.
2.12 State and prove the law of distribution under addition for vectors.
2.13 The scalar product provides a means to find the magnitude of a vector. Comment.
2.14 The necessary and sufficient condition for the orthogonality of two vectors **A** and **B** is that their scalar product is zero. Discuss.
2.15 Obtain an expression for the scalar product of two vectors in terms of their components.
2.16 How will you use the concept of scalar product to find the angle between two vectors?
2.17 Define vector product of two vectors **A** and **B** and give two examples of physical quantities where this concept is employed.
2.18 Bring out the difference between scalar and vector products.
2.19 The cross or vector product of two vectors is not commutative. Discuss.
2.20 Discuss the convention used for defining the direction of area vectors.
2.21 How can one use the knowledge of vectors to find the area of a parallelogram?
2.22 Show that for given three vectors **A, B** and **C**,

$$\mathbf{C} \times (\mathbf{A} + \mathbf{B}) = \mathbf{C} \times \mathbf{A} + \mathbf{C} \times \mathbf{B}.$$

2.23 Starting from the vectors expressed in terms of their components, show that the cross product of two vectors can be expressed as a determinant.
2.24 Define scalar triple product and obtain an expression for it.
2.25 Show that:
(a) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$.
(b) $\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{B} \times \mathbf{A}) = \mathbf{A} \cdot (\mathbf{C} \times \mathbf{B})$.
2.26 Sometimes the scalar triple product of three vectors **A, B** and **C** is written as (**A B C**). Justify this form of the expression.
2.27 Bring out the meaning of scalar triple product as the volume of a parallelepiped.
2.28 Depending on the choice of the order of vectors used in defining the scalar triple product for the volume of a parallelepiped, it may come out to be a positive or negative quantity. What is the significance assigned to this aspect?
2.29 If, for non-zero vectors **A, B** and **C**, $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = 0$, then three vectors are coplanar, Justify.
2.30 Justify the choice of area as a vector and volume as a scalar quantity.
2.31 Prove that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.
2.32 The vector $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is coplanar with **B** and **C**. Discuss.
2.33 What are rotational vectors? Give three examples of such vectors.
2.34 Discuss the convention used for the representation of rotational vectors taking angular displacement as an example.
2.35 Show that angular displacement vectors corresponding to infinitesimal rotation are commutative under addition.
2.36 Prove that vectors representing finite rotations are not commutative under addition. In view of this result, comment on the validity of the name 'vectors' for finite rotations.
2.37 Show that angular velocity, angular momentum and torque are rotational vectors.
2.38 Justify the name 'axial vectors' for rotational vectors and 'radial vectors' for the so-called polar vectors.
2.39 Discuss the rotation of a rectangular cartesian coordinate system around the z-axis.

2.40 Bring out the difference between the operations: (i) reflection in a plane, and (ii) inversion. (Remember that inversion is also called reflection in the origin.)
2.41 Distinguish a scalar from a pseudoscalar and a vector from a pseudovector, citing one example for each case.
2.42 Can there be an equation in which the left-hand side involves a polar vector and the right-hand side is some function of a pseudovector? Justify your answer.
2.43 The vector product of two polar vectors is a pseudovector. Comment.
2.44 What is meant by the field of any physical quantity? Give examples of vector and scalar fields.
2.45 Define the gradient of a scalar field. If **r** is the position vector of any particle, find grad$(1/r)$.
2.46 Explain clearly the divergence and curl of a vector field. Obtain expressions in general orthogonal coordinates, for
(a) div **F** and (b) curl **F**
2.47 Explain the following terms:
(a) line integral
(b) surface integral
(c) volume integral
2.48 Define solenoidal and irrotational vectors. Is magnetic induction vector **B** solenoidal?
2.49 Give Laplacian operator in Cartesian coordinates. Is it a scalar operator?
2.50 Enunciate Gauss's theorem. Give its proof.
2.51 State and prove Stoke's theorem.

PROBLEMS

2.1 Given $\mathbf{A} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$. Determine: (i) $\mathbf{A} + \mathbf{B}$, (ii) $\mathbf{A} - \mathbf{B}$, (iii) $\mathbf{A} \cdot \mathbf{B}$, (iv) the angle between these vectors and (v) $(\mathbf{A} \times \mathbf{B})$.
Ans. (i) $5\mathbf{i} - \mathbf{j} + \mathbf{k}$, (ii) $\mathbf{i} - 7\mathbf{j} + 9\mathbf{k}$, (iii) -26, (iv) $133^\circ\ 10'$, (v) $\mathbf{i} + 22\mathbf{j} + 17\mathbf{k}$
2.2 $\mathbf{A} = \mathbf{i} + \mathbf{j}$, $\mathbf{B} = \mathbf{j} + \mathbf{k}$, $\mathbf{C} = \mathbf{i} + \mathbf{k}$, determine: (i) $|\mathbf{A} + \mathbf{B}|$, (ii) $|\mathbf{A} + \mathbf{B} + \mathbf{C}|$, (iii) $|\mathbf{A} \cdot (\mathbf{B} + \mathbf{C})|$, (iv) $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$ and (v) $|\mathbf{A} \times (\mathbf{B} \times \mathbf{C})|$. *Ans.* (i) $\sqrt{6}$, (ii) $2\sqrt{3}$, (iii) 2, (iv) 2, (v) $\sqrt{2}$.
2.3 Draw three vectors of equal magnitude such that their resultant is zero.
2.4 Show that for two arbitrary vectors **A** and **B**

$$|\mathbf{A} - \mathbf{B}| \le (\mathbf{A} + \mathbf{B}) \le |\mathbf{A} + \mathbf{B}|$$

2.5 A particle is under the influence of three accelerations given by $\mathbf{A}_1 = 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$, $\mathbf{A}_2 = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, $\mathbf{A}_3 = 3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$. Find the unit vector along the resultant acceleration.
Ans. $\frac{1}{\sqrt{53}}(6\mathbf{i} + 4\mathbf{j} - \mathbf{k})$
2.6 Two vectors **A** and **B** are such that $|\mathbf{A} + \mathbf{B}| = |\mathbf{A} - \mathbf{B}|$. show that the vectors are perpendicular to each other.
2.7 A body is under the influence of three forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ whose unit vectors are

$$\hat{\mathbf{F}}_1 = \frac{1}{\sqrt{6}}(-2\mathbf{i} + \mathbf{j} - \mathbf{k}),\ \hat{\mathbf{F}}_2 = \frac{1}{\sqrt{14}}(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$$

and $\hat{\mathbf{F}}_3 = \frac{1}{\sqrt{14}}(+\,2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$. Find the magnitudes of these forces such that the resultant force is given by $\mathbf{F} = (3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$ N.
Ans. $\mathbf{F}_1 = 2$N, $\mathbf{F}_2 = 1$N; $\mathbf{F}_3 = 3$N

2.8 Prove that $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = A^2 - B^2$.

2.9 Given two vectors such that $\mathbf{A} + \mathbf{B} = 2\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ and $\mathbf{A} - \mathbf{B} = 4\mathbf{i} + 2\mathbf{j} - 10\mathbf{k}$. Find **A** and **B** and also the angle between these two vectors. *Ans.* $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$
$\mathbf{B} = -\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$
$\theta = 137° \, 36'$

2.10 Prove the law of cosines for a plane triangle, i.e.

$$C^2 = A^2 + B^2 - 2AB \cos \theta$$

2.11 Find the unit vector $\hat{\mathbf{C}}$ perpendicular to $\mathbf{A} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{B} = \mathbf{i} - \mathbf{j} + \mathbf{k}$.
Ans. $\hat{\mathbf{C}} = \frac{1}{\sqrt{2}} (\mathbf{j} + \mathbf{k})$

2.12 Find a unit vector in the *yz*-plane such that it is perpendicular to vector $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.
Ans. $\pm \frac{1}{\sqrt{2}} (\mathbf{j} - \mathbf{k})$

2.13 Determine the unit vector normal to the plane defined by the vectors $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{B} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. *Ans.* $-\frac{1}{\sqrt{2}} (\mathbf{j} + \mathbf{k})$

2.14 The motion of a particle along a circular path of radius b is given by

$$\mathbf{r} = b\,(\sin \omega t\mathbf{i} + \cos \omega t\mathbf{j}).$$

where ω is constant angular velocity. Find an expression for linear velocity **v** and show that it is perpendicular to **r**. *Ans.* $\mathbf{v} = b\omega\,(\cos \omega t\mathbf{i} - \sin \omega t\mathbf{j})$ $\mathbf{r} \,.\, \mathbf{v} = 0$

2.15 Find the work done in causing a displacement $\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ m, with the help of force $\mathbf{F} = (3\mathbf{i} - \mathbf{j} + \mathbf{k})$ N. *Ans.* 5 J

2.16 The cosines of the angles subtended by a vector with the three coordinate axes x, y, z are called its direction consines. If the three angles referred to here are α, β and γ, then show that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

2.17 Show that the vector $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ makes the same angle with each of the three axes. Also determine this angle.
(*Hint:* Remember, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$) *Ans.* 54° 45′

2.18 Both the ends of a massless string are tied to a metre rod, which is hung horizontally. A mass of 100 g is attached with a thread to the string at some point and it is found that the mass hangs in such a way that the acute angles made by the segments of the string with the thread are 60° and 45°. Determine the tension on the two parts of the string. *Ans.* 89.67 g-wt; 73.21 g-wt

2.19 A force of magnitude 10 N is acting along the line making equal angles with the three coordinates. Evaluate its components along the three axes.
Ans. (5.77 **i**, 5.77 **j**, 5.77 **k**) N

2.20 Show that, for any two arbitrary vectors **A** and **B**, $(\mathbf{A} + \mathbf{B}) \times (\mathbf{A} - \mathbf{B}) = -2\mathbf{A} \times \mathbf{B}$

2.21 Find the area of the parallelogram defined by the vectors $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, given that the magnitudes are in metres. *Ans.* $8\,(\mathbf{i} + \mathbf{j} + \mathbf{k})$ m^2

2.22 A force $\mathbf{F} = (4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k})$ N is acting at the point $\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ m. Determine the torque about the origin and about the point (3, 0, 0).
Ans. $\mathbf{\Gamma}_1 = -(\mathbf{i} + 10\mathbf{j} + 23\mathbf{k})$ Nm;
$\mathbf{\Gamma}_2 = -(\mathbf{i} + 4\mathbf{j} + 8\mathbf{k})$ Nm

2.23 When a positively charged particle having charge q and velocity **v** passes through a magnetic induction **B**, it experiences a force given by the formula

$$\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B})$$

In a particular set of experiments, it was found that

For $\mathbf{v} = v\mathbf{i}$, $\quad \mathbf{F} = q\,(4\mathbf{k} - 5\mathbf{j})\,v$

For $\mathbf{v} = v\mathbf{j}$, $\mathbf{F} = q\,(5\mathbf{i} - 2\mathbf{k})\,v$
For $\mathbf{v} = v\mathbf{k}$, $\mathbf{F} = q\,(2\mathbf{j} - 4\mathbf{i})\,v$

Evaluate the magnetic induction **B**. *Ans.* $\mathbf{B} = 2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

2.24 Prove that:

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \cdot \mathbf{B})\,(\mathbf{A} \cdot \mathbf{B}) = A^2\,B^2$$

2.25 The vectors for the three edges of a parallelopiped are given to be $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{B} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{C} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$. Determine the area of the parallelogram formed by the first two vectors and also the volume of the parallelepiped.

Ans. $(-11\mathbf{i} + 2\mathbf{j} + 7\mathbf{k})$; 12 units

2.26 Given nonzero, noncoplanar vectors **A, B** and **C**, which are used to define the vectors

$$\mathbf{A}' = \frac{\mathbf{B} \times \mathbf{C}}{\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})'}\ \mathbf{B}' = \frac{\mathbf{C} \times \mathbf{A}}{\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})'} \text{ and } \mathbf{C}' = \frac{\mathbf{A} \times \mathbf{B}}{\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})}$$

Show that the vectors **A′**, **B'** and **C′** are reciprocal to the vectors **A**, **B** and **C** respectively. Also comment on the directions of **A′**, **B′** and **C′**.

2.27 Prove that the volume of the parallelepiped defined by the reciprocal vectors **A′**, **B′** and **C′** is inverse of the volume defined by the vectors **A, B** and **C** used in the definition of reciprocal vectors.

2.28 A room has dimensions 5m × 4m × 3m. Express various corners of the room in terms of vector distances with respect to one corner taken as the origin of coordinate system. Also, write down the areas of the floor and the roof as vectors. Use vector formalism to determine the volume of the room.

Ans. Volume $= (5\mathbf{i} \times 4\mathbf{j}) \cdot 3\mathbf{k}$
$= 60\ \text{m}^3$ (See Fig. P 2.28)

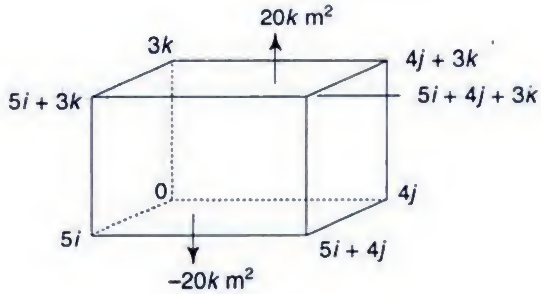


Fig. P 2.28

2.29 Show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$.

2.30 Prove that for three vectors **A, B** and **C**,

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} + (\mathbf{B} \times \mathbf{C}) \times \mathbf{A} + (\mathbf{C} \times \mathbf{A}) \times \mathbf{B} = \mathbf{O}.$$

2.31 According to Biot-Savart law, the magnetic induction **B** at a point *P* due to charge *Q* moving with velocity **v** is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi} Q \frac{\mathbf{v} \times \mathbf{r}}{r^3}$$

where **r** is the position vector from the charge to point *P*. Show that the Lorentz force on a charge Q' passing through *P* at velocity $\mathbf{v}'$ will be

$$\mathbf{F} = \frac{\mu_0}{4\pi} \cdot \frac{Q \cdot Q'}{r^3}\,[\mathbf{v}' \times (\mathbf{v} \times \mathbf{r})]$$

Calculate the force if Q' is moving parallel to and with same speed as Q.

$$Ans.\ \mathbf{F} = -\frac{\mu_0}{4\pi}\cdot\frac{QQ'}{r^3}\mathbf{v}^2\,\mathbf{r}$$

2.32 A symmetrical top is spinning around its axis in such a way that its angular velocity is given by $\omega = 6\mathbf{i} + 8\mathbf{k}$. Find the angles subtended by this axis with the x, y and z-axes.

Ans. 53° 8′, 90°, 36° 52′

2.33 Suppose **i, j** and **k** are the vectors with respect to a coordinate system. Obtain the expressions for the unit vectors $\mathbf{i}'$, $\mathbf{j}'$ and $\mathbf{k}'$ with respect to another coordinate system rotated about the z-axis through an angle θ. Hence show that the magnitude of these vectors is still unity.

Ans. $\mathbf{i}' = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$

$\mathbf{j}' = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$

$\mathbf{k}' = \mathbf{k}$

2.34 The unit vector **i** in a coordinate system obtained by rotation through θ of the original coordinate system about its z-axis is related to **i** and **j** through $\mathbf{i}' = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$. If two successive rotations through θ_1 and θ_2 are considered then show that

$$\cos(\theta_2 - \theta_1) = \cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1$$

using the above relationship.

2.35 The magnetic scalar potential of a current distribution is given by

$$\phi_m = \left(\frac{\mu_0}{4\pi}\right)\frac{\mathbf{m}\cdot\mathbf{r}}{r^3}$$

where **m** is the dipole moment of the current distribution. Show that

$$\begin{aligned}\mathbf{B} &= -\nabla\phi_m \\ &= \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right)\end{aligned}$$

2.36 Show that from one of Maxwell's equations, $\nabla\cdot\mathbf{B} = 0$, one can express $\mathbf{B} = \nabla\times\mathbf{A}$ where **A**, is the vector potential. Show that for a constant and uniform magnetic field **B**, the vector potential can be chosen in the form $\mathbf{A} = \frac{1}{2}(\mathbf{B}\times\mathbf{r})$, where **r** is a vector from the origin to the field point.

2.37 A single point charge q, situated at the origin creates an electric field

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

where r is the spherical coordinate and ε_0 is the permittivity of free space. Prove by direct calculation that the equations of electrostatics

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho$$

$$\nabla\times\mathbf{E} = 0$$

follow from the electric field of a point charge.

2.38 The charge and current densities satisfy the equation of continuity

$$\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0$$

which implies that the total charge of any closed system is conserved. Obtain the equation of continuity.

3

Coordinate Systems and Kinematics

3.1 INTRODUCTION

The description of the motion of a body or particle requires the knowledge of the relationship of three independent quantities: mass, length and time. Such a complete description involving dynamics is discussed in the next chapter. It is, however, apparent that the study of motion of a body does involve the relationship of space and time, which constitutes the subject matter of kinematics. The apparent form of these relationships depends on the coordinate system which one uses. It is instructive, therefore, to develop the relationship of various physical quantities with coordinates and time using different coordinate systems to bring out the concepts involved. The knowledge of vector algebra is, of course, assumed.

Though the basic concepts of the properties of space are discussed in the next two chapters, we assume here that space is flat, three-dimensional and Euclidean.

In general, three coordinate systems are used:

1. Rectangular or cartesian,
2. Spherical polar, and
3. cylindrical.

We will develop various relationships in kinematics for various physical quantities in this chapter using only cartesian and spherical coordinate systems.

The various functions of space and time that one comes across in mechanics and which will be dealt with in this chapter are:

1. Displacement,
2. area,
3. volume,
4. velocity,
5. acceleration, and
6. solid angle.

3.2 RECTANGULAR CARTESIAN COORDINATE SYSTEM

In this system the three dimensions are represented by three axes x, y, z perpendicular to each other. The coordinates of any point in space are taken as distances from

the origin along the three axes and written as (x, y, z). This coordinate system is generally used where no special symmetry is involved.

The direction of x-, y- and z-axes can be chosen in two different ways, i.e. right-handed and the left-handed cartesian coordinate systems. In the right-handed system [Fig. 3.1(a)], x-, y- and z-axes are so oriented that when the x-axis is rotated anticlockwise through 90° to take the position of the y-axis, the z-axis coincides with the direction in which a right-handed screw with such a rotation would move. One can visualize the situation by stretching the central finger, thumb and forefinger of the right hand at 90° to each other and taking x, y and z directions along these directions, respectively. An alternative choice of such a coordinate system is shown in Fig. 3.1(b).

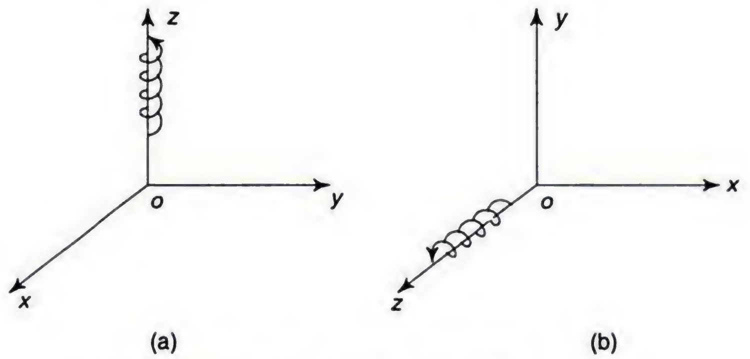


Fig. 3.1 The right-handed cartesian coordinate system

In the left-handed rectangular cartesian coordinate system, the clockwise rotation of the x-axis towards the y-axis through 90° produces left-handed screw-rotation advancing along the z-axis, as shown in Fig. 3.2. The left-handed system can be obtained from the right-handed system by changing the direction of one of the three coordinates. Compare Fig. 3.2 with Fig. 3.1 (a). If one stretches the central finger, thumb and forefinger of the left-hand pointing along the x-, y- and z-axes respectively, it gives the left-handed system.

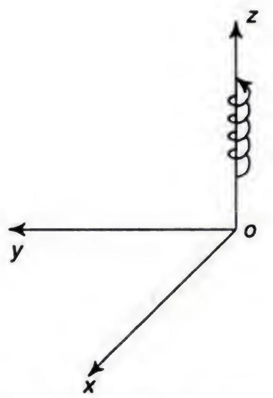


Fig. 3.2 The left-handed cartesian coordinate system

Generally, the right-handed system is preferred for use. Accordingly, we will now discuss various relationships in the right-handed rectangular cartesian coordinate system.

(a) Displacement

Let a point A with coordinates (x, y, z) be situated in a right-handed rectangular coordinate system, as shown in Fig. 3.3. We express vectorially its displacement from O as $\mathbf{r}$ which is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{3.1}$$

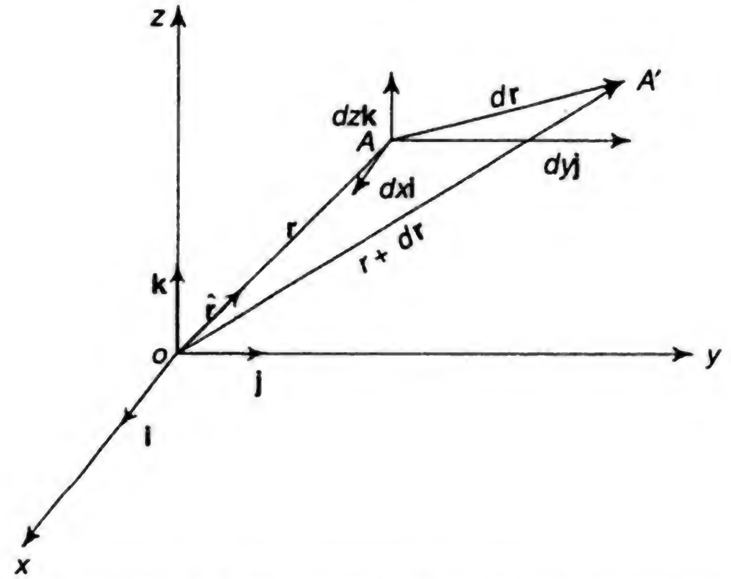


Fig. 3.3 Increment $d\mathbf{r}$ in vector $\mathbf{r}$ in the right-handed rectangular cartesian coordinate system

We can write the vectorial increment $d\mathbf{r}$ in $\mathbf{r}$, from Eq. (3.1), as

$$\begin{aligned} d\mathbf{r} &= d\mathbf{x} + d\mathbf{y} + d\mathbf{z} \\ &= dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \end{aligned} \tag{3.2}$$

Often, $d\mathbf{r}$ is expressed as $d\mathbf{s}$, i.e.

$$d\mathbf{r} \equiv d\mathbf{s} \tag{3.3}$$

EXAMPLE 3.1

We now show, by a simple example, the power of the method of vectors in the case of displacement. Figure 3.4 shows a rectangular parallelepiped element $ABCDEFGH$ in a rectangular coordinate system. It is apparent from the figure that $|d\mathbf{s}| = |d\mathbf{l}|$ and is given by

$$|d\mathbf{s}| = |d\mathbf{l}| = [(dx)^2 + (dy)^2 + (dz)^2]^{1/2} \tag{3.4}$$

However, the directions of $d\mathbf{s}$ and $d\mathbf{l}$ are different. This can be seen if we write them vectorially. Then

$$d\mathbf{s} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \tag{3.5}$$

On the other hand,

$$d\mathbf{l} = -dx\mathbf{i} - \text{dyb}\mathbf{j} + dz\mathbf{k} \tag{3.6}$$

Obviously, while $d\mathbf{s} \neq d\mathbf{l}$, $|d\mathbf{s}| = |d\mathbf{l}|$.

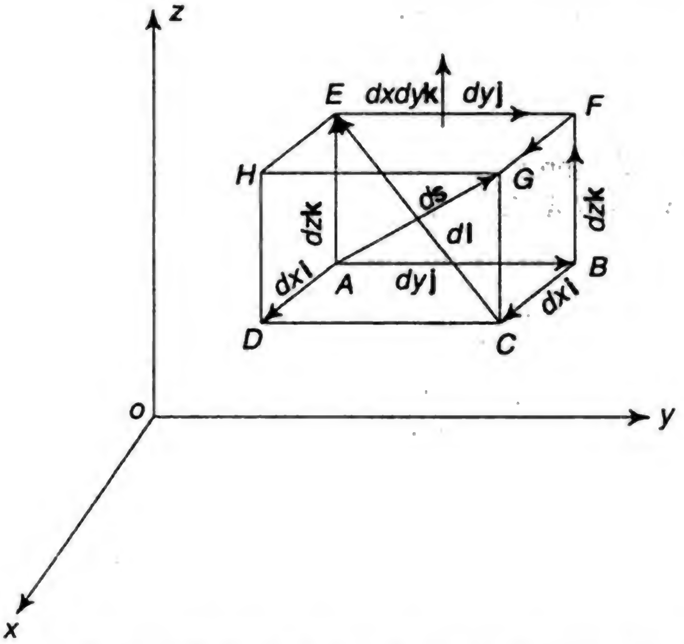


Fig. 3.4 A volume element in right-handed cartesian coordinate system

(b) Area

Referring to Fig. 3.4, where the sides are perpendicular to each other, the areas of various surfaces are:

$$(dA)_{xy} = \text{Area of } EFGH = dx\, dy \tag{3.7}$$

$$(dA)_{yz} = \text{Area of } DCGH = dy\, dz \tag{3.8}$$

$$(dA)_{zx} = \text{Area of } CBFG = dz\, dx \tag{3.9}$$

As stated in the previous chapter, we can express these areas vectorially. It is a general convention to represent area by a vector perpendicular to the surface and of magnitude equal to the value of area. For an enclosed volume, as shown in Fig. 3.4, the vectors are taken in such a manner that these are along the normal pointing outward from the closed surface.

Thus in the above case

$$\begin{aligned}(d\mathbf{A})_{xy} &= d\mathbf{x} \times d\mathbf{y}\\ &= (\mathbf{i} \times \mathbf{j})\, dx\, dy\end{aligned} \tag{3.10}$$

$$\begin{aligned}(d\mathbf{A})_{yz} &= d\mathbf{y} \times d\mathbf{z}\\ &= (\mathbf{j} \times \mathbf{k})\, dy\, dz\end{aligned} \tag{3.11}$$

and

$$(d\mathbf{A})_{zx} = (\mathbf{k} \times \mathbf{i})\, dz\, dx \tag{3.12}$$

What is the significance of expressing areas in this manner? Let us take the case of $(d\mathbf{A})_{xy} = (\mathbf{i} \times \mathbf{j})\, dx\, dy$ (Fig. 3.4). The direction of the vector $\mathbf{i} \times \mathbf{j}$ is along the direction of + $\mathbf{k}$ i.e. along the z-axis. This means that the vector $(d\mathbf{A})_{xy}$ represents a magnitude of $dx\, dy$ with a direction which is perpendicular to the area $dx\, dy$ and points in the outward direction. Also, from the definition of a vector product, the direction of the vector representing the area $(d\mathbf{A})_{xy}$ is such that the rotation of x towards y corresponds to the forward motion of a right-handed screw moving along $(d\mathbf{A})_{xy}$. One can extend these arguments to the area $(d\mathbf{A})_{yz}$ and $(d\mathbf{A})_{zx}$ etc.

The various properties of area in vector notation have been discussed in detail in Sec. 2.3.2.

(c) Volume

As discussed in the previous chapter, volume can be expressed as scalar triple product of the vectors **A, B, C** representing the three edges of the parallelepiped i.e.

$$\text{Volume} = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \tag{3.13}$$

The advantage of using vector notation for volume is that the angles between the sides of the parallelopiped are automatically taken into account and the relative sense of the vectors is also fixed.

Referring to Fig. 3.4, the volume of the volume-element in rectangular cartesian coordinate is given by

$$\begin{aligned} dV &= (d\mathbf{x} \times d\mathbf{y}) \cdot d\mathbf{z} \\ &= (\mathbf{i} \times \mathbf{j})\, dx\, dy \cdot dz\, \mathbf{k} \\ &= dx\, dy\, dz \end{aligned} \tag{3.14}$$

(d) Velocity

Velocity is defined as the rate of change of displacement. Differentiating Eq. (3.2) with respect to time, we can write velocity **v** as

$$\begin{aligned} \mathbf{v} &= (d/dt)\, \mathbf{r} \\ &= (d/dt)\, (r\hat{\mathbf{r}}) \\ &= (dr/dt)\, \hat{\mathbf{r}} + r(d\hat{\mathbf{r}}/dt) \end{aligned} \tag{3.15}$$

Thus velocity consists of contribution from a change of $|\mathbf{r}|$ in the direction of unit vector $\hat{\mathbf{r}}$ (first part) and another factor due to a change of the unit vector $\hat{\mathbf{r}}$ itself (second part). We shall discuss its implication further in the case of spherical polar coordinates.

Also, from Eq. (3.2), by differentiating with respect to time, we get

$$\begin{aligned} \mathbf{v} &= (d\mathbf{x}/dt) + (d\mathbf{y}/dt) + (d\mathbf{z}/dt) \\ &= \dot{\mathbf{x}} + \dot{\mathbf{y}} + \dot{\mathbf{z}} \\ &= \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \end{aligned} \tag{3.16}$$

The magnitude of **v** is given by

$$|\mathbf{v}| = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \tag{3.17}$$

(e) Acceleration

Acceleration is the rate of change of velocity. Differentiating Eq. (3.16) with respect to time, we get acceleration **a** as

$$\begin{aligned} \mathbf{a} = d\mathbf{v}/dt &= (d/dt)\, (\dot{\mathbf{x}}) + (d/dt)\, (\dot{\mathbf{y}}) + (d/dt)\, (\dot{\mathbf{z}}) \\ &= \ddot{\mathbf{x}} + \ddot{\mathbf{y}} + \ddot{\mathbf{z}} \\ &= \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \end{aligned} \tag{3.18}$$

The magnitude of **a** is given by

$$|\mathbf{a}| = (\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2)^{1/2} \tag{3.19}$$

EXAMPLE 3.2

The motion of a particle is described by the equations

$$x = 4 \sin 2t, \quad y = 4 \cos 2t, \quad z = 6t$$

Find the velocity and acceleration of the particle if the coordinates are expressed in metres.

Solution

The components of the radius vector of the particle are given by

$$x = 4 \sin 2t,\ y = 4 \cos 2t,\ z = 6t$$

Differentiating with respect to time, we get

$$\dot{x} = 8 \cos 2t$$
$$\dot{y} = -8 \sin 2t$$
$$\dot{z} = 6$$
$$\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$$
$$= 8 \cos 2t\mathbf{i} - 8 \sin 2t\mathbf{j} + 6\mathbf{k}$$
$$|\mathbf{v}| = [\dot{x}^2 + \dot{y}^2 + \dot{z}^2]^{1/2} \text{ m/s}$$
$$= 10 \text{ m/s}$$

Differentiating the expression for velocity with respect to time again, we have

$$\ddot{x} = -16 \sin 2t$$
$$\ddot{y} = -16 \cos 2t$$
$$\ddot{z} = 0$$

Therefore

$$\mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}$$
$$= -16(\sin 2t\mathbf{i} + \cos 2t\mathbf{j}) \text{ m/s}^2$$

and

$$|\mathbf{a}| = (\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2)^{1/2} \text{ m/s}^2$$
$$= 16 \text{ m/s}^2$$

3.3 SPHERICAL POLAR COORDINATES

The spherical polar coordinates derive their name from the fact that they represent the coordinates of points on the surface of a sphere in a convenient form. The coordinates of a point, say P in this system are represented by the radial vector $\mathbf{r}$; the zenith, colatitude or polar angle θ; and azimuthal or longitudinal angle ϕ as shown in Fig. 3.5. These coordinates are related to the rectangular coordinates x, y, and z through

$$x = r \sin \theta \cos \varphi \tag{3.20a}$$
$$y = r \sin \theta \sin \varphi \tag{3.20b}$$
$$z = r \cos \theta \tag{3.20c}$$

These relationships can be understood by realising that vector $\mathbf{r}$, represented by OP, may not be, in general, in the xy-plane. The line PL is drawn normal to the xy-plane so that its length is given by $r \cos \theta$, where θ is the angle which $\mathbf{r}$ makes with the z-axis. The line OL thus represents the projection of OP in the xy-plane. The angle φ is called the azimuthal angle which OL makes with the x-axis, in the xy-plane. Then evidently,

$$OL = \rho = r \sin \theta$$

and $\qquad x = r \sin \theta \cos \phi \qquad$ and $\qquad y = r \sin \theta \cos \phi$

hence the above relations.

The vector $\overrightarrow{OP}$ representing $\mathbf{r}$, the normal $\overrightarrow{PL}$ and projection $\overrightarrow{OL}$ all lie in the same plane in space.

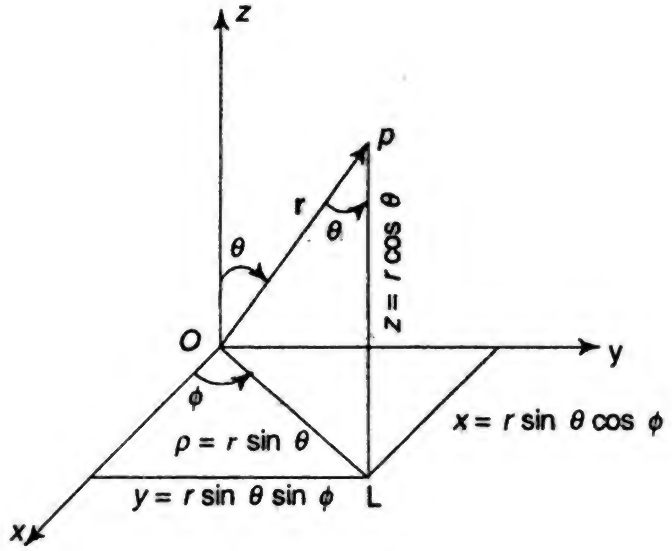


Fig. 3.5 Relationship between the spherical polar coordinates and the rectangular cartesian coordinates

The spherical coordinates are very convenient in those problems of physics where there is no preferred direction and the force in the physical problem is spherically symmetrical. Examples of such cases are:

1. Coulomb force due to a point charge, and
2. gravitational force due to a point mass,
   which are also examples of central forces.

EXAMPLE 3.3

Starting from the relation between rectangular, cartesian and spherical polar coordinates, Eq. (3.20) show that the angle θ_{12} between the radii to the points (R, θ_1, φ_1) and (R, θ_2, φ_2) on a sphere is given by,

$$\cos\theta_{12} = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2)$$

Solution

Suppose the radial vectors for the two arbitrary points P_1 and P_2 are $\mathbf{r}_1$ and $\mathbf{r}_2$. From the vector algebra, it is known that in cartesian coordinates,

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = |\mathbf{r}_1|\,|\mathbf{r}_2| \cos\theta_{12}$$

Hence

$$\cos\theta_{12} = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|}$$

The spherical polar coordinates of the two points are:
(R, θ_1, φ_1) and (R, θ_2, φ_2) so that,

$$\begin{aligned} x_1 &= R\sin\theta_1\cos\varphi_1 & x_2 &= R\sin\theta_2\cos\varphi_2 \\ y_1 &= R\sin\theta_1\sin\varphi_1 & y_2 &= R\sin\theta_2\sin\varphi_2 \\ z_1 &= R\cos\theta_1 & z_2 &= R\cos\theta_2 \\ |\mathbf{r}_1| &= R & |\mathbf{r}_2| &= R \end{aligned}$$

$$\cos\theta_{12} = \frac{\begin{array}{c} R^2\sin\theta_1\,\sin\theta_2\,\cos\varphi_1\,\cos\varphi_2\;+ \\ R^2\sin\theta_1\,\sin\theta_2\,\sin\varphi_1\,\sin\varphi_2 + R^2\cos\theta_1\,\cos\theta_2 \end{array}}{R^2}$$

$$= \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 (\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2)$$
$$= \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2)$$

It may be remarked that the curved distance between these two points will be $R\theta_{12}$.

3.3.1 Two-Dimensional System

(a) Displacement and Velocity

In a two-dimensional case, the motion takes place in a plane. Let us consider the case of motion in the xy-plane. Suppose the position of a particle at time t is represented by P (Fig. 3.6) such that $\overrightarrow{OP} = \mathbf{r}$ and $\angle XOP = \theta$. It may be noted that the angle θ in this case is different from the one used in Fig. 3.5. As a matter of fact, the two-dimentional case in the xy-plane corresponds to the situation where $\mathbf{r} = \rho$. The polar angle θ given in Fig. 3.5 is then $\pi/2$ and angle θ of Fig. 3.6 is the same as the azimuthal angle φ in Fig. 3.5.

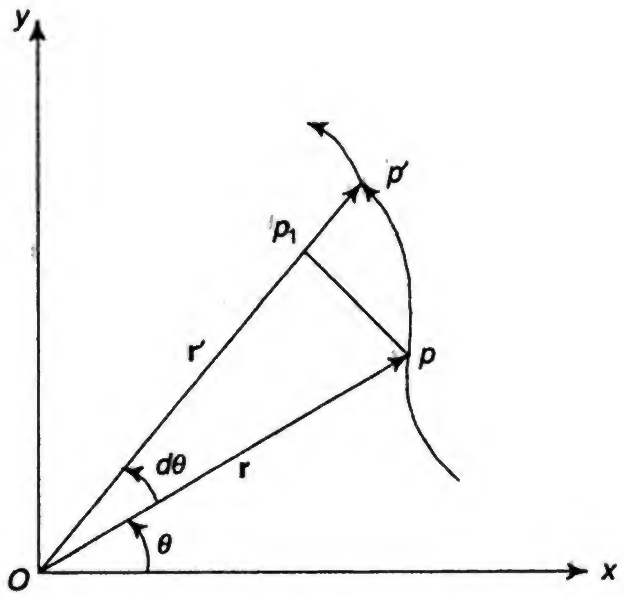


Fig. 3.6 Representation of planer motion

Let the path of the particle in the xy-plane be represented by $\overrightarrow{PP'}$. Thus the position at time t' is P' whose coordinates are r' and $\theta + d\theta$ where $d\theta$ is a small increment in the angle. We see that vectorially,

$$\overrightarrow{OP'} = \overrightarrow{OP} + \overrightarrow{PP'} \tag{3.21a}$$

Here $\overrightarrow{PP'}$ represents the vectorial change of $\mathbf{r}$ and can be represented by $d\mathbf{r}$. Hence Eq. (3.21a) can also be written as

$$\mathbf{r}' = \mathbf{r} + d\mathbf{r} \tag{3.21b}$$

Now let us draw $\overrightarrow{PP_1}$ perpendicular on $\overrightarrow{OP'}$. If $d\theta$ is very small, then $\overrightarrow{PP'}$, $\overrightarrow{PP_1}$ and $\overrightarrow{P_1P'}$ may be taken as straight lines; and the following relation holds good

$$\overrightarrow{PP'} = \overrightarrow{PP_1} + \overrightarrow{P_1P'}$$

Remembering that $\overrightarrow{P_1 P'}$ represents the increase in **r**, along its own direction, it may be represented as $dr\hat{\mathbf{r}}$ and should be distinguished from $d\mathbf{r}$.

Furthermore, $\overrightarrow{PP_1}$ may be taken as the very small arc of circle a of radius **r** so that, $\overrightarrow{PP_1} = \mathbf{r}d\theta$. Hence above equation may be rewritten as

$$d\mathbf{r} = rd\boldsymbol{\theta} + dr\hat{\mathbf{r}} \tag{3.22}$$

It may, again, be emphsised that $d\mathbf{r}$ represents the total vectorial, displacement; $dr\hat{\mathbf{r}}$ is the vectorial displacement along **r**; and $rd\theta$ is the vectorial displacement along the direction of increment of θ. It is easy to see, therefore, that we can write

$$(d/dt)\ \mathbf{r} = \hat{\mathbf{r}}dr/dt + rd\theta/dt$$

or

$$\mathbf{v} = \mathbf{v}_r + \mathbf{v}_\theta \tag{3.23}$$

where **v** is the total velocity;

$$\mathbf{v}_r = \hat{\mathbf{r}}dr/dt \tag{3.24}$$

is the velocity along the direction of **r** and

$$\mathrm{v}_\theta = rd_\theta/dt \tag{3.25}$$

is the velocity along the direction of increase in θ.

We can represent the above results in a more quantitative and elegant manner, by introducing the concept of unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. The idea of unit vector $\hat{\mathbf{r}}$ along **r** has been introduced earlier, and $\hat{\boldsymbol{\theta}}$ is the unit vector along the direction of increase of θ (Fig. 3.7). The direction of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ will be perpendicular to each other, as shown in Fig. 3.7. It may be mentioned that the direction of $\hat{\boldsymbol{\theta}}$ is taken in the xy plane, as it represents the direction of the motion of radial vector, when only θ is changing and $|\mathbf{r}|$ is constant. In other words, it represents the direction of $rd\theta$.

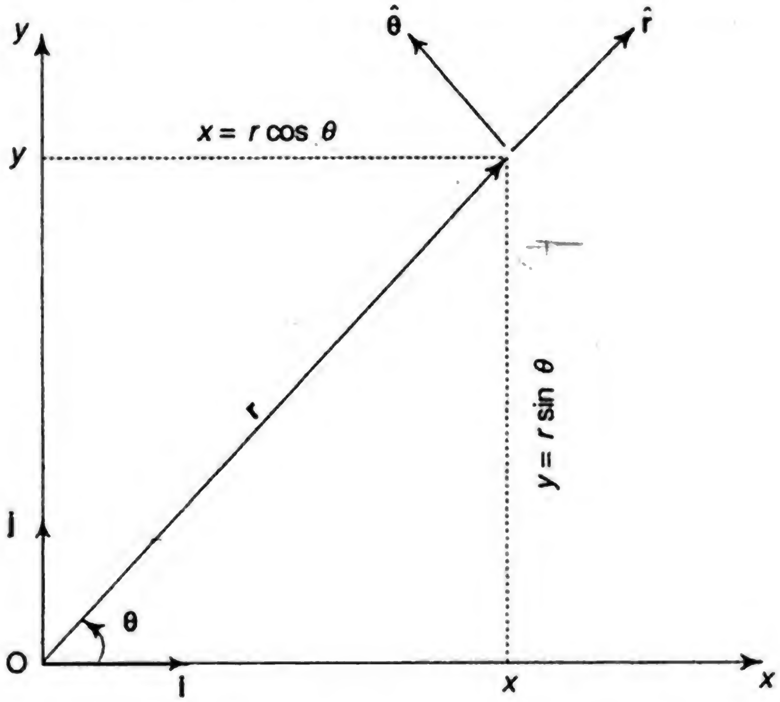


Fig. 3.7 The unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in the planar motion. Here **i** and **j** are unit vectors along x- and y-directions

We represent in Fig. 3.8 the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ and the changes in these vectors i.e. $d\hat{\mathbf{r}}$ and $d\hat{\boldsymbol{\theta}}$. In Fig. 3.8 (a), we represent the change in $\hat{\mathbf{r}}$ with θ, the direction of θ remaining fixed. As we are talking of unit vector $\hat{\mathbf{r}}$ the magnitude of $\hat{\mathbf{r}}$ remains the same, both for θ and $\theta + d\theta$. The direction of $\hat{\boldsymbol{\theta}}$ is perpendicular to $\hat{\mathbf{r}}$. In Fig. 3.8 (b), we represent the change in $\hat{\boldsymbol{\theta}}$ with θ.

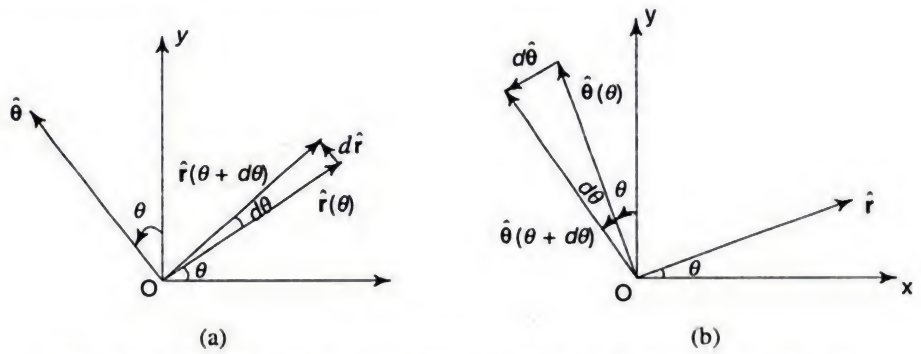


Fig. 3.8 Increments in the unit vectors (a) $\hat{\mathbf{r}}$ and (b) $\hat{\boldsymbol{\theta}}$ due to increase in θ

Further, realising that the unit vectors **i** and **j** for the two-dimensional cartesian system will be along x- and y-axes (Fig. 3.7), we see that

$$\hat{\mathbf{r}} = \mathbf{i} \cos \theta + \mathbf{j} \sin \theta \tag{3.26}$$

and

$$\hat{\boldsymbol{\theta}} = -\,\mathbf{i} \sin \theta + \mathbf{j} \cos \theta \tag{3.27}$$

Differentiating these with respect to θ, we see that

$$d\hat{\mathbf{r}}/d\theta = -\,\mathbf{i} \sin \theta + \mathbf{j} \cos \theta = \hat{\boldsymbol{\theta}} \tag{3.28}$$

and

$$d\hat{\boldsymbol{\theta}}/d\theta = -\,\mathbf{i} \cos \theta - \mathbf{j} \sin \theta = -\,\hat{\mathbf{r}} \tag{3.29}$$

Physically, $d\hat{\mathbf{r}}/d\theta$ in Eq. (3.28) represents the rate of change of $\hat{\mathbf{r}}$ with θ, when $\hat{\boldsymbol{\theta}}$ is fixed. Since the magnitude of $\hat{\mathbf{r}}$ is unity, $d\hat{\mathbf{r}}/d\theta$ is the rate of change of direction of $\hat{\mathbf{r}}$ with θ. This means that the path of **r** has circular motion in the xy-plane. Similarly, Eq. (3.29) represents the change of $\hat{\boldsymbol{\theta}}$ with θ, as shown in Fig. 3.8 (b). Here again, only the direction of $\hat{\boldsymbol{\theta}}$ is changing and $d\hat{\boldsymbol{\theta}}/d\theta$ represents the rate of change of the direction of angular rotation with angle. This is possible if the particle is not rotating in one plane and the plane of rotation itself is changing as the magnitude of θ is changed.

Referring to Fig. 3.7, it can be seen that the magnitudes r, x and y are related by the equations

$$x = r \cos \theta \tag{3.30a}$$

and

$$y = r \sin \theta \tag{3.30b}$$

The components of velocity are obtained by differentiating the above equations with respect to time, so that

$$\dot{x} = \dot{r} \cos \theta - r \sin\theta\dot{\theta} \tag{3.31a}$$

and

$$\dot{y} = \dot{r} \sin \theta + r \cos \theta\dot{\theta} \tag{3.31b}$$

Solving for $\dot{r}$ and $\dot{\theta}$, we get

$$\dot{r} = \dot{x}\cos\theta + \dot{y}\sin\theta$$

$$= \frac{\dot{x}x + \dot{y}y}{r} = \frac{\dot{x}x + \dot{y}y}{(x^2 + y^2)^{1/2}} \tag{3.32}$$

and

$$\dot{\theta} = \frac{-\dot{x}\sin\theta + \dot{y}\cos\theta}{r}$$

$$= \frac{\dot{y}x - \dot{x}y}{r^2} = \frac{\dot{y}x - \dot{x}y}{x^2 + y^2} \tag{3.33}$$

A comparison of Eqs (3.32) and (3.33) with Eqs (3.26) and (3.27) shows that $\dot{r}$ and $r\dot{\theta}$ are obtained by taking components of $\dot{x}$ and $\dot{y}$ along $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ respectively. This means that $\dot{r}$ represents velocity along the radial unit vector $\hat{\mathbf{r}}$ and $r\dot{\theta}$ along $\hat{\boldsymbol{\theta}}$, their directions being perpendicular to each other.

The velocity vector, however, can be written using Eq. (3.15) as

$$\mathbf{v} = \hat{\mathbf{r}}\,(dr/dt) + r(d\hat{\mathbf{r}}/dt)$$

This can be expressed from Eq. (3.28) as

$$\mathbf{v} = \hat{\mathbf{r}}\,(dr/dt) + r(d\hat{\mathbf{r}}/d\theta)\,(d\theta/dt)$$

$$= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$= \mathbf{v}_r + \mathbf{v}_\theta \tag{3.34}$$

This is the same result, as in Eq. (3.23), but obtained more elegantly. The quantity $\mathbf{v}_r$ is called the radial velocity and corresponds to the change in the magnitude of $\mathbf{r}$ only ($|\theta|$ = constant). The quantity $\mathbf{v}_\theta$ corresponds to the change in θ, where $|\mathbf{r}|$ is constant. It is called transverse or tangential velocity and represents the motion on the arc of a circle.

(b) Acceleration

The components of acceleration $\ddot{x}$ and $\ddot{y}$, along x and y directions are found by differentiating the expressions for $\dot{x}$ and $\dot{y}$ [Eq. (3.31)] with respect to time, i.e.

$$\ddot{x} = (\ddot{r} - r\dot{\theta}^2)\cos\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin\theta$$

and

$$\ddot{y} = (\ddot{r} - r\dot{\theta}^2)\sin\theta + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\cos\theta$$

The expression for acceleration $\mathbf{a}$ and its magnitude can then be obtained from Eqs (3.18) and (3.19) by putting $\ddot{z} = 0$.

We can also obtain $\mathbf{a}$ by differentiating $\mathbf{v}$ in Eq. (3.34) so that

$$\mathbf{a} = d\mathbf{v}/dt = (d/dt)\,(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}})$$

$$= \ddot{r}\hat{\mathbf{r}} + \dot{r}(d\hat{\mathbf{r}}/dt) + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}(d\hat{\boldsymbol{\theta}}/dt)$$

$$= \ddot{r}\hat{\mathbf{r}} + \dot{r}(d\hat{\mathbf{r}}/d\theta)\,(d\theta/dt) + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\,(d\hat{\boldsymbol{\theta}}/d\theta)\,(d\theta/dt)$$

$$= \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}}$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}}$$

$$= \mathbf{a}_r + \mathbf{a}_\theta \tag{3.35}$$

where

$$|\mathbf{a}_r| = \ddot{r} - r\dot{\theta}^2 \tag{3.36a}$$

and

$$|\mathbf{a}_\theta| = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{3.36b}$$

Extending the relationships expressed by Eqs (3.32) and (3.33) for the components of velocity $|\mathbf{v}_r|$ and $|\mathbf{v}_\theta|$, we can write for $|\mathbf{a}_r|$ and $|\mathbf{a}_\theta|$

$$|\mathbf{a}_r| = \ddot{x}\cos\theta + \ddot{y}\sin\theta \tag{3.37a}$$

and

$$|\mathbf{a}_\theta| = -\ddot{x}\sin\theta + \ddot{y}\cos\theta \tag{3.37b}$$

The form of these expressions shows that the components $|\mathbf{a}_r|$ and $|\mathbf{a}_\theta|$ of acceleration are the algebraic sum of the components of $\ddot{x}$ and $\ddot{y}$ along $\mathbf{r}$ and θ respectively. The quantities $\mathbf{a}_r$ and $\mathbf{a}_\theta$ are the radial and transverse parts of the acceleration in the directions of $\mathbf{r}$ and θ respectively, and are perpendicular to each other.

The terms in the expressions for $|\mathbf{a}_r|$ and $|\mathbf{a}_\theta|$ deserve further discussion to bring out the physical significance of these two components. In the expression for $|\mathbf{a}_r|$ [Eq. (3.36a)] the quantity $\ddot{r}$ denotes the linear acceleration due to the change in the magnitude of $\mathbf{r}$ and is directed away from the centre (positive sign). The quantity $\mathbf{r}\dot{\theta}^2$ denotes the centripetal acceleration due to change in θ and is directed towards centre (negative sign). In the expression for $|\mathbf{a}_\theta|$, the quantity $r\ddot{\theta}$ is due to angular acceleration and $2\dot{r}\dot{\theta}$ is a term arising from the interaction of changes in both r and θ. This term looks similar to the Coriolis acceleration discussed in Chapter 10, but actually this arises because of the interaction of linear and angular velocities present in curvilinear motion. Obviously, this cannot be strictly called Coriolis acceleration in the present case because it is generally used for fictitious acceleration occurring in the case of rotating frames of reference.

EXAMPLE 3.4

A particle moves in a plane with constant radial velocity 25 m/s and constant angular velocity 5 rad/s. Obtain the expressions for velocity and acceleration of the particle if time is counted from $r = 0$ and $\theta = 0$. Also, determine their magnitudes at $t = 2$s.

Solution

Taking the radial vector as $r\hat{\mathbf{r}}$ and angular displacement as $\theta\hat{\boldsymbol{\theta}}$ at time t, we have the magnitude of radial velocity as $\dot{r} = 25$ m/s and $\ddot{r} = 0$.

Magnitude of angular velocity $\dot{\theta} = 5$ rad/s and $\ddot{\theta} = 0$.

Further

$$r = \dot{r}t = 25t \text{ m}$$
$$\theta = \dot{\theta}t = 5\,t \text{ rad}$$

Now

Velocity, $\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$

$= (25\hat{\mathbf{r}} + 125t\hat{\boldsymbol{\theta}})$ m/s

Acceleration, $\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$

$= [(0 - 625t)\,\hat{\mathbf{r}} + (0 + 250)\,\hat{\boldsymbol{\theta}}]$ m/s^2

$= [-625t\hat{\mathbf{r}} + 250\hat{\boldsymbol{\theta}}]$ m/s^2

At $t = 2$s,

$\mathbf{v} = [25\hat{\mathbf{r}} + 250\hat{\boldsymbol{\theta}}]$ m/s

$$|\mathbf{v}| = (25^2 + 250^2)^{1/2} \text{ m/s}$$
$$= 251.2 \text{ m/s}$$
$$\mathbf{a} = [-1250\hat{\mathbf{r}} + 250\hat{\boldsymbol{\theta}}] \text{ m/s}^2$$

so that $|\mathbf{a}| = [(-1250)^2 + (250)^2]^{1/2} \text{ m/s}^2 = 1274.7 \text{ m/s}^2$

EXAMPLE 3.5

The path of motion of planets around the sun is elliptical with the sun at one of the foci. If a coordinate system is defined with the sun at the origin and the major axis as the polar axis, then the position coordinates r and θ of the planet are related through

$$r = a(1 - \varepsilon^2)/(1 - \varepsilon \cos \theta)$$

where ε is the eccentricity of the ellipse. Find expressions for velocity and acceleration of the planet.

Solution

The trajectory of the plant around the sun is shown in Fig. 3.9. The plane polar coordinates r and θ defining the position of the planet at any time t are related to each other through

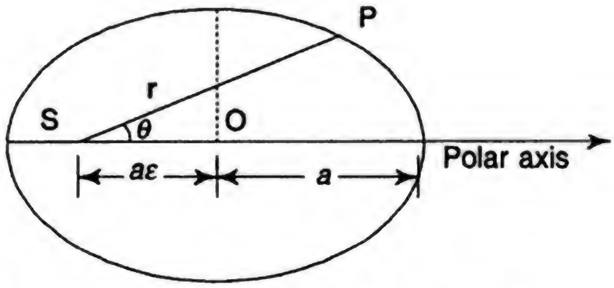


Fig. 3.9 Trajectory of the planet around the sun

$$r = a(1 - \varepsilon^2)/(1 - \varepsilon \cos \theta)$$

Both r and θ are functions of time. Differentiating both sides with respect to time, we get

$$\dot{r} = -a\varepsilon(1 - \varepsilon^2)\frac{\sin \theta\dot{\theta}}{(1 - \varepsilon \cos \theta)^2}$$

Further differentiation gives

$$\ddot{r} = -a\varepsilon(1 - \varepsilon^2)\left[\frac{\cos \theta\dot{\theta}^2}{(1 - \varepsilon\cos\theta)^2} - \frac{2\sin\theta\dot{\theta}\varepsilon\sin\theta\dot{\theta}}{(1 - \varepsilon\cos\theta)^3} + \frac{\sin\theta\ddot{\theta}}{(1 - \varepsilon\cos\theta)^2}\right]$$

$$= -\frac{a\varepsilon(1 - \varepsilon^2)}{(1 - \varepsilon\cos\theta)^3}[\cos\theta\dot{\theta}^2(1 - \varepsilon\cos\theta) - 2\varepsilon\sin^2\theta\dot{\theta}^2 +$$
$$\sin\theta\ddot{\theta}(1 - \varepsilon\cos\theta)]$$

$$= -\frac{a\varepsilon(1 - \varepsilon^2)}{(1 - \varepsilon\cos\theta)^3}[(\cos\theta - \varepsilon\cos^2\theta - 2\varepsilon\sin^2\theta)\dot{\theta}^2 +$$
$$\sin\theta\ddot{\theta}(1 - \varepsilon\cos\theta)]$$

$$= -\frac{a\varepsilon(1-\varepsilon^2)}{(1-\varepsilon\cos\theta)^3}\,[(\cos\theta-\varepsilon-\varepsilon\sin^2\theta)\,\dot{\theta}^2+\sin\theta\times$$

$$(1-\varepsilon\cos\theta)\ddot{\theta}]$$

Now from Eq. (3.34),

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$= -a\varepsilon(1-\varepsilon^2)\,\frac{\sin\theta\dot{\theta}}{(1-\varepsilon\cos\theta)^2}\,\hat{\mathbf{r}} + \frac{a(1-\varepsilon^2)}{1-\varepsilon\cos\theta}\,\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$= \frac{a(1-\varepsilon^2)}{(1-\varepsilon\cos\theta)^2}\,\dot{\theta}\,[-\varepsilon\sin\theta\,\hat{\mathbf{r}} + (1-\varepsilon\cos\theta)\,\hat{\boldsymbol{\theta}}]$$

From Eq. (3.36), we have

$$|\mathbf{a}_r| = \ddot{r} - r\dot{\theta}^2$$

$$= -\frac{a\varepsilon(1-\varepsilon^2)}{(1-\varepsilon\cos\theta)^3}\,[(\cos\theta-\varepsilon-\varepsilon\sin^2\theta)\,\dot{\theta}^2+\sin\theta\times$$

$$(1-\varepsilon\cos\theta)\ddot{\theta}] - \frac{a(1-\varepsilon^2)}{(1-\varepsilon\cos\theta)}\,\dot{\theta}^2$$

$$= -\frac{\varepsilon(1-\varepsilon^2)}{(1-\varepsilon\cos\theta)^3}\,[(\varepsilon\cos\theta-\varepsilon^2-\varepsilon^2\sin^2\theta+1+\varepsilon^2\cos^2\theta-$$

$$2\varepsilon\cos\theta)\,\dot{\theta}^2+\varepsilon\sin\theta\,(1-\varepsilon\cos\theta)\ddot{\theta}]$$

$$= -\frac{a(1-\varepsilon^2)}{(1-\varepsilon\cos\theta)^3}\,[(1-2\varepsilon^2\sin^2\theta-\varepsilon\cos\theta)\,\dot{\theta}^2+$$

$$\varepsilon\sin\theta\times(1-\varepsilon\cos\theta)\,\ddot{\theta}]$$

and

$$|\mathbf{a}_\theta| = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$= -\frac{a(1-\varepsilon^2)\ddot{\theta}}{(1-\varepsilon\cos\theta)} - 2a\varepsilon(1-\varepsilon^2)\,\frac{\sin\theta\dot{\theta}^2}{(1-\varepsilon\cos\theta)^2}$$

$$= \frac{a(1-\varepsilon^2)}{(1-\varepsilon\cos\theta)^2}\,[(1-\varepsilon\cos\theta)\,\ddot{\theta} - 2\,\varepsilon\sin\theta\,\dot{\theta}^2]$$

Hence

$$\mathbf{a} = |\mathbf{a}_r|\,\hat{\mathbf{r}} + |\,\mathbf{a}_\theta|\,\hat{\boldsymbol{\theta}}$$

$$= \frac{-a(1-\varepsilon^2)}{(1-\varepsilon\cos\theta)^3}\,\{[(1-2\varepsilon^2\sin^2\theta-\varepsilon\cos\theta)\,\dot{\theta}^2+\varepsilon\sin\theta\times$$

$$(1-\varepsilon\cos\theta)\,\ddot{\theta}]\,\hat{\mathbf{r}} + [2\varepsilon\sin\theta\,(1-\varepsilon\cos\theta)\dot{\theta}^2-(1-\varepsilon\cos\theta)^2\,\ddot{\theta}]\,\hat{\boldsymbol{\theta}}\}$$

3.3.2 Three-Dimensional System

(a) Displacement

We can illustrate the various concepts involved in displacement in the three-dimensional spherical coordinate system, by considering the line-elements in a volume

element, as shown in Fig. 3.10, each line-element representing the displacement in a direction normal to the other two displacements.

Before discussing these line-elements, it is pertinent to point out a few features in Fig. 3.10.

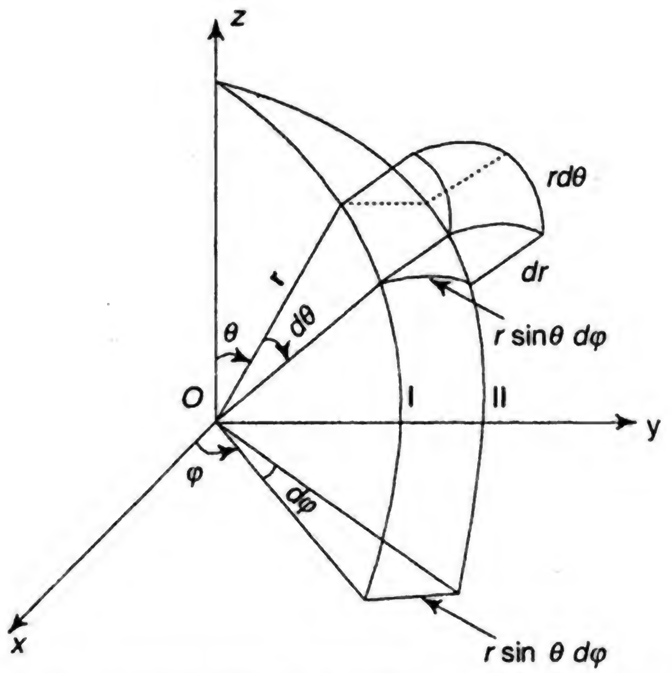


Fig. 3.10 Volume element in spherical polar coordinates

The circular boundary I, drawn here encloses a plane in which r, $r \sin\theta$ and z lie. The circular boundary II encloses a plane at a distance $r \sin\theta d\varphi$ behind it. It may be noted that the value of θ varies for different points of boundaries.

The three line-elements of the volume elements are:

1. dr,
2. $rd\theta$, and
3. $r \sin\theta d\varphi$

which are perpendicular to each other.

We consider these as our orthogonal (perpendicular) directions. Vectorially, these elements may be represented as

$$d\mathbf{r} = dr\hat{\mathbf{r}} \tag{3.38a}$$

$$rd\theta = (rd\theta)\,\hat{\boldsymbol{\theta}} \tag{3.38b}$$

$$r \sin\theta d\boldsymbol{\varphi} = (\rho d\varphi)\,\hat{\boldsymbol{\varphi}} \tag{3.38c}$$

where $\rho \equiv r \sin\theta$. The unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$ are the vectors indicating the three orthogonal directions of these line-elements.

The vector $\hat{\mathbf{r}}$ denotes a unit vector along the direction of increasing r while $\hat{\boldsymbol{\theta}}$ is a unit vector along the direction of increasing θ and is perpendicular to $\hat{\mathbf{r}}$. The vector $\hat{\boldsymbol{\varphi}}$ denotes a unit vector along the direction of increase in φ and is perpendicular to both $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$.

The three elements $d\mathbf{r}$, $rd\theta$ and $r\sin\theta d\varphi$ are along the three directions $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$ respectively. Referring to Fig. 3.11, we write

$$\mathbf{r}' = \mathbf{r} + d\mathbf{r} \tag{3.39}$$

where

$$d\mathbf{r} = dr\hat{\mathbf{r}} + (rd\theta)\,\hat{\boldsymbol{\theta}} + (r\sin\,\theta d\varphi)\hat{\boldsymbol{\varphi}} \tag{3.40}$$

and

$$\begin{aligned}(d\mathbf{r})^2 &= d\mathbf{r}\cdot d\mathbf{r}\\ &= (dr)^2 + (rd\theta)^2 + (r\sin\,\theta\,d\varphi)^2\end{aligned} \tag{3.41}$$

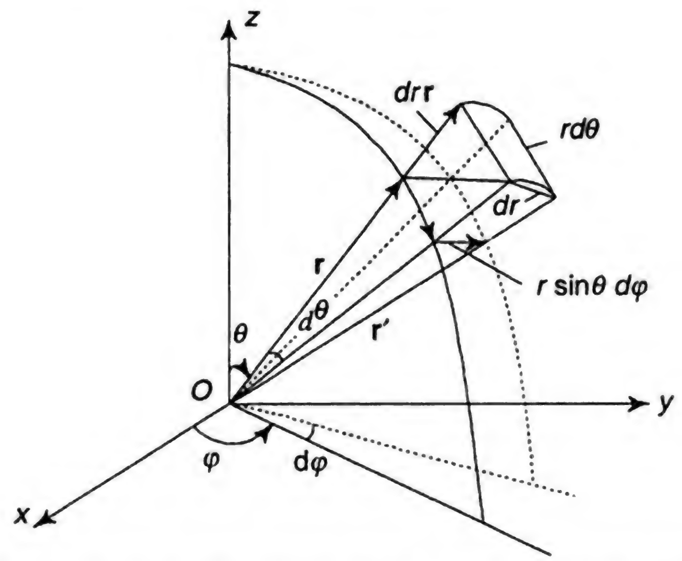


Fig 3.11 The line elements in their vector notations and the total displacement $d\mathbf{r}$ in the spherical polar coordinates

To find expressions for $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$, we refer to Fig. 3.3 where **i, j** and **k** are unit vectors along x-, y- and z-axes. Then

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{3.42}$$

Using Eq. (3.20) we can write this equation as

$$\mathbf{r} = (r\sin\,\theta\cos\,\varphi)\,\mathbf{i} + (r\sin\,\theta\sin\,\varphi)\,\mathbf{j} + (r\cos\,\theta)\,\mathbf{k} \tag{3.43}$$

The unit vector $\hat{\mathbf{r}}$ can be written as

$$\begin{aligned}\hat{\mathbf{r}} &= \mathbf{r}/r\\ &= (\sin\,\theta\cos\,\varphi)\,\mathbf{i} + (\sin\,\theta\sin\,\varphi)\,\mathbf{j} + (\cos\,\theta)\,\mathbf{k}\end{aligned} \tag{3.44}$$

We can similarly obtain expressions for $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$ by realising (as also discussed in the two-dimensional case) that the direction of $\hat{\boldsymbol{\theta}}$ is at right angles to $\hat{\mathbf{r}}$ in the plane of **r** and **z**.

Accordingly, we can write

$$\begin{aligned}\hat{\boldsymbol{\theta}} &= \sin\,(\pi/2 + \theta)\cos\,\varphi\mathbf{i} + \sin\,(\pi/2 + \theta)\sin\,\varphi\mathbf{j} + \cos\,(\pi/2 + \theta)\,\mathbf{k}\\ &= (\cos\,\theta\cos\,\varphi)\,\mathbf{i} + (\cos\,\theta\sin\,\varphi)\,\mathbf{j} - (\sin\,\theta)\,\mathbf{k}\end{aligned} \tag{3.45}$$

On the other hand, $\hat{\boldsymbol{\varphi}}$ is perpendicular to $\boldsymbol{\rho}$ or ($\mathbf{r}\sin\,\theta$) in the xy plane. We, therefore, express $\boldsymbol{\rho}$ vectorially as

$$\boldsymbol{\rho} = (\rho\cos\,\varphi)\,\mathbf{i} + (\rho\sin\,\varphi)\mathbf{j} \tag{3.46}$$

or

$$\hat{\rho} \equiv \frac{\rho}{\rho} = (\cos\varphi)\,\mathbf{i} + (\sin\varphi)\,\mathbf{j} \tag{3.47}$$

As $\hat{\varphi}$ and $\hat{\rho}$ are perpendicular to each other, the expressions for $\hat{\varphi}$ can be obtained by changing the angle φ to $(\varphi + \pi/2)$ in the expression for $\hat{\rho}$.

Hence

$$\hat{\varphi} = \cos(\varphi + \pi/2)\,\mathbf{i} + \sin(\theta + \pi/2)\,\mathbf{j}$$

or

$$\hat{\varphi} = -(\sin\varphi)\,\mathbf{i} + (\cos\varphi)\,\mathbf{j} \tag{3.48}$$

We can further see that

$$\partial\hat{\mathbf{r}}/\partial\theta = (\cos\theta\cos\varphi)\,\mathbf{i} + (\cos\theta\sin\varphi)\,\mathbf{j} - (\sin\theta)\,\mathbf{k} = \hat{\theta}$$

and

$$\partial\hat{\mathbf{r}}/\partial\varphi = (-\sin\theta\sin\varphi)\,\mathbf{i} + (\sin\theta\cos\varphi)\,\mathbf{j} \tag{3.49}$$

$$= \sin\theta\,(-\sin\varphi\mathbf{i} + \cos\varphi\mathbf{j}) = \sin\theta\hat{\varphi} \tag{3.50}$$

Again,

$$\partial\hat{\theta}/\partial\theta = -(\sin\theta\cos\varphi)\,\mathbf{i} - (\sin\theta\sin\varphi)\,\mathbf{j} - (\cos\theta)\,\mathbf{k} = -\hat{\mathbf{r}} \tag{3.51}$$

We can similarly show that

$$\partial\hat{\theta}/\partial\varphi = \cos\theta\hat{\varphi} \tag{3.52}$$

$$\partial\hat{\varphi}/\partial\varphi = -(\cos\varphi\,\mathbf{i}) - (\sin\varphi\mathbf{j}) = -\hat{\rho} \tag{3.53}$$

and

$$\partial\hat{\varphi}/\partial\theta = 0$$

We will make use of the above relationships to find the expressions for velocity, acceleration and other physical quantities in three-dimensional space.

(b) Velocity

We can now find the expression for velocity using Eq. (3.40).

$$\mathbf{v} = (d\mathbf{r}/dt) = (dr/dt)\,\hat{\mathbf{r}} + r(d\theta/dt)\,\hat{\theta} + r\sin\theta\,(d\varphi/dt)\,\hat{\varphi} \tag{3.54}$$

$$= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\varphi}\hat{\varphi} \tag{3.55}$$

and

$$v^2 = \mathbf{v}\cdot\mathbf{v}$$

$$= \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2$$

$$= v_r^2 + v_\theta^2 + v_\varphi^2 \tag{3.56}$$

where

$$|\mathbf{v}_r| = \dot{r},\ |\mathbf{v}_\theta| = r\dot{\theta} \quad\text{and}\quad |\mathbf{v}_\varphi| = r\sin\theta\dot{\varphi} \tag{3.57}$$

It is instructive to note that $\mathbf{v}_r$ is the velocity along $\hat{\mathbf{r}}$, $\mathbf{v}_\theta$ is the velocity along $\hat{\theta}$, and $\mathbf{v}_\varphi$ along $\hat{\varphi}$. In polar coordinates the three velocity vectors $\mathbf{v}_r$, $\mathbf{v}_\theta$ and $\mathbf{v}_\varphi$ act as three orthogonal components of the velocity $\mathbf{v}$ in the same manner as $\mathbf{v}_x$, $\mathbf{v}_y$ and $\mathbf{v}_z$ are orthogonal components in the rectangular cartesian coordinate system.

In order to obtain the relations between the velocity components in rectangular cartesian coordinates and spherical polar coordinates, we differentiate both sides of Eq. (3.20) with respect to time. This gives

$$\dot{x} = \dot{r} \sin\theta \cos\varphi + r\dot{\theta} \cos\theta \cos\varphi - r\dot{\varphi} \sin\theta \sin\varphi \tag{3.58a}$$

$$\dot{y} = \dot{r} \sin\theta \sin\varphi + r\dot{\theta} \cos\theta \sin\varphi + r\dot{\varphi} \sin\theta \cos\varphi \tag{3.58b}$$

and

$$\dot{z} = \dot{r} \cos\theta - r\dot{\theta} \sin\theta \tag{3.58c}$$

From these equations, it can be proved that

$$\dot{r} = \frac{x\dot{x} + y\dot{y} + z\dot{z}}{(x^2 + y^2 + z^2)^{1/2}} \tag{3.59}$$

$$\dot{\theta} = \frac{(x\dot{x} + y\dot{y})z/p - \dot{z}p}{(x^2 + y^2 + z^2)} \tag{3.60}$$

and

$$\dot{\varphi} = \frac{x\dot{y} - y\dot{x}}{(x^2 + y^2)} \tag{3.61}$$

where

$$p = (x^2 + y^2)^{1/2}$$

(c) Acceleration

We recall from Eq. (3.55) that

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + \mathbf{r} \sin\theta\dot{\varphi}\hat{\boldsymbol{\varphi}}$$

Differentiating both sides, we obtain the expression for acceleration $\mathbf{a}$ as

$$\begin{aligned}
\mathbf{a} &= d\mathbf{v}/dt \\
&= (d/dt)\,(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\varphi}\hat{\boldsymbol{\varphi}}) \\
&= \ddot{r}\hat{\mathbf{r}} + \dot{r}\,(d\hat{\mathbf{r}}/dt) + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\,(d\hat{\boldsymbol{\theta}}/dt) + \dot{r}\sin\theta\,\dot{\varphi}\hat{\boldsymbol{\varphi}} + \\
&\quad r\cos\theta\dot{\theta}\,\dot{\varphi}\hat{\boldsymbol{\varphi}} + r\sin\theta\,\ddot{\varphi}\hat{\boldsymbol{\varphi}} + r\sin\theta\dot{\varphi}\;d\hat{\boldsymbol{\varphi}}/dt
\end{aligned} \tag{3.62}$$

Now

$$\begin{aligned}
d\hat{\mathbf{r}}(\theta, \varphi)/dt &= (\partial\hat{\mathbf{r}}/\partial\theta)\,(\partial\theta/dt) + (\partial\hat{\mathbf{r}}/\partial\varphi)\,(\partial\varphi/\partial t) \\
&= \dot{\theta}\hat{\boldsymbol{\theta}} + \dot{\varphi}\sin\theta\hat{\boldsymbol{\varphi}}
\end{aligned} \tag{3.63}$$

$$\begin{aligned}
d\hat{\boldsymbol{\theta}}(\theta, \varphi)/dt &= (\partial\hat{\boldsymbol{\theta}}/\partial\theta)\left(\frac{\partial\theta}{\partial t}\right) + \left(\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\phi}\right)\left(\frac{\partial\phi}{\partial t}\right) \\
&= -\dot{\theta}\hat{\mathbf{r}} + \dot{\varphi}\cos\theta\,\hat{\boldsymbol{\varphi}}
\end{aligned} \tag{3.64}$$

and

$$\begin{aligned}
d\hat{\boldsymbol{\varphi}}/dt &= (\partial\hat{\boldsymbol{\varphi}}/\partial\varphi)\,(\partial\varphi/\partial t) \\
&= -\dot{\varphi}\sin\theta\hat{\mathbf{r}} - \dot{\varphi}\cos\theta\hat{\boldsymbol{\theta}}
\end{aligned} \tag{3.65}$$

Here we have used

$$\begin{aligned}
\partial\theta/\partial t &= d\theta/dt = \dot{\theta} \\
\partial\phi/\partial t &= d\varphi/dt = \dot{\varphi}
\end{aligned} \tag{3.66}$$

which implies that θ and φ are explicit functions of time.
Hence

$$
\begin{aligned}
\mathbf{a} &= \ddot{r}\mathbf{r} + \dot{r}(\dot{\theta}\hat{\boldsymbol{\theta}} + \sin\theta\dot{\varphi}\hat{\boldsymbol{\varphi}}) + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} \\
&\quad + r\dot{\theta}(-\dot{\theta}\mathbf{r} + \cos\theta\,\dot{\varphi}\hat{\boldsymbol{\theta}}) + \dot{r}\sin\theta\dot{\varphi}\hat{\boldsymbol{\varphi}} + r\cos\theta\,\dot{\theta}\,\dot{\varphi}\hat{\boldsymbol{\varphi}} \\
&\quad + \ddot{\varphi}\sin\theta\hat{\boldsymbol{\varphi}} + r\sin\theta\dot{\varphi}(-\sin\theta\dot{\varphi}\hat{\mathbf{r}} - \cos\theta\dot{\varphi}\hat{\boldsymbol{\theta}}) \\
&= (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\varphi}^2)\hat{\mathbf{r}} \\
&\quad + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\dot{\varphi}^2)\,\hat{\boldsymbol{\theta}} \\
&\quad + (2\dot{r}\sin\theta\dot{\varphi} + 2r\cos\theta\dot{\varphi}\dot{\theta} + r\sin\theta\ddot{\varphi})\,\hat{\boldsymbol{\varphi}} \\
&= \mathbf{ar} + \mathbf{a}_\theta + \mathbf{a}_\varphi
\end{aligned}
\tag{3.67}
$$

where

$$
\begin{aligned}
\mathbf{a}_r &= (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\varphi}^2)\,\hat{\mathbf{r}} \\
\mathbf{a}_\theta &= (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\dot{\varphi}^2)\,\hat{\boldsymbol{\theta}} \\
\mathbf{a}_\varphi &= (2\dot{r}\sin\theta\dot{\varphi} + 2r\cos\theta\,\dot{\theta}\,\dot{\varphi} + r\sin\theta\ddot{\varphi})\,\hat{\boldsymbol{\varphi}}
\end{aligned}
\tag{3.68}
$$

It is easy to see from the above that $\mathbf{a}_r$ is acceleration along $\hat{\mathbf{r}}$, $\mathbf{a}_\theta$ along $\hat{\boldsymbol{\theta}}$ and $\mathbf{a}_\varphi$ along $\hat{\boldsymbol{\varphi}}$.

(d) Area

The value of the area will depend on the side of the volume element being considered. Let us consider the following three cases.

Case I: $|\mathbf{r}|$ is constant but θ and φ are variable (Fig. 3.12). The surface element is enclosed by the vectors $r\,d\theta$ and $r\sin\theta d\varphi$. Therefore, the area is given by

$$
\begin{aligned}
d\mathbf{A}_1 &= (rd\boldsymbol{\theta}) \times (r\sin\theta\, d\boldsymbol{\varphi}) \\
&= (rd\theta\hat{\boldsymbol{\theta}}) \times (r\sin\theta\, d\varphi\hat{\boldsymbol{\varphi}}) \\
&= r^2\sin\theta d\theta\, d\varphi\,(\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\varphi}}) \\
&= r^2\sin\theta d\theta\, d\varphi\hat{\mathbf{r}}
\end{aligned}
\tag{3.69}
$$

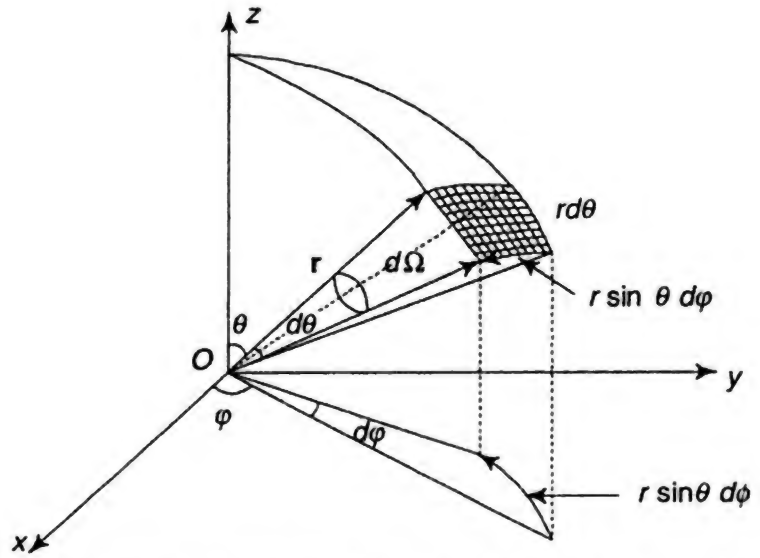


Fig. 3.12 Element of area perpendicular to **r** and the solid angle **Ω** subtended by it at the origin

Here we have made use of the fact that $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$ constitute a right-handed orthogonal system so that $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\varphi}} = \hat{\mathbf{r}}$. The area vector $d\mathbf{A}_1$ has magnitude $r^2 \sin\theta d\theta d\varphi$ and is along $\hat{\mathbf{r}}$. Therefore, the surface element is perpendicular to $\mathbf{r}$.

Case II: θ is constant but r and φ are variable (Fig. 3.13).

The vectors defining this surface element are $r \sin\theta\, d\varphi$ and $d\mathbf{r}$, so that

$$\begin{aligned} d\mathbf{A}_2 &= (r \sin\theta\, d\boldsymbol{\varphi}) \times (d\mathbf{r}) \\ &= (\mathbf{r} \sin\theta\, d\varphi\hat{\boldsymbol{\varphi}}) \times (dr\hat{\mathbf{r}}) \\ &= r \sin\theta\, dr\, d\varphi\ (\hat{\boldsymbol{\varphi}} \times \hat{\mathbf{r}}) \\ &= r \sin\theta\, dr\, d\varphi\hat{\boldsymbol{\theta}} \end{aligned} \tag{3.70}$$

Where we have used the fact that $\hat{\boldsymbol{\varphi}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}$. Since the vector representing $d\mathbf{A}_2$ is along $\hat{\boldsymbol{\theta}}$, this area is in the same plane as $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$. Also, for this plane, $d\theta = 0$ but $d\varphi$ is finite.

Case III: φ is constant but $\mathbf{r}$ and θ are variable (Fig. 3.14). From the figure, we see that the sides of this surface element are given by vectors $d\mathbf{r}$ and $rd\theta$.

Accordingly,

$$\begin{aligned} d\mathbf{A}_3 &= (d\mathbf{r}) \times (rd\theta) \\ &= (dr\,\hat{\mathbf{r}}) \times (rd\theta\hat{\boldsymbol{\theta}}) \\ &= rdr\, d\theta\ (\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}) \\ &= r\, dr\, d\theta\, \hat{\boldsymbol{\varphi}} \end{aligned} \tag{3.71}$$

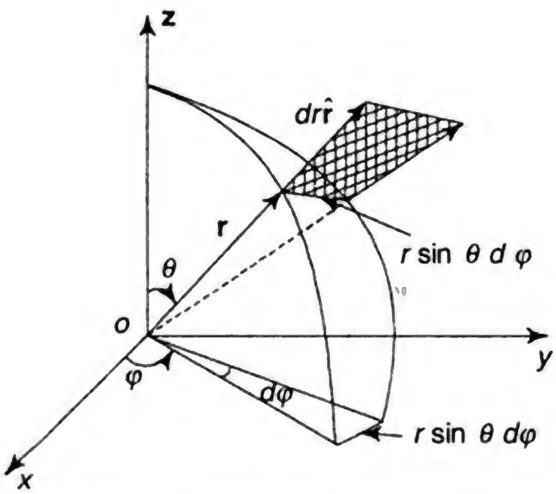


Fig. 3.13 Element of area for constant θ

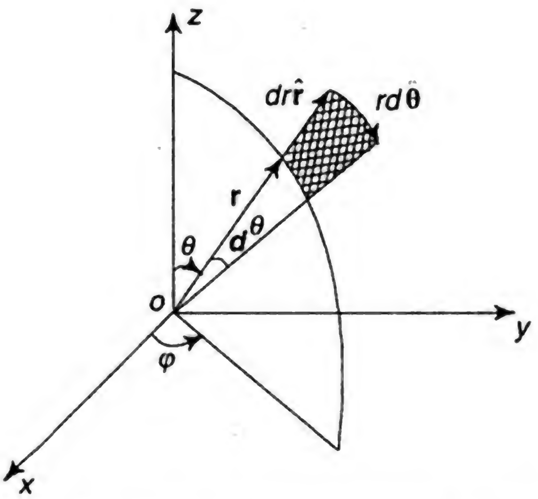


Fig. 3.14 Element of area corresponding to constant ϕ

The direction of the vector representing this area is along increasing φ and hence the area is in the plane defined by $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. In this case, the angular spread $d\boldsymbol{\theta}$ is finite but $d\varphi = 0$.

(e) Volume Element

The vectors defining the volume element are $d\mathbf{r}$, $rd\boldsymbol{\theta}$ and $r \sin\theta d\boldsymbol{\varphi}$. Hence the volume element will be given by

$$dV = (d\mathbf{r} \times rd\boldsymbol{\theta}) \cdot r \sin\theta d\boldsymbol{\varphi}$$

$$= (dr\,\hat{\mathbf{r}} \times r d\theta\hat{\boldsymbol{\theta}}) \cdot r \sin\theta\, d\varphi\, \hat{\boldsymbol{\varphi}}$$
$$= r^2 \sin \theta dr\, d\theta\, d\varphi\, (\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}) \cdot \hat{\boldsymbol{\varphi}}$$
$$= r^2 \sin\theta\, dr\, a\theta\, d\varphi\, \hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\varphi}}$$
$$= r^2 \sin\theta\, dr\, d\theta\, d\varphi \tag{3.72}$$

(f) Solid Angle

Solid angle is a measure of divergence at the meeting point of three dimensions and is the cone subtended at that point by the straight lines from all the points on the boundaries of the given surface. The solid angle made by a surface at a point is obtained by dividing the normal component of the area by the square of the distance of the surface from the point, i.e.

$$d\Omega = dA/r^2 \tag{3.73}$$

This is measured in steradians (sr).

Out of the cases of areas discussed in (d) above, only area $d\mathbf{A}_1$ is perpendicular to **r**, while other two areas are coplanar with **r**, therefore, solid angle will be defined only for $d\mathbf{A}_1$ and no solid angle is subtended by area elements $d\mathbf{A}_2$ and $d\mathbf{A}_3$ at the point 0. For $d\mathbf{A}_1$.

$$d\Omega = \frac{d\mathrm{A}_1}{r^2} = \frac{r^2 \sin\theta\, d\theta d\varphi}{r^2}$$
$$= \sin\theta\, d\theta\, d\varphi \tag{3.74}$$

One can, of course, have areas making angle α with the plane of $d\mathbf{A}_1$. In such a case, the solid angle is given by

$$d\Omega' = \frac{d\mathrm{A}_1 \cos\alpha}{r^2} = d\Omega \cos\alpha \tag{3.75}$$

Example 3.6

The motion of a particle is observed for 10 s and is found to be in accord with the following equations

$$r = R \text{ (const)}, \quad \theta = (\pi/12)\, t \quad \text{and} \quad \varphi = \pi t$$

Find the velocity and acceleration of the particle at an arbitrary time $t \leq 10$ s.

Solution

Here the components of the position vector of the particle at any time t are given to be

$$r = R, \quad \theta = (\pi/12)t \quad \text{and} \quad \varphi = \pi t$$

Differentiating with respect to time, we have

$$\dot{r} = 0, \quad \dot{\theta} = \pi/12 \quad \text{and} \quad \dot{\varphi} = \pi$$

Further differentiation with respect to time gives

$$\ddot{r} = 0, \quad \ddot{\theta} = 0 \quad \text{and} \quad \ddot{\varphi} = 0$$

Now

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r \sin\theta\, \dot{\varphi}\hat{\boldsymbol{\theta}}$$
$$= 0 + R(\pi/12)\, \hat{\boldsymbol{\theta}} + R \sin(\pi t/12)\, \pi\hat{\boldsymbol{\varphi}}$$
$$= (\pi R/12)\, \hat{\boldsymbol{\theta}} + \pi R \sin(\pi t/12)\, \hat{\boldsymbol{\varphi}}$$

$$|\mathbf{v}| = [(\pi R/12)^2 + [\pi R \sin(\pi t/12)]^2]^{1/2}$$
$$= (\pi R/12)\,[1 + 144 \sin^2(\pi t/12]^{1/2}$$

Also,

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\varphi}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\varphi}^2)\hat{\boldsymbol{\theta}}$$
$$+ (2\dot{r}\sin\theta\dot{\varphi} + 2r\cos\theta\dot{\theta}\dot{\varphi} + r\sin\theta\ddot{\varphi})\,\hat{\boldsymbol{\varphi}}$$
$$= \left[0 - R\frac{\pi^2}{144} - R\sin^2\left(\frac{\pi t}{12}\right)\pi^2\right]\hat{\mathbf{r}} + \left[0 + 0 - R\sin\left(\frac{\pi t}{12}\right)\times\right.$$
$$\left.\cos\left(\frac{\pi t}{12}\right)\pi^2\right]\hat{\boldsymbol{\theta}} + \left[0 + 2R\cos\left(\frac{\pi t}{12}\right)\cdot\frac{\pi^2}{12} + 0\right]\hat{\boldsymbol{\varphi}}$$
$$= \left[-\frac{\pi^2 R}{144} - \pi^2 R\sin^2\left(\frac{\pi t}{12}\right)\right]\hat{\mathbf{r}} - \left[\pi^2 R\sin\left(\frac{\pi t}{12}\right)\cos\left(\frac{\pi t}{12}\right)\right]\hat{\boldsymbol{\theta}}$$
$$+ \left[\frac{\pi^2 R}{6}\cos\left(\frac{\pi t}{12}\right)\right]\hat{\boldsymbol{\varphi}}$$

$$\therefore \qquad |\mathbf{a}| = \frac{\pi^2 R}{144}\left\{\left[1 + 144\sin^2\left(\frac{\pi t}{12}\right)\right]^2 + \left[144\sin\left(\frac{\pi t}{12}\right)\cos(\pi t/12)\right]^2\right.$$
$$\left. + \left[24\cos\left(\frac{\pi t}{12}\right)\right]^2\right\}^{1/2}$$
$$= \frac{\pi^2 R}{144}\left[577 + 20448\sin^2\left(\frac{\pi t}{12}\right)\right]^{1/2}$$

EXAMPLE 3.7

In spherical polar coordinates θ can have values from 0 to π and φ from 0 to 2π. Starting from the definition of the elementary solid angle given in the text, show that the solid angle subtended by a hemisphere of radius R at the centre is 2π.

Solution

The surface of a sphere is always perpendicular to its radius and, therefore, any portion of it will subtend a solid angle at the centre of the sphere. Taking the centre as the origin of the spherical polar coordinates, we have.

Element of area for constant $R = d\mathbf{A}_1 = R^2 \sin\theta\, d\theta d\varphi\hat{\mathbf{r}}$

Now for a hemisphere, θ has values from 0 to $\pi/2$ and φ from 0 to 2π so that area of surface of the hemisphere is given by

$$\text{Area} = \int dA = \int_0^{\pi}\int_0^{2\pi} R^2 \sin\theta\, d\theta d\varphi$$
$$= 2\pi R^2\,[-\cos\theta]_0^{\pi/2}$$

$$= -2\pi R^2 \cdot (0 - 1)$$
$$= 2\pi R^2$$

Alternatively, θ and φ both may change from θ to π, with the same results.

The solid angle subtended at the centre is given by

$$\Omega = \frac{A}{R^2} = 2\pi$$

QUESTIONS

3.1 Comment on the need of mass, time and space as fundamental quantities in mechanics. Discuss the possibility of using force as fundamental quantity in place of mass.

3.2 'In Newtonian mechanics space is taken to be three-dimensional and not four.' Discuss.

3.3 What are left- and right-handed cartesian coordinates?

3.4 Show that in rectangular cartesian coordinate system small increment $d\mathbf{r}$ in $\mathbf{r}$ is given by $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$.

3.5 Justify the choice of area as a vector and volume as a scalar quantity.

3.6 Obtain expressions for area and volume elements in rectangular cartesian coordinates.

3.7 Show that the velocity vector does not necessarily point in the same direction as the displacement vector.

3.8 What are spherical coordinates? How are these related to the rectangular cartesian coordinates?

3.9 Account for the statement: 'The kinematics of a charged particle moving in the field of another point charge can be treated as a problem with spherical symmetry'.

3.10 Show that in plane polar coordinates the total vectorial displacement is equal to the sum of vectorial displacements along $\mathbf{r}$ and $\boldsymbol{\theta}$, i.e. $d\mathbf{r} = dr\hat{\mathbf{r}} + rd\boldsymbol{\theta}$.

3.11 Discuss the meaning of $\boldsymbol{\theta}$ and $d\boldsymbol{\theta}$ in plane polar coordinates.

3.12 Define unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in planer motion in terms of their cartesian counterparts, i.e. $\mathbf{i}$ and $\mathbf{j}$. Also, show that $d\hat{\mathbf{r}}/d\theta = \hat{\boldsymbol{\theta}}$ and $d\hat{\boldsymbol{\theta}}/d\theta = -\hat{\mathbf{r}}$.

3.13 Starting from the expressions (3.26) and (3.27) for $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, show that

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}} \quad \text{and} \quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\hat{\mathbf{r}}$$

3.14 For planar motion, $x = r\cos\theta$ and $y = r\sin\theta$. Prove that $\dot{r} = (x\dot{x} + y\dot{y})/r$ and $r\dot{\theta} = (x\dot{y} - y\dot{x})/r$

3.15 Prove the following equalities for the motion of a particle in a plane:

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$
$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + \dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$$

3.16 Justify the terms radial and transverse (tangential) components for $\mathbf{v}_r$ and $\mathbf{v}_\theta$ as well as $\mathbf{a}_r$ and $\mathbf{a}_\theta$.

3.17 Discuss the physical significance of various terms appearing in the expression for acceleration (Question 3.15) in two dimensional motion.

3.18 Show that in spherical polar coordinates $d(\mathbf{r}) = dr\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + (r\sin\theta\, d\varphi)\hat{\boldsymbol{\varphi}}$, where the symbols have their ususal meaning.

3.19 Derive the relationships between unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in rectangular cartesian coordinate system and $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$ in spherical polar coordinates.

3.20 Starting from the expressions for $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$ in terms of the unit vectors **i, j, k**, work out the expression for $\partial\hat{\mathbf{r}}/\partial\theta$, $\partial\hat{\mathbf{r}}/\partial\varphi$, $\partial\hat{\boldsymbol{\theta}}/\partial\theta$, $\partial\hat{\boldsymbol{\theta}}/\partial\varphi$, $\partial\hat{\boldsymbol{\varphi}}/\partial\theta$ and $\partial\hat{\boldsymbol{\varphi}}/\partial\varphi$.

3.21 Show that the velocity of a particle moving in three-dimensional space can by written as

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\varphi}\,\hat{\boldsymbol{\varphi}}$$

where r, θ, φ are spherical polar coordinates of the particle.

3.22 Starting from the relationship between rectangular cartesian and spherical polar coordinates of a point, derive expressions for $\dot{r}$, $\dot{\theta}$ and $\dot{\varphi}$ in terms of x, y, z and $\dot{x}, \dot{y}, \dot{z}$.

3.23 Starting from the expression for velocity in question, find an expression for acceleration and write it in the following form

$$\mathbf{a} = a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\varphi\,\hat{\boldsymbol{\varphi}}$$

3.24 In spherical polar coordinates, one can talk about three different types of real elements. But only one of these forms a solid angle at the origin. Obtain expression for these quantities and discuss the above statement.

3.25 Define a steradian.

PROBLEMS

3.1 The motion of a particle can be expressed in terms of the following parametric equations:

$$x = 5t - 6, \quad y = 2\cos 3t, \quad z = 2\sin 3t$$

Show that the magnitudes of its velocity and acceleration are 7.81 and 18 units respectively.

3.2 Circular motion of a particle around the origin may be described by $r = b$, where b is the radius of the circle. Show that for this case

$$\mathbf{v} = b\omega\hat{\boldsymbol{\theta}} \quad \text{and} \quad a = -\omega^2 b\hat{\mathbf{r}} + b\alpha\hat{\boldsymbol{\theta}}$$

where $\omega = \dot{\theta}$ and $\alpha = \ddot{\theta}$. Discuss the nature of different terms obtained.

3.3 The velocity of a particle moving in the xy-plane is given by $\mathbf{v} = \mathbf{v}_x\,\mathbf{i} + \mathbf{v}_y\,\mathbf{j}$, at any instant when its radius vector $\mathbf{r}$ makes an angle θ with the x-axis. Show that in the polar coordinates the velocity can be expressed as

$$\mathbf{v} = (v_x\cos\theta + v_y\sin\theta)\hat{\mathbf{r}} + (v_y\cos\theta - v_x\sin\theta)\,\hat{\boldsymbol{\theta}}$$

[*Hint*: From Eqs (3.26) and (3.27) it can be shown that $\mathbf{i} = \cos\theta\ \hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$, $\mathbf{j} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}$. Substitute these values and rearrange the terms.]

3.4 A bead moves along the spoke of a cycle wheel at constant speed of 10 cm/s while the wheel is rotated about its axis at uniform angular velocity of 10 π rad/s. Find the velocity and acceleration of the bead if it were at the origin at $t = 0$ and also the spoke carrying it were along the x-axis *Ans.* $\mathbf{v} = 10\hat{\mathbf{r}} + 100\pi t\ \hat{\boldsymbol{\theta}}$

$\mathbf{a} = -1000\pi^2 t\hat{\mathbf{r}} + 200\pi\hat{\boldsymbol{\theta}}$

3.5 The motion of a particle is described by $r = be^{\omega t}$ and $\theta = \omega t$ as plane polar parameters. Obtain expressions for its velocity and acceleration assuming b and ω to be constant

Ans. $\mathbf{v} = b\,\omega e^{\omega t}\,(\hat{\mathbf{r}} + \hat{\boldsymbol{\theta}})$

$\mathbf{a} = 2b\omega^2 e^{\omega t}\,\hat{\boldsymbol{\theta}}$

3.6 Starting from the expressions for unit vectors in the spherical polar coordinate system, prove that these constitute an orthonormal set.

3.7 Find the length of an arc element on the surface of a sphere of radius R.

Ans. $ds = R[d\theta^2 + \sin^2\theta\, d\varphi^2]^{1/2}$

3.8 The position of a particle at time t is given by

$$r = R, \quad \theta = \theta_0 \sin \omega t, \quad \varphi = 2\,\omega t$$

Find expressions for velocity and acceleration.

$$\textit{Ans. } \mathbf{v} = \omega R\,[\theta_0 \cos \omega t\,\hat{\boldsymbol{\theta}} + 2 \sin(\theta_0 \sin \omega t)\,\hat{\boldsymbol{\varphi}}]$$

$$\mathbf{a} = -\omega^2 R\,[\theta_0^2 \cos^2 \omega t + 4 \sin^2(\theta_0 \sin\omega t)]\hat{\mathbf{r}} + [\theta_0 \sin \omega t$$

$$+ 2 \sin(2\theta_0 \sin \omega t)]\,\hat{\boldsymbol{\theta}} - [4\theta_0 \cos \omega t \cos(\theta_0 \sin \omega t)]\hat{\boldsymbol{\varphi}}]\}]$$

3.9 The parity of a physical quantity is said to be even or odd depending on whether its sign remains unchanged or is altered on inversion of coordinates, i.e. transformations $x \to -x$, $y \to -y$ and $z \to -z$. Find the equivalent transformations in spherical polar coordinates and hence show that $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$ have odd parity whereas the parity of $\hat{\boldsymbol{\theta}}$ is even.

Ans. $r \to r, \theta \to \pi - \theta, \varphi \to \pi + \varphi$

3.10 Show that the solid angle subtended by a ring element cut from a sphere of radius R is given by

$$d\,\Omega = 4\pi \sin(\theta/2) \cos(\theta/2)\, d\theta$$

where θ is the angle between the normal through the centre of the ring and the line joining the centre of the sphere with a point on the internal circumference and $d\theta$ is the angular width of the curved ring element.

4

Particle Dynamics

We have discussed in the previous chapter the relationship of various coordinates among themselves and with time without any reference to forces operating on a particle, i.e. the part of mechanics called kinematics. In the present chapter, we will discuss the relationship of coordinates and time with the forces operating on one particle (or a body) or many particles (or bodies). The topics falling under the purview of this discussion constitute dynamics of a particle or particles.

4.1 NEWTON'S LAWS OF MOTION

In his development of mechanics, Newton introduced three laws of motion as axioms. These laws are essentially based on different observations on the earth carried over centuries and the Kepler's three laws of planetary motion which, in turn, were developed from the observations made on the motion of planets. Since then, these laws have been successfully used to explain various astronomical and terrestrial phenomena of motion of bodies. As such, the Newton's laws form the core of classical dynamics.

These laws are stated below:

1. A body continues in a state of rest or of uniform motion in a straight line, unless it is compelled to change that state by force impressed upon it.
2. The rate of change of momentum is directly proportional to the impressed force and is in the direction in which the force acts.
3. To every action, there is always a reaction which is equal in magnitude but opposite in direction to the action.

In the statement of the second law of motion, momentum $\mathbf{p}$ is given by

$$\mathbf{p} = m\mathbf{v} = m \frac{d\mathbf{r}}{dt} \tag{4.1}$$

where m is the mass of the body and $\mathbf{v}$ is its velocity. The second law of motion can, therefore, be expressed as:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$= m \frac{d\mathbf{v}}{dt}$$

$$= m \frac{d^2\mathbf{r}}{dt^2} \tag{4.2}$$

where **F** is the force applied to the body and $d^2\mathbf{r}/dt^2$ is the acceleration. Here, it is assumed that the mass of the body is constant and does not change with motion.

From the above, it is evident that if $\mathbf{F} = 0$,

$$\frac{d\mathbf{p}}{dt} = 0$$

or

$$m\frac{d\mathbf{v}}{dt} = 0$$

or

$$\mathbf{v} = \text{const.}$$

The constancy of **v** implies constancy of speed as well as of direction. This is, in fact, the statement of the first law of motion.

In the third law of motion, action means the force due to, say body one (1) on body two (2), and may be expressed as $\mathbf{F}_{12}$. The reaction, then means the force due to body two (2) on body one (1) and may be expressed as $\mathbf{F}_{21}$.

The third law may then be expressed as

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \tag{4.3}$$

We know from experience that when we apply 'force', the body changes its motion and the change is along the direction of force. Hence, force should be a vector quantity. It was after a lot of discussion, that it was agreed that it is the rate of change of momentum, $d\mathbf{p}/dt$ which is proportional to force **F** and not the total change of momentum $d\mathbf{p}$. That no higher powers were used for m or $d\mathbf{v}/dt$, shows that physicists tend to adopt, as far as possible, the simplest possible assumptions. It is from the second law of motion that mass attains its character of inertia, i.e. greater the mass, greater the force required to give it a certain acceleration. While the unit of mass in any system is taken arbitrarily, the unit of force is then defined from it using Eq. (4.2). Thus in SI units, unit of the mass—kilogram is defined arbitrarily and Newton, the unit of force is obtained from it.

The third law of motion was the greatest contribution of Newton according to Earnst Mach, the noted philosopher. This was because of the introduction of the concept of reaction by Newton. Though this law was based on 'experience', it required a great insight to get the exact relationship of action and reaction.

As shown above, the first law is contained in the second law. Still the independent statement of the first law is necessary as it defines the zero force making it clear that it is $d\mathbf{v}/dt$ which becomes zero for zero force. Further, the third law is in accord with the second law because when no external force is acting on the system (1, 2), there should be no net acceleration of the whole system (1, 2). Assuming the presence of action $\mathbf{F}_{12}$ and reaction $\mathbf{F}_{21}$, the total force can be written as

$$\mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

or

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

However, from the second law, one cannot automatically get $\mathbf{F}_{12} = -\mathbf{F}_{21}$. It required Newton's genius to define action and reaction specifically in conformity with the experience.

EXAMPLE 4.1

The position vector of a particle of mass m moving under the influence of a force is given by

$$\mathbf{r} = A \sin \omega t\mathbf{i} + B \cos \omega t\mathbf{j}$$

Find expressions for its momentum and force.

Solution

The position vector of the particle depends on time through the relation

$$\mathbf{r} = A \sin \omega t\mathbf{i} + b \cos \omega t\mathbf{j}$$

Therefore, the velocity of the particle at time t will be given by

$$\mathbf{v} = \dot{\mathbf{r}} = A\omega \cos \omega t\mathbf{i} - B\omega \sin \omega t\mathbf{j}$$

Accordingly, the momentum of the particle is

$$\mathbf{p} = m\mathbf{v} = m\omega(A \cos \omega t\mathbf{i} - B \sin \omega t\mathbf{j})$$

The force acting on the particle,

$$\begin{aligned}\mathbf{F} = d\mathbf{p}/dt &= m\omega(-A\omega \sin \omega t\mathbf{i} - B\omega \cos \omega i\mathbf{j} \\ &= -m\omega^2 (A \sin \omega t\mathbf{i} + B \sin \omega t\mathbf{j}) \\ &= -m\omega^2\mathbf{r}\end{aligned}$$

The force is proportional to the position vector $\mathbf{r}$ of the particle. The negative sign shows that the force is directed towards the origin of the coordinate system. This is the equation for simple harmonic motion.

4.2 DYNAMICAL CONCEPTS

The fundamental concepts of length, time and mass have been discussed earlier. These ideas have in turn been used to define other physical quantities in classical mechanics. These are called derived quantities. We have already dealt with one such quantity—force. The other quantities are linear momentum, impulse, angular momentum, torque, work and energy. These are not just arbitrary functions of length, time and mass, but correspond closely to real situations that we come across in the physical world. The discussion of these concepts constitutes the contents of this section

(a) Linear Momentum

As mentioned in Sec. 4.1, linear momentum $\mathbf{p}$ of a mass moving with velocity $\mathbf{v}$ is given by

$$\mathbf{p} = m\mathbf{v} \tag{4.4}$$

Evidently $\mathbf{p}$ is a vector quantity. We have already seen that the concept of linear momentum is connected with the concept of force. A body moving with a larger linear momentum requires a larger force to stop it. Its special importance lies in expressing the fact that a large mass with a small velocity can have the same momentum as a small mass with a large velocity. It, therefore, conveys the experimental fact that it is mass multiplied by velocity which determines the force to stop it and not mass or velocity alone.

(b) Impulse

The total change of linear momentum in a given impact is called impulse, i.e.

$$\Delta \mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1$$
$$= m(\mathbf{v}_2 - \mathbf{v}_1) \tag{4.5a}$$

Like momentum, impulse is also a vector quantity. Now from the second law of motion

$$\mathbf{F} = d\mathbf{p}/dt$$

or
$$\int_{t_1}^{t_2} \mathbf{F} dt = \int_{t_1}^{t_2} d\mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1 = \Delta \mathbf{p}$$

If $\mathbf{F}$ is constant, then

$$\int_{t_1}^{t_2} \mathbf{F} dt = \mathbf{F}(t_2 - t_1) = \mathbf{F} \cdot \Delta t = \Delta \mathbf{p} \tag{4.5b}$$

Impulse is, therefore, given by the constant or average force multiplied with the duration of the impact. This brings out the physical meaning of the term 'impulse', which finds application in phenomena involving an impact for short times, such as hitting of a ball, collision of marbles, etc.

(c) Angular Momentum

This is a concept which is applied in the case of a point mass revolving around an axis or an extended body rotating around an axis passing through the body itself.

The angular momentum $\mathbf{L}$ of a mass-point m revolving around an axis is defined by

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \tag{4.6}$$

where $\mathbf{p}$ is the linear momentum of the mass point m and $\mathbf{r}$ is the radial vector from the axis of rotation to the mass point as shown in Fig. 4.1.

In view of the fact that $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, the quantity $\mathbf{L}$ is also called moment of momentum and is a pseudo-vector. It is related to the rotatory motion in the same manner, as the linear momentum is connected with linear motion. From the definition, as given in Eq. (4.6) it is clear that the angular momentum vector $\mathbf{L}$ is perpendicular to both the radial vector and linear momentum vector. From the definition as given in Eq. (4.6), it is evident that if $\mathbf{r}$ and $\mathbf{p}$ are in the same direction, the angular momentum $\mathbf{L}$ is zero.

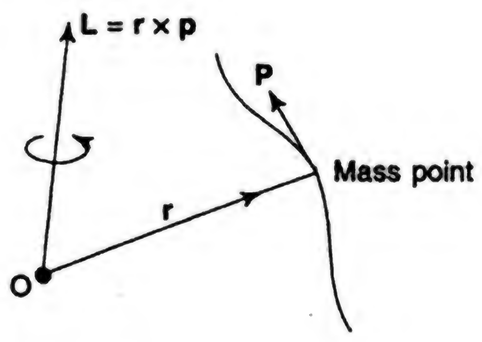


Fig. 4.1 The geometrical representations of the relation $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

The concept of angular momentum is useful in describing rotatory motion.

(d) Torque

It is common experience that if one wants to make a body rotate or revolve around an axis, one has to apply a force $\mathbf{F}$ at a point at a distance $\mathbf{r}$ from the axis so that $\mathbf{F}$

has a component perpendicular to **r**. It is also observed that if the force **F** is along **r** itself, then the body moves along **F** but does not rotate. Hence, for the rotation of the body, a physical operation is required to be impressed on the body; involving **F** and **r**, with **F** having a component perpendicular to **r**. The physical quantity involved in such an operation is called torque and is defined as

$$\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F} \tag{4.7a}$$

The torque $\mathbf{\Gamma}$ is an axial vector or a pseudo-vector and plays the same role in rotatory motion as force **F** does in linear motion. To confirm this, we note that angular momentum is defined by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

On differentiation this leads to

$$d\mathbf{L}/dt = \mathbf{r} \times d\mathbf{p}/dt + \frac{d\mathbf{r}}{dt} \times \mathbf{p}$$

But

$$\frac{d\mathbf{r}}{dt}\,\mathbf{p} = \frac{d\mathbf{r}}{dt} \times m\,\frac{d\mathbf{r}}{dt} = 0$$

Hence

$$\begin{aligned} d\mathbf{L}/dt &= \mathbf{r} \times d\mathbf{p}/dt \\ &= \mathbf{r} \times \mathbf{F} = \mathbf{\Gamma} \end{aligned} \tag{4.7b}$$

Hence, the rate of change of angular momentum gives torque $\mathbf{\Gamma}$. This provides a definition of $\mathbf{\Gamma}$ and also the relationship of $\mathbf{\Gamma}$ with **L**. One may, therefore, define torque as the rate of change of angular momentum in the same manner as force was defined as the rate of change of linear momentum. Because of Eq. (4.7a), the torque may also be defined as the moment of force, as angular momentum was defined as the moment of momentum.

It may be emphasised here that the direction of rotation of the body is vectorially represented along the same direction as $d\mathbf{L}/dt$ or $\mathbf{\Gamma}$.

It is easily seen from Eq. (4.7b) that for $\mathbf{\Gamma} = 0$,

$$\frac{d\mathbf{L}}{dt} = 0$$

i.e. **L** = constant. This means that in the absence of an external torque, the angular momentum of a body is conserved. We will discuss this aspect in detail in Sec. 4.5 and Chapter 5.

(e) Work

We have already discussed the concept of impulse as force integrated over time. Another even more useful concept is that of force integrated over space. Therefore one defines a quantity which is a multiplication of force and distance. A vector quantity of this type called torque has already been defined as one such quantity. However, as we have seen, this gives rise to rotation.

We now define a scalar quantity, which is a dot product of force and displacement and which we call it work; it is defined as

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} dW = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} \tag{4.8}$$

where $d\mathbf{r}$ is the displacement. The quantity

$$\mathbf{F} \cdot d\mathbf{r} = (F \cos \theta)\, dr \tag{4.9}$$

corresponds to the algebraic multiplication of displacement *(dr)* and the component of the force, along displacement, i.e. ($F \cos \theta$) so that Eq. (4.8) defines work *(W)* done by the force **F** on the body as it moves the body from position $\mathbf{r}_1$ to $\mathbf{r}_2$. Being a dot product of the two vectors, the physical quantity W is a scalar quantity.

Why do we call this quantity work? This nomenclature is also based on the experience of the human mind. In common language, work means an accumulated effort when force has been used in doing a certain job. It remained a debatable point whether $F\,dt$ should be called work or $\mathbf{F} \cdot d\mathbf{r}$ should be called work. Again, it is a tribute to the insight of the physicists of the earlier years that work was defined as in Eq. (4.8). This is in accord with the intuitive feeling that one gets tired on moving a body along a certain distance, using force. Such an effort gives the feeling of having worked more than simply applying the force on a stationary object for a given time. Equation (4.8) not only conforms to the intuitive experience of the human mind but also leads to the proper relationship of the fundamental properties of the physical world, as we shall see subsequently.

If the angle between force and displacement is less than $\pi/2$, work is positive. This means that force is doing work on the body in displacing it. On the other hand, if the angle is more than $\pi/2$, then the work is negative. This corresponds to the situation that the force is being applied to stop the body or the body is doing work against the force. Thus positive work means work on the body and negative work means by the body.

(f) Energy

Energy associated with a body is the capacity of the body to do work. A moving body, say (1) can make another body, say (2) move from the position of rest. In other words, it can exert force. The total work which body (1) can do before stopping depends on the initial velocity of body (1). This capacity of the moving body to do work due to its motion is called the energy of the body due to motion or the kinetic energy.

Alternatively, a body can do work because of its position in a field. The corresponding capacity to do work is called potential energy. It is the work, which can be done by the body in moving from a given position in the field to the reference point of zero potential energy.

(a) *Kinetic energy:* Suppose a body with mass m is travelling with velocity $\mathbf{v}$. As discussed above, the capacity to do work against an external force, before it stops, is called its kinetic energy. Its magnitude will be the same as work required to be done on the body in moving it from the position of rest to the state of velocity $\mathbf{v}$ by an external force.

Let a force **F** be applied to the body at rest. Then the work done to move the body by a small displacement $d\mathbf{r}$ is given by

$$dW = \mathbf{F}(r) \cdot d\mathbf{r}$$

Now

$$\mathbf{F}(r) = \lim_{\Delta t \to 0} m \frac{\Delta \mathbf{v}}{\Delta t}$$

Hence

$$dW = \lim_{\Delta t \to 0} m\frac{\Delta \mathbf{v}}{\Delta t} \cdot \Delta \mathbf{r}$$

$$= \lim_{\Delta t \to 0} m\Delta \mathbf{v} \cdot \frac{\Delta \mathbf{r}}{\Delta t}$$

$$= md\mathbf{v} \cdot \mathbf{v}$$

$$= m\mathbf{v} \cdot d\mathbf{v}$$

The kinetic energy of the body is, then, given by

$$KE = \int_0^v dW = \int_0^v m(\mathbf{v} \cdot d\mathbf{v}) = \frac{1}{2}mv^2 \tag{4.9}$$

If the moving body is under continuous influence of force **F**, the work done in moving it from position $\mathbf{r}_1$ to $\mathbf{r}_2$, will be

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} \tag{4.10a}$$

As a result of the continuous effect of force, the velocity of the body will change from $\mathbf{v}_1$ to $\mathbf{v}_2$ so that,

$$W = \int_{\mathbf{v}_1}^{\mathbf{v}_2} m\mathbf{v} \cdot d\mathbf{v}$$

$$= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$= K_2 - K_1 \tag{4.10b}$$

Here K_1 and K_2 are the kinetic energies of the body at positions $\mathbf{r}_1$ and $\mathbf{r}_2$. Equation (4.10b) is generally referred to as work-energy theorem. If $|\mathbf{v}_2| < |\mathbf{v}_1|$ then $K_2 - K_1$ is negative. This corresponds to the situation when the work is being done by the body against an external force, say, friction. On the other hand, if $|\mathbf{v}_2| > |\mathbf{v}_1|$, the work is done by the external force on the body, resulting in increase in its kinetic energy.

(b) *Potential energy:* As mentioned earlier, the potential energy of a body at a point is its capacity to do work, due to its position in a field, in moving from the given position to the reference point corresponding to zero potential energy.

Such a capacity of the body to do work because of its position arises because of the existence of field. Some examples are: (i) gravitational field, (ii) electric field, and (iii) the field due to tension in stretched spring. The first two fields are, without any tangible contact between the body in question and the source of the field while in the last case the tension of the stretched spring at different points provides the field. We will not discuss the mechanism of the intangible fields in the first two cases. It is sufficient to say that we know experimentally that such fields exist. We shall illustrate the concept of potential energy by a few examples.

Let a mass m be situated at height x from the surface of the earth, Fig. 4.1. The gravitational force acting on the body is attractive and hence negative and for any point x above ground it can be written as

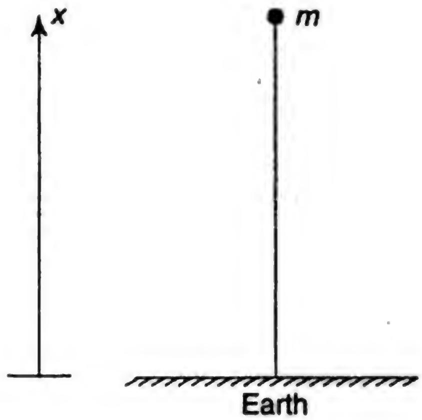


Fig. 4.2 The mass at a distance x from the earth

$$\mathbf{F}(x) = -\frac{GM}{(R+x)^2} m\hat{\mathbf{x}}$$

$$\approx -\frac{GM}{R^2} m\hat{\mathbf{x}} = -mg\hat{\mathbf{x}} \tag{4.11}$$

Here R *is* the radius of the earth, M the mass of the earth, G the gravitational constant and g the acceleration due to gravity. The negative sign shows that $\mathbf{F}(x)$ and $\mathbf{x}$ are in opposite directions. Further, $(R + x)$ has been approximated by R because $x << R$.

The work done by the gravitational force as the body falls a small distance $d\mathbf{x}$ at $\mathbf{x}$ is given by

$$dW = -mg\hat{\mathbf{x}} \cdot d\mathbf{x}$$

It should be noted that the displacement $d\mathbf{x}$ is in the same direction as $\mathbf{F}$.

The total work done by the gravitational field as the body falls from a height x to ground will be

$$U(x) = \int_x^0 dW = \int_x^0 \mathbf{F}(x)\cdot d\mathbf{x} = \int_x^0 -mg\hat{\mathbf{x}}\cdot d\mathbf{x} = mgx \tag{4.12}$$

Therefore, the potential energy of the body, $U(x)$ at the height x is given by mgx. It is instructive to note that

$$-\frac{\partial U(x)}{\partial x} = -mg = |\mathbf{F}(x)| \tag{4.13}$$

In fact, this gives the general relation between potential energy and force.

Another example to illustrate the concept of potential energy can be obtained from the electric field between two similar (say positive) electric charges q_1 and q_2. The reference point is now infinity, where there is no potential energy.

For this case, the potential energy may be defined as the work done on the charge q_1 as it is moved in the field of q_2 from infinity to position $|\mathbf{r}|$, Fig. 4.3. Now the force of repulsion between two charges separated by distance $\mathbf{r}$ is given by

$$\mathbf{F}(r) = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \tag{4.14}$$

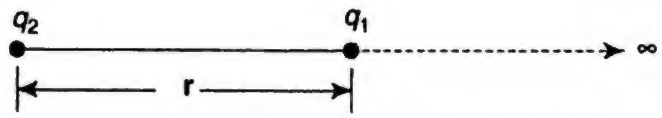


Fig. 4.3 Charges q_1 and q_1 separated by a distance r

where ε_0 is the permitivity of the medium.

The work done on the charge to give it a small displacement $d\mathbf{r}$ near $\mathbf{r}$ is given by

$$dW = \mathbf{F}(r)\cdot d\mathbf{r}$$

$$= \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\,\hat{\mathbf{r}}\cdot d\mathbf{r}$$

Hence the total work done on the charge q_1 to move it from ∞ to $\mathbf{r}$ (which is equal to the potential energy of the charge) is given by

$$U(r) = -\int_{\infty}^{r} \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\,\hat{\mathbf{r}}\cdot d\mathbf{r}$$

$$U(r) = -\int_{\infty}^{r} \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\,dr$$

$$= \frac{q_1 q_2}{4\pi\varepsilon_0 r}\left|\frac{1}{r}\right|_{\infty}^{r}$$

$$= \frac{q_1 q_2}{4\pi\varepsilon_0 r} \tag{4.15}$$

Again $$-\frac{\partial U(r)}{\partial r} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} = F(r) \tag{4.16}$$

As the third example, consider the case of a stretched spring, Fig. 4.4. Suppose the equilibrium position of the spring is 0 and it is stretched through length x. Due to elasticity a restoring force comes into play, which is proportional to the displacement $\mathbf{x}$ from the equilibrium position, i.e.

$$\mathbf{F}(x) = -k\mathbf{x} \tag{4.17}$$

Therefore, the work done on the spring in extending it from $\mathbf{x}$ to $(\mathbf{x} + d\mathbf{x})$ is given by

$$dW = \mathbf{F}(x)\cdot d\mathbf{x}$$

$$= -k\mathbf{x}\cdot d\mathbf{x}$$

Since the directions of restoring force and displacement are opposite to each other, the total work done in causing the displacement $\mathbf{x}$ will be

$$U(x) = \int_0^x -k\mathbf{x}\cdot d\mathbf{x} = -\int_0^x kx\,dx$$

$$= \frac{1}{2}kx^2 \tag{4.18}$$

Again, we see that

$$\frac{-\partial U(x)}{\partial x} = -kx = F(x) \tag{4.19}$$

We learn from these three examples that the potential energy U of a body at a point is calculated by finding out the total work done in moving the system from the reference position of zero potential to the point under consideration. In this calculation force should be written with the proper sign.

We have also learnt that

$$\mathbf{F}(x) = -\frac{\partial U(x)}{\partial x}\hat{\mathbf{x}}$$

or, in general

$$\mathbf{F}(r) = -\left(\frac{\partial U}{\partial x}\right)\hat{\mathbf{r}} = -\nabla U \tag{4.20}$$

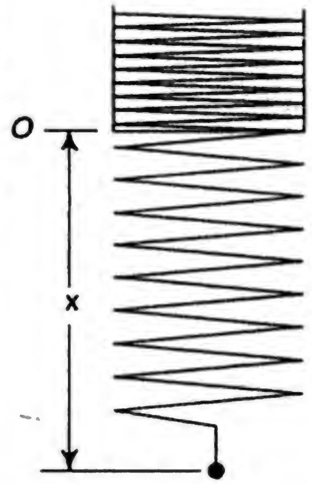


Fig. 4.4 The position of a stretched spring

The term ∇U is called the gradient of U and is expressed as:

$$\nabla U = \mathbf{i}\frac{\partial U}{\partial x} + \mathbf{j}\frac{\partial U}{\partial y} + \mathbf{k}\frac{\partial U}{\partial z} = \frac{\partial U}{\partial r}\hat{\mathbf{r}} \tag{4.21}$$

It may be pointed out that though U itself is a scalar quantity, ∇U is a vector quantity because it denotes the rate of change of U in the direction of $d\mathbf{r}$.

(g) Conservative Forces

Forces represented by Eq. (4.20) have a special property of being conservative, i.e. the total energy (sum of potential and kinetic energy) of the body located in their force-field remains constant.

This can be illustrated by considering the simple example of the mass held at a certain height x_1 from the ground. At point x_1, the body is at rest so there is no kinetic energy associated with the body. As calculated earlier, the potential energy of the body at height x_1 from the ground is given by mgx_1. Hence the total energy at point x_1 above the ground is given by

Total energy of the body at point x_1

$$= \text{kinetic energy} + \text{potential energy}$$
$$= 0 + mgx_1 = mgx_1 \tag{4.22}$$

Let the body move down from height x_1 to x_2. The potential energy of the body will now be given by mgx_2. If the velocity of the body at height x_2 is $\mathbf{v}$, then its kinetic energy, as given by Eq. (4.9) will be

$$KE = \int_0^v m\mathbf{v}\cdot d\mathbf{v} = \frac{1}{2}mv^2 \tag{4.23}$$

Therefore, the total energy of the body at x_2 is given by $(mgx_2 + \frac{1}{2}mv^2)$.

Now from Eq. (4.10a), the kinetic energy gained from x_1, to x_2 is given by

$$KE = \int_{x_1}^{x_2} \mathbf{F}\cdot d\mathbf{x}$$

$$= -\int_{x_1}^{x_2} mg\,dx$$

$$= mg\,[x_1 - x_2]$$

Accordingly

$$mg[x_1 - x_2] = \frac{1}{2}mv^2$$

or

$$mgx_1 = mgx_2 + \frac{1}{2}mv^2 \tag{4.24}$$

This equation implies that the total energy of the body at x_2 is equal to its total energy at x_1, i.e. the total energy of the body is the same at all values of x.

Next, consider the example of the electrostatic field between two positive charges. Suppose charge q_2 is stationary and another charge q_1 can move in the electrostatic field of q_2. The electrostatic force on q_1 due to the field of q_2 is given by

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\,\hat{\mathbf{r}}$$

Suppose that q_1 has velocity $\mathbf{v}_1$ at $|\mathbf{r}_1|$ and $\mathbf{v}_2$ at $|\mathbf{r}_2|$ (Fig. 4.5), then from Eqs (4.9) and (4.10a) we have

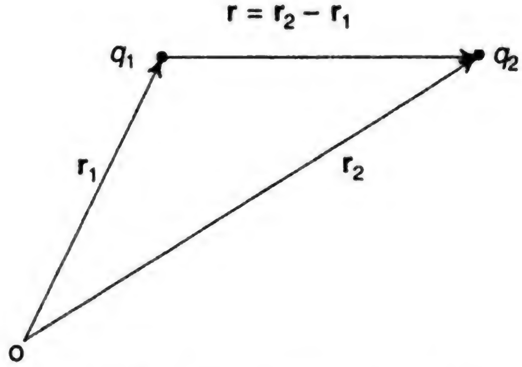


Fig. 4.5 Two charges at $\mathbf{r}_1$ and $\mathbf{r}_2$

$$KE = \int_{r_1}^{r_2} \mathbf{F}\cdot d\mathbf{r} = \int_{v_1}^{v_2} m\mathbf{v}\cdot d\mathbf{v}$$

or

$$\int_{r_1}^{r_2} \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\,dr = \int_{v_1}^{v_2} mv\,dv$$

or

$$\frac{q_1 q_2}{4\pi\varepsilon_0 r_1} - \frac{q_1 q_2}{4\pi\varepsilon_0 r_2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

or

$$\frac{q_1 q_2}{4\pi\varepsilon_0 r_1} + \frac{1}{2}mv_1^2 = \frac{q_1 q_2}{4\pi\varepsilon_0 r_2} + \frac{1}{2}mv_2^2 \tag{4.25}$$

Here $\frac{q_1 q_2}{4\pi\varepsilon_0 r_1}$ and $\frac{q_1 q_2}{4\pi\varepsilon_0 r_2}$ are potential energies of q_1 in the field of q_2 at points $|\mathbf{r}_1|$ and $|\mathbf{r}_2|$, while $\frac{1}{2}mv_1^2$ and $\frac{1}{2}mv_2^2$ are the corresponding kinetic energies.

It again shows that the sum of potential and kinetic energies of a charged body moving under the influence of an electrostatic field is always the same.

The forces, which can be derived from a position-dependent potential [as in Eq. (4.20)] are called conservative forces, because they lead to the conservation of the total energy of the body. It may be emphasised that the energy of the body is conserved only if the forces are derivable from a position dependent potential. If the potential is velocity-dependent, then the forces are not conservative. Such a situation can arise in frictional forces or electromagnetic forces, arising from moving charges. Forces arising from velocity dependent potentials are called nonconservative forces.

We have seen that the difference of the potential energy of a body at two different points in the force field is given by

$$U_1 - U_2 = \int_1^2 \mathbf{F}(r)\cdot d(\mathbf{r}) \tag{4.26}$$

What happens if starting from point (1), we bring the body back to point (1) in a loop as shown in Fig. 4.6?

First, it should be realised that if we take the body from point 1 to point 2, whether by path C or path C', the work done is the same as seen from Eq. (4.26). For example, considering, say the case of the gravitational field, the work done due to gravity is the same between two heights, by whatever method we arrive from one height to the other.

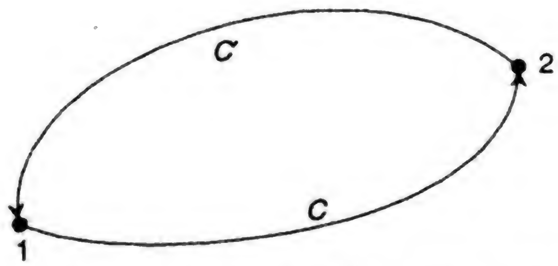


Fig. 4.6 Illustration of two paths C and C' for a point moving in a field between points 1 and 2

This means that considering Fig. 4.6, the work done to go from point 1 to point 2 via path C will be the same but opposite to the work done to move from point 2 to point 1 via path C', i.e.

$$\int_{C(1)}^{(2)} \mathbf{F}(r)\cdot d\mathbf{r} = -\int_{C'(2)}^{(1)} \mathbf{F}(r)\cdot d\mathbf{r}$$

or

$$\int_{C(1)}^{(2)} \mathbf{F}(r)\cdot d(\mathbf{r}) + \int_{C(2)}^{(1)} \mathbf{F}(r)\cdot d(\mathbf{r}) = 0 \tag{4.27}$$

In other words, the work done to bring the body back to point 1 after going through a loop is zero or

$$\oint \mathbf{F}(r) \cdot d(\mathbf{r}) = 0 \tag{4.28}$$

In fact, this is taken as the condition for a force to be conservative.

The following comments are pertinent in connection with the concept of potential energy in a conservative force field:

1. The work done by a conservative force, Eq. (4.26) always results in the decrease of potential energy.
2. It is the difference in potential energy rather than the absolute value of potential energy, which is important. The choice of a reference point or point of zero potential energy is arbitrary.

Although Eq. (4.26) linking the potential energy difference and work is like the Eq. (4.10b) representing work-energy principle, Eq. (4.26) is valid only for conservative forces whereas Eq. (4.10b) holds for all type of forces.

EXAMPLE 4.2

A particle of 0.010 kg moves along the curve

$$\mathbf{r} = [(10t^3 - 5t^2)\mathbf{i} + 5t^2\,\mathbf{j} + (t^2 - 5)\mathbf{k}]\ \text{m}$$

Determine the angular momentum about the origin of the coordinate system and the torque acting on it at $t = 1$ s.

Solution

The position vector of the particle is defined by

$$\mathbf{r} = [10t^3 - 5t^2)\mathbf{i} + 5t^2\,\mathbf{j} + (t^2 - 5)\mathbf{k}]\ \text{m}$$

Therefore, velocity and acceleration will be given by

$$\dot{\mathbf{r}} = [(30t^2 - 10t)\mathbf{i} + 10t\mathbf{j} + 2t\,\mathbf{k}]\ \text{m s}^{-1}$$

$$\ddot{\mathbf{r}} = [(60t - 10)\mathbf{i} + 10\mathbf{j} + 2\mathbf{k}]\ \text{m s}^{-2}$$

Since mass of the particle is 0.010 kg, the momentum of the particle is

$$\mathbf{p} = m\dot{\mathbf{r}} = [0.1(3t^2 - t)\mathbf{i} + 0.1t\,\mathbf{j} + .02t\,\mathbf{k}]\ \text{kg m s}^{-1}$$

and force acting on it is

$$\mathbf{F} = m\ddot{\mathbf{r}} = [0.1\,(6t - 1)\,\mathbf{i} + 0.1\,\mathbf{j} + 0.02\,\mathbf{k}]\ \text{N}$$

At $t = 1$ s, the position vector $\mathbf{r}$, linear momentum $\mathbf{p}$ and force $\mathbf{F}$ are:

$$\begin{aligned}
\mathbf{r} &= [(10 - 5)\mathbf{i} + 5\mathbf{j} + (1 - 5)\mathbf{k}]\ \text{m} \\
&= (5\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})\ \text{m} \\
\mathbf{p} &= [0.1(3 - 1)\mathbf{i} + 0.1\mathbf{j} + 0.02\mathbf{k}]\ \text{kg m s}^{-1} \\
&= (0.2\mathbf{i} + 0.1\mathbf{j} + 0.02\mathbf{k})\ \text{kg m s}^{-1} \\
\mathbf{F} &= [0.1(6 - 1)\mathbf{i} + 0.1\mathbf{j} + 0.02\mathbf{k}]\ \text{N} \\
&= (0.5\mathbf{i} + 0.1\mathbf{j} + 0.02\mathbf{k})\ \text{N}
\end{aligned}$$

The angular momentum of the particle will be

$$\begin{aligned}
\mathbf{L} &= \mathbf{r} \times \mathbf{p} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 5 & -4 \\ 0.2 & 0.1 & 0.02 \end{vmatrix} \\
&= \mathbf{i}(0.1 + 0.4) + \mathbf{j}(-0.8 - 0.10) + \mathbf{k}(0.5 - 1.0) \\
&= (0.\,5\mathbf{i} - 0.9\mathbf{j} - 0.5\mathbf{k})\ \text{kg m}^2\ \text{s}^{-1}
\end{aligned}$$

The torque on the particle is

$$\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 5 & -4 \\ 0.5 & 0.1 & 0.02 \end{vmatrix}$$

$$= \mathbf{i}(0.1 + 0.4) + \mathbf{j}(-2.0 - 0.1) + \mathbf{k}(0.5 - 2.5)$$

$$= (0.5\mathbf{i} - 2.1\mathbf{j} - 2.0\mathbf{k}) \text{ N m}$$

EXAMPLE 4.3

A body of mass 1 kg having velocity $\mathbf{v}_1 = (5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k})$ m/s at $\mathbf{r}_1 = (4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k})$ m is moved to position $\mathbf{r}_2 = (5\mathbf{i} + 8\mathbf{j} + \mathbf{k})$ m along a straight line by force $\mathbf{F} = (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$ N. Find the change in the magnitude of its velocity when it moves from $\mathbf{r}_1$ to $\mathbf{r}_2$.

Solution

The force acting on the particle is

$$\mathbf{F} = (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \text{ N}$$

The particle moves in a straight line from $\mathbf{r}_1 = 4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$ to $\mathbf{r}_2 = 5\mathbf{i} + 8\mathbf{j} + \mathbf{k}$. Therefore, the displacement of the particle is

$$\Delta(\mathbf{r}) = \mathbf{r}_2 - \mathbf{r}_1 = (5 - 4)\mathbf{k} + (8 - 6)\mathbf{j} + (1 + 2)\mathbf{k}$$

$$= (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \text{ m}$$

Work done in moving the particle,

$$W = \mathbf{F} \cdot \Delta\mathbf{r}$$

$$= (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

$$= (2 - 6 + 12) \text{ J}$$

$$= 8 \text{ J}$$

Now, from Eq. (4. 10b), work done on the particle is related to its kinetic energies at $\mathbf{r}_1$ and $\mathbf{r}_2$ through

$$W = K_2 - K_1$$

Therefore

$$K_2 = W + K_1$$

But $$K_1 = \frac{1}{2}mv_1^2$$

$$= \frac{1}{2}mv_1 \cdot v_1$$

$$= \frac{1}{2} \times 1 \times (5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) \cdot (5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k})$$

$$= \frac{1}{2}(25 + 16 + 9)\text{J}$$

$$= 25 \text{ J}$$

$$K_2 = (8 + 25) \text{ J}$$

$$= 33 \text{ J}$$

Since

$$K_2 = \frac{1}{2}mv_2^2$$

$$v_2^2 = \frac{2K_2}{m}$$

$$= \frac{2 \times 33}{1}$$

$$= 66(\text{m s}^{-1})^2$$

Therefore

$$|\mathbf{v}_2| = \sqrt{66}\ \text{m s}^{-1}$$

$$= 8.12\ \text{m s}^{-1}$$

Also

$$|\mathbf{v}_1| = (\mathbf{v}_1 \cdot \mathbf{v}_1)^{1/2}$$

$$= (25 + 16 + 9)^{1/2} = (50)^{1/2} = 7.07\ \text{m s}^{-1}$$

$$|\mathbf{v}_2| - |\mathbf{v}_1| = 8.12 - 7.07 = 1.05\ \text{m s}^{-1}$$

EXAMPLE 4.4

A force is said to be conservative if $\oint \mathbf{F} \cdot d\mathbf{r} = 0$. Show that this condition can also be written as curl $\mathbf{F} = 0$.

Solution

By definition, a force is conservative if the work done by it around a closed path is zero, i.e.

$$\oint \mathbf{F}(r) \cdot d\mathbf{r} = 0$$

In vector calculus it is well-known that the integral of a vector quantity along a closed path can be converted into an integral over the surface enclosed by the closed path through Stokes theorem. Hence

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int_s \nabla \times \mathbf{F} \cdot d\mathbf{S} \tag{4.29}$$

$\nabla \times \mathbf{F}$ is called the curl of $\mathbf{F}$. Combining the above two relations for a conservative force, we have

$$\int_s (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0 \tag{4.30}$$

Since the element of the surface area is quite arbitrary, the above relation implies that the integrand must be zero, i.e.

$$\nabla \times \mathbf{F} = 0 \tag{4.31}$$

This relation is taken as a necessary and sufficient condition for a force to be conservative and can alternatively be written as

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} = 0 \tag{4.32}$$

We show below that a central force is conservative in nature. Let the central force $\mathbf{F} = F_r\hat{\mathbf{r}}$ be acting on a particle; where F_r is a function of position vector $\mathbf{r}$ and $\hat{\mathbf{r}}$ is a unit vector along the radius vector $\mathbf{r}$. When a particle moves from point 1 to 2, Fig. Ex 4.4, the work done by the central force is

$$W_{12} = \int_1^2 \mathbf{F}\cdot d\mathbf{S} = \int_1^2 F_r\hat{\mathbf{r}}\cdot d\mathbf{S}$$

$$= \int_1^2 F_r\, dr$$

As F_r is a function of r only, its integral will be a function of r. Thus,

$$W_{12} = \int_1^2 F_r\, dr = \phi|_1^2 = \phi_2 - \phi_1$$

The work done only depends on the position of points 1 and 2 and not on the path followed. This establishes the fact that the central force is conservative in nature.

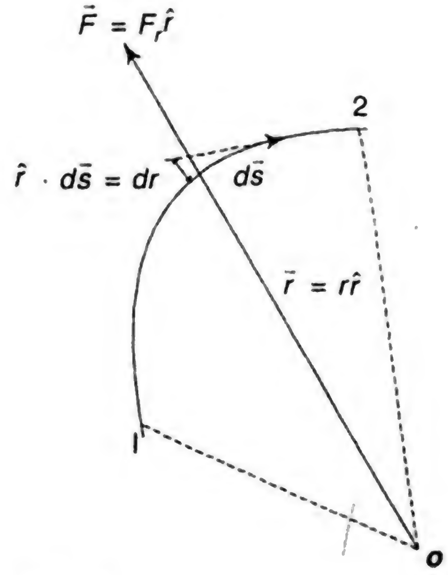


Fig. Ex 4.7

Alternatively,

$$\nabla\times\mathbf{F} = \nabla\times F_r\hat{\mathbf{r}} = \nabla\times F_r\frac{\mathbf{r}}{r}$$

$$= \frac{F_r}{r}\nabla\times\mathbf{r} = 0$$

This clearly shows that the central force is conservative in nature.

EXAMPLE 4.5

Prove that the electrostatic force between two charges is conservative. Also, obtain an expression for the potential energy of two charges.

Solution

The electrostatic force between two charges q_1 and q_2 is given by

$$\mathbf{F}(r) = \frac{q_1q_2}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

By definition,

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{r}$$

Therefore

$$\mathbf{F} = \frac{q_1q_2}{4\pi\varepsilon_0 r^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

or $$\nabla\times\mathbf{F} = \frac{q_1q_2}{4\pi\varepsilon_0}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x/r^3 & y/r^3 & z/r^3 \end{vmatrix}$$

$$= \frac{q_1 q_2}{4\pi\varepsilon_0}\left[\mathbf{i}\left\{\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^3}\right)\right\} + \mathbf{j}\left\{\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) - \frac{\partial}{\partial x}\left(\frac{z}{r^3}\right)\right\}\right.$$

$$\left. + \mathbf{k}\left\{\frac{\partial}{\partial x}\left(\frac{y}{r^3}\right) - \frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right\}\right]$$

Since $\mathbf{r} = (x^2 + y^2 + z^2)^{1/2}$, we have

$$\frac{\partial}{\partial x}\left(\frac{y}{r^3}\right) = \frac{\partial}{\partial x}\frac{y}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= y(-3/2)\,(x^2 + y^2 + z^2)^{-5/2}\,2x$$

$$= -y\,\frac{3x}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= -y\,\frac{3x}{r^5}$$

Similarly

$$\frac{\partial}{\partial x}\left(\frac{z}{r^3}\right) = -z\left(\frac{3x}{r^5}\right); \quad \frac{\partial}{\partial y}\left(\frac{x}{r^3}\right) = -x\left(\frac{3y}{r^5}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right) = -z\left(\frac{3y}{r^5}\right); \quad \frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) = -x\left(\frac{3z}{r^5}\right)$$

$$\frac{\partial}{\partial z}\left(\frac{y}{r^3}\right) = -y\left(\frac{3z}{r^5}\right)$$

Therefore

$$\nabla \times \mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0}\left[\mathbf{i}\left\{-\frac{3yz}{r^5} + \frac{3yz}{r^5}\right\} + \mathbf{j}\left\{-\frac{3zx}{r^5} + \frac{3zx}{r^5}\right\} + \mathbf{k}\left\{-\frac{3xy}{r^5} + \frac{3xy}{r^5}\right\}\right]$$

$$= \frac{q_1 q_2}{4\pi\varepsilon_0}[0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}]$$

$$= 0$$

Hence electrostatic force is conservative in nature.

Since the electrostatic force is conservative, the potential energy can be obtained from

$$U = -\int \mathbf{F}\cdot d\mathbf{r}$$

It is easier to evaluate this integral in spherical polar coordinates because $\mathbf{F}$ is radial in nature. Let the initial coordinates of q_2 be $(r_0, \theta_0, \varphi_0)$ and the final coordinates (r, θ, φ). The charge can be moved from point 1 to point 2 along the path shown in Fig. 4.8. Along P_1 only $\mathbf{r}$ is changing and θ, φ are constant so that

$$\mathbf{ds} \equiv d\mathbf{r}$$

$$= dr\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta d\varphi\,\hat{\boldsymbol{\varphi}}$$

$$= dr\hat{\mathbf{r}}$$

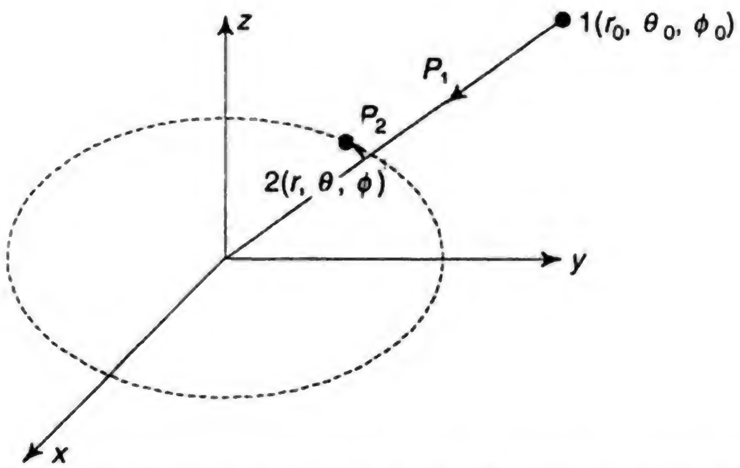


Fig. 4.8 Representation of two points 1 and 2 at $1(r_0, \theta_0, \phi_0)$ and $2(r, \theta, \phi)$; the location of the points at (r_0, θ_0, ϕ) and (r, θ, ϕ)

Along P_2, r is constant and θ, φ are changing, therefore,

$$\mathbf{ds} = d\mathbf{r} = rd\theta\ \hat{\boldsymbol{\theta}} + r \sin\ \theta d\varphi\, \hat{\boldsymbol{\varphi}}$$

Hence

$$U = -\int_{1}^{2} \mathbf{F} \cdot \mathbf{ds}$$

$$= -\int_{P_1} \mathbf{F} \cdot \mathbf{ds} - \int_{P_2} \mathbf{F} \cdot \mathbf{ds}$$

$$= -\int_{r_0}^{r} \mathbf{F} \cdot \hat{\mathbf{r}} dr - \int_{\theta_0, \phi_0}^{\theta, \phi} \mathbf{F} \cdot (r d\theta \hat{\boldsymbol{\theta}} + r \sin\theta\ d\varphi\, \hat{\boldsymbol{\varphi}})$$

Since

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \qquad \text{and} \quad \mathbf{F} \cdot \hat{\boldsymbol{\theta}} = \mathbf{F} \cdot \hat{\boldsymbol{\varphi}} = 0$$

Accordingly

$$U = -\frac{q_1 q_2}{4\pi\varepsilon_0} \int_{r_0}^{r} \frac{\hat{\mathbf{r}}}{r^2} \cdot \hat{\mathbf{r}} dr = +\frac{q_1 q_2}{4\pi\varepsilon_0} \left[\frac{1}{r}\right]_{r_0}^{r}$$

or

$$U = \frac{q_1 q_2}{4\pi\varepsilon_0} \left(\frac{1}{r} - \frac{1}{r_0}\right)$$

The potential energy of q_2 in the field of q_1 is defined with reference to the situation when q_2 is initially at infinity, i.e. $\mathbf{r} = \infty$. For this case. $(r_0, \theta_0, \varphi_0)$ are replaced by $(\infty, \theta_0, \varphi_0)$. Substituting this in the above relationship, we get the expression for potential energy as

$$U = \frac{q_1 q_2}{4\pi\varepsilon_0 r}$$

4.3 MECHANICS OF A SYSTEM OF PARTICLES

Till now we have considered a single body for kinematic or dynamic problems. Though the underlying relationships governing a system of many particles are the

same as for a single particle, we require to introduce some new concepts to handle problems of many particles more conveniently. A few such concepts are (i) centre of mass (ii) centre of gravity and (iii) moment of inertia. Also, the expressions for Eqs (4.4) and (4.6) for linear and angular motion are now different.

(a) Centre of Mass

The centre of mass of a body is defined as a point inside the body so that the whole mass of the body can be considered to act on that point, for the purpose of calculating the effect of an external force on the motion of the body.

How do we find such a point? Formally, the centre of mass of a body is derived in such a manner that the moment of the mass of the whole body acting at the centre of mass, about any reference point, outside or inside the body is equal to the sum of the moments of the various mass points in the body about the reference point.

The moment of mass of a body, about any reference point is defined as the product of the mass of the body and the radial vector of the mass. One can, therefore, define the centre of mass by the equation

$$M\mathbf{R} = \sum_i m_i \mathbf{r}_i \tag{4.33}$$

where M *is* the total mass of the body, $\mathbf{R}$ is the radial vector of the centre of mass and m_i is the mass of the ith mass point whose radial vector is $\mathbf{r}_i$. The radial vectors are measured with reference to the origin of the coordinate system. Obviously,

$$M = \sum_i m_i$$

If the centre of the coordinate system is the centre of mass itself, then evidently $|\mathbf{R}| = 0$. Hence Eq. (4.33) becomes

$$\sum_i m_i \mathbf{r}_i = 0 \tag{4.34}$$

where $\mathbf{r}_i$ is now the radial vector of the ith mass point from the centre of mass. This provides a practical method of defining the centre of mass, according to which the centre of mass is a point in space so that the vector sum of the moments of mass points around that point is zero.

The centre of mass of a system is unique since it depends on the distribution of its mass and, as such, is independent of any coordinate system used to define it.

The position vector $\mathbf{R}$ of the centre of mass of a system of particles is defined as the average of the radius vectors of the particles, weighted in proportion to their masses. So

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \mathbf{r}_i}{M} \tag{4.34a}$$

Differentiating with respect to time, we obtain the centre of mass velocity

$$\dot{\mathbf{R}} = \frac{\sum_i m_i \dot{\mathbf{r}}_i}{\sum_i m_i} = \frac{\sum_i m_i \mathbf{v}_i}{\sum_i m_i} \tag{4.34b}$$

Now $\sum_i m_i \mathbf{v}_i$ is just the total momentum of the system. Since the total momentum is constant when no external forces are acting, we get

$$\dot{\mathbf{R}} = \text{constant} \tag{4.34c}$$

Thus, the centre of mass (CM) moves with constant velocity in the absence of external forces. The CM frame acts as an inertial frame of reference when no external forces are acting. This property is put to good use in the solution of collision of particles and decay in flight of unstable particles as discussed in Chapters 7 and 12.

The total linear momentum of the system is

$$\mathbf{P} = \sum m_i \dot{\mathbf{r}}_i = \frac{d}{dt}\sum_i m_i \mathbf{r}_i$$

$$= \frac{d}{dt}(M\mathbf{R}) = M\dot{\mathbf{R}}$$

and it becomes

$$\dot{\mathbf{P}} = \mathbf{F}^{\text{ext}} \tag{4.36}$$

Hence, if the external force acts on a system of many particles, the acceleration takes place as if the whole mass of the system was concentrated at the centre of mass. The motion of the centre of mass is independent of the internal forces between the constituent particles of the system, since the forces between any two particles are equal and opposite according to Newton's third law of motion. If the total external force acting on the system is zero, then the total linear momentum of the system is conserved.

The position of the CM of the system may be obtained from Eq. (4.34a). Thus,

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} \tag{4.34a}$$

Now, $\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$

and $\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$

where X, Y, and Z are the Cartesian coordinates of the center of mass. Therefore,

$$X = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

$$Y = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

$$Z = \frac{\sum_i m_i z_i}{\sum_i m_i}$$

For a continuous distribution of mass like in a rigid body, the sign of summation is replaced by that of integration over the total volume of the body. For a small volume element dV, the mass

$$dm = \rho\, dV$$

where ρ is the mass density of the body. If $\mathbf{R}$ is the position vector of the centre of mass of the body, we have

$$\mathbf{R} = \frac{\sum \mathbf{r}dm}{M} = \frac{\int_v \mathbf{r}dm}{\int_v dm}$$

and the coordinates of the CM are

$$X = \frac{\int_v x dm}{\int_v dm} = \frac{\int_v x\rho dV}{\int_v \rho dV};$$

$$Y = \frac{\int_v y dm}{\int_v dm} = \frac{\int_v y\rho dV}{\int_v \rho dV};$$

$$Z = \frac{\int_v z dm}{\int_v dm} = \frac{\int_v z\rho dV}{\int_v \rho dV}.$$

If a force acts on the ith mass point, its value can be written as

$$\mathbf{F}_i = m_i \ddot{\mathbf{r}}_i$$

Therefore, the total force will be given by

$$\begin{aligned}
\mathbf{F} &= \sum_i \mathbf{F}_i = \sum_i m_i \ddot{\mathbf{r}}_i \\
&= \sum_i m_i \frac{d^2}{dt^2}\mathbf{r}_i \\
&= \frac{d^2}{dt^2}\left(\sum_i m_i r_i\right) \\
&= \frac{d^2}{dt^2} M\mathbf{R} = M\frac{d^2\mathbf{R}}{dt^2} = M\ddot{\mathbf{R}}
\end{aligned} \tag{4.35}$$

Hence, if a force acts on an extended rigid body (whose mass points do not change their relative distances), the acceleration of the body takes place as if the whole mass of the body was concentrated at the centre of mass. The quantity $d^2\mathbf{R}/dt^2$ is the acceleration of the centre of mass.

(b) Centre of Gravity

The centre of gravity of a body is defined as the point at which the total weight of the body can be assumed to be acting. Formally, the centre of gravity of a body is a point in the body so that the moment of the whole weight of the body assumed to be concentrated at the centre of gravity around any reference point outside or inside the body is equal to the sum of the moments of the weights of the various mass points in the body. Since the distance involved in the definition of moment is perpendicular to force, the expression defining centre of gravity can be written as

$$\mathbf{R} \times M\mathbf{g} = \sum_i (\mathbf{r}_i \times m_i\mathbf{g})$$

Since $\mathbf{g}$ is essentially constant in magnitude and direction for all the points, we can write

$$M\mathbf{R} \times \mathbf{g} = \left(\sum_i m_i \mathbf{r}_i\right) \times \mathbf{g}$$

or

$$M\mathbf{R} = \sum_i m_i \mathbf{r}_i \tag{4.37}$$

which is the same as the definition for the centre of mass, i.e. the centre of gravity is the same as the centre of mass. It is easy to see that the above equation can also be written as

$$MX = \sum_i m_i x_i$$

$$MY = \sum_i m_i y_i$$

$$MZ = \sum_i m_i z_i \tag{4.38a}$$

where X, Y, and Z are the three coordinates of the centre of gravity and x_i, y_i, and z_i are the three coordinates of the ith mass point. Then

$$\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} \tag{4.38b}$$

and

$$\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k} \tag{4.38c}$$

These relationships are found helpful in the calculation of coordinates of the centre of mass or centre of gravity. This concept of the centre of gravity is illustrated in Fig. 4.9.

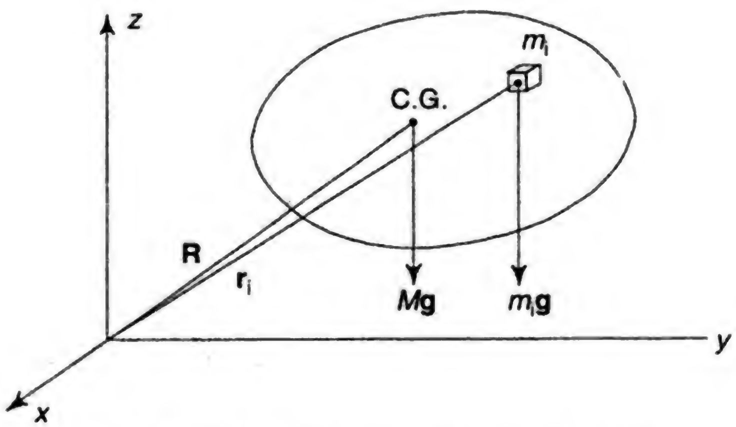


Fig. 4.9 Illustration of centre of gravity

EXAMPLE 4.6

Centre of mass

A quadrilateral ABCD has masses 1,2,3, and 4 gm located at its vertices A(–1,–2,2), B(3,2,–1), C(1,–2,4), and D(3,1,2). Find the coordinates of the centre of mass.

Solution

The position vectors of the masses 1,2,3, and 4 gm, respectively are

$$\mathbf{r}_1 = -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{r}_2 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$
$$\mathbf{r}_3 = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$
$$\mathbf{r}_4 = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

The position vector **R** of C.M. is

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i}$$

$$= \frac{\mathbf{i}(-1+6+3+12) + \mathbf{j}(-2+4-6+4) + \mathbf{k}(2-2+12+8)}{1+2+3+4}$$

$$= \frac{20\mathbf{i} + 0\mathbf{j} + 20\mathbf{k}}{10} = 2\mathbf{i} + 0\mathbf{j} + 2\mathbf{k}$$

Thus, the coordinates of C.M. are (2,0.2)

EXAMPLE 4.7

Three particles of masses 2,1,3 gm, respectively, have position vectors

$$\mathbf{r}_1 = 5t\mathbf{i} - 2t^2\mathbf{j} + (3t - 2)\mathbf{k},$$
$$\mathbf{r}_2 = (2t - 3)\mathbf{i} + (12 - 5t^2)\mathbf{j} + (4 + 6t - 3t^3)\mathbf{k}$$
$$\mathbf{r}_3 = (2t - 1)\mathbf{i} + (t^2 + 2)\mathbf{j} - t^3\mathbf{k}$$

where t is the time. Find

(a) the coordinates of C.M. at $t = 1$

(b) the velocity of the C.M. at $t = 1$; and

(c) the total linear momentum of the system at $t = 1$.

Solution

(a) The position **R** of the centre of mass

$$R = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i}$$

$$= \frac{(10t + 2t - 3 + 6t - 3)\mathbf{i} + (-4t^2 + 12 - 5t^2 + 3t^2 + 6)\mathbf{j} + (6t - 4 + 4 + 6t - 3t^3 - 3t^3)\mathbf{k}}{1+2+3}$$

$$= (3t - 1)\mathbf{i} + (3 - t^2)\mathbf{j} + (2t - t^3)\mathbf{k}$$

R at time $t = 1$ is given by $\mathbf{R} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$; and its coordinates are (2,2,–1)

(b) The velocity of the centre of mass **R** is $\dot{\mathbf{R}} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

(c) The linear momentum $M\dot{\mathbf{R}} = \sum_i m_i \dot{\mathbf{r}}_i = 18\mathbf{i} - 12\mathbf{j} - 6\mathbf{k}$

EXAMPLE 4.8

If the centre of mass of three particles of masses 1,2, and 3 gm be at a point (1,–2,3), then where should a fourth particle of mass 4 gm be placed so that the combined centre of mass may be at the point (1,1,1)?

Solution

If (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) are the positions of the three masses 1,2, and 3 gm, then $1 = \dfrac{x_1 + 2x_2 + 3x_3}{1+2+3}$

or $$x_1 + 2x_2 + 3x_3 = 6 \tag{1}$$

If the fourth particle is placed at the position (x_4, y_4, z_4), then the x-coordinate of the resulting C.M. is $1 = \dfrac{x_1 + 2x_2 + 3x_3 + 4x_4}{10}$

or $$x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \tag{2}$$

Subtracting Eq. (1) from (2), we get

$$4x_4 = 4 \quad \text{or} \quad x_4 = 1 \tag{3}$$

Similarly, calculating the y_4 and z_4 coordinates, we get

$$y_4 = 5.5;\ z_4 = -2$$

Thus, the fourth particle is to placed at (1,5.5,–2).

(c) Angular Momentum of a System of Particles

As defined earlier, the angular momentum of the *i*th mass point is given by

$$\mathbf{L}_i = \mathbf{r}_i \times \mathbf{p}_i$$

In a system of particles, the total angular momentum **L** of the whole system can be written as

$$\mathbf{L} = \sum_i \mathbf{L}_i = \sum_i (\mathbf{r}_i \times \mathbf{p}_i) \tag{4.39}$$

It is, of course, evident that the angular momenta are added vectorially. Further, the above relationship can be written as

$$\mathbf{L} = \sum_i m_i (\mathbf{r}_i \times \mathbf{v}_i) \tag{4.40}$$

But $\mathbf{v}_i = \boldsymbol{\omega}_i \times \mathbf{r}_i$, where $\boldsymbol{\omega}_i$ is the angular velocity of the *i*th point. Using the theorem that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

we can write Eq. (4.40) as

$$\mathbf{L} = \sum_i m_i [\boldsymbol{\omega}_i (\mathbf{r}_i \cdot \mathbf{r}_i) - \mathbf{r}_i (\mathbf{r}_i \cdot \boldsymbol{\omega}_i)]$$

$$= \sum_i m_i r_i^2 \boldsymbol{\omega}_i - \sum_i m_i \mathbf{r}_i (\mathbf{r}_i \cdot \boldsymbol{\omega}_i) \tag{4.41}$$

In a special case, when $\mathbf{r}_i$ and $\boldsymbol{\omega}_i$ are perpendicular to each other, $\mathbf{r}_i \cdot \boldsymbol{\omega}_i = 0$. In a rigid body such a situation can arise if the body is rotating around a fixed axis. Then $\boldsymbol{\omega}_i$ are the same for all the particles and may be put as $\boldsymbol{\omega}$. Then

$$\mathbf{L} = \sum_i (m_i r_i^2)\boldsymbol{\omega}$$
$$= I\boldsymbol{\omega} \tag{4.42}$$

where

$$I \equiv \sum_i m_i r_i^2$$

is called the moment of inertia around the axis of rotation. It is a constant for the body for a given axis of rotation. We will discuss the general case in Chapter 8.

One can also calculate the expression for torque for a system of particles as follows:

$$\boldsymbol{\Gamma} = \frac{d\mathbf{L}}{dt} = \frac{d}{dt}\sum_i \mathbf{r}_i \times \mathbf{p}_i$$
$$= \sum_i \dot{\mathbf{r}}_i \times \mathbf{p}_i + \sum_i \mathbf{r}_i \times \dot{\mathbf{p}}_i \tag{4.43}$$

But

$$\sum_i \dot{\mathbf{r}}_i \times \mathbf{p}_i = \sum_i \dot{\mathbf{r}}_i \times m_i \dot{\mathbf{r}}_i$$
$$= \sum_i m_i \dot{\mathbf{r}} \times \dot{\mathbf{r}}_i \tag{4.44a}$$
$$= 0$$

and

$$\sum_i \mathbf{r}_i \times \dot{\mathbf{p}}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i \tag{4.44b}$$

Hence

$$\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}$$

In any system of particles the total force acting on any particle is a vector sum of external forces and internal forces. The internal forces may arise from inter-particle forces. The total internal force on ith particle will be given by $\sum_{j \neq i} \mathbf{F}_{ij}^{\text{int}}$, where j denotes a particle other than the ith particle, on which the force is being calculated. Hence total force $\mathbf{F}_{\text{total}}$ is the sum of the external force $\mathbf{F}_i^{\text{ext}}$ and internal forces $\sum_{j \neq i} \mathbf{F}_{ij}^{\text{int}}$.

Hence

$$\mathbf{F}^{\text{total}} = \mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{ij}^{\text{int}} \tag{4.45}$$

Therefore, Eq. (4.43) may be written as

$$\boldsymbol{\Gamma} = \sum_i \mathbf{r}_i \times \left(\mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{ij}^{\text{int}} \right)$$
$$= \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} + \sum_j{}' \mathbf{r}_i \times \sum_j{}' \mathbf{F}_{ij}^{\text{int}} \tag{4.46}$$

The symbol $\sum_j{}'$ denotes summation over all j's, except for $j \neq i$, i.e. we assume

$\mathbf{F}_{ii} = 0$. This is justified because we know from experience that a stable particle does not create any motion due to any internal forces.

Now

$$\sum_i \sum_j{}' \mathbf{r}_i \times \mathbf{F}_{ij}^{\text{int}} = \sum_j \sum_i{}' \mathbf{r}_j \times \mathbf{F}_{ji}^{\text{int}} \tag{4.47}$$

The right-hand side of the above equation has been obtained by changing i to j and j to i on the left-hand side. This is justified because i and j are both summation indices and can have all the values. Therefore, we can write

$$\sum_{ij}{}' \mathbf{r}_i \times \mathbf{F}_{ij}^{\text{int}} = \frac{1}{2} \sum_{ij}{}' \left(\mathbf{r}_i \times \mathbf{F}_{ij}^{\text{int}} + \mathbf{r}_j \times \mathbf{F}_{ji}^{\text{int}} \right) \tag{4.48}$$

Here $\sum_{ij}{}'$ stands for summation over i and j, but excluding ii and jj terms. Since from Newton's third law,

$$\mathbf{F}_{ij}^{\text{int}} = -\mathbf{F}_{ji}^{\text{int}}$$

we have

$$\sum_{ij}{}' \mathbf{r}_i \times \mathbf{F}_{ij}^{\text{int}} = \frac{1}{2} \sum_{ij}{}' \left[(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}^{\text{int}} \right] \tag{4.49}$$

$$= \frac{1}{2} \sum_{ij}{}' \mathbf{r}_{ij} \times \mathbf{F}_{ij}^{\text{int}}$$

where $\mathbf{r}_{ij}$ is the radial vector between i and j mass points and $\mathbf{F}_{ij}^{\text{int}}$ is the internal force between them. The force $\mathbf{F}_{ij}^{\text{int}}$ is naturally acting along $\mathbf{r}_{ij}$, because it arises out of the interaction between the particles i and j. We are, of course, assuming that the forces between the particles are central and are, therefore, along $\mathbf{r}_{ij}$. It can be seen that,

$$\sum_{ij}{}' \mathbf{r}_{ij} \times \mathbf{F}_{ij}^{\text{int}} = 0 \tag{4.50}$$

In this equation, there will occur pairs of $\mathbf{r}_{ij} \times \mathbf{F}_{ij}^{\text{int}}$, with opposite signs, hence the total sum will be zero. Physically this means that there cannot arise any torque from the internal forces of a stable system of particles.

Therefore

$$\boldsymbol{\Gamma} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \sum_i \boldsymbol{\Gamma}_i = \frac{d\mathbf{L}}{dt} \tag{4.51}$$

If the total torque acting on the system is zero, that is

$$\boldsymbol{\Gamma} = \frac{d\mathbf{L}}{dt} = 0 \qquad \text{or} \qquad \mathbf{L} = \text{const}$$

Thus, if the vector sum of all the external torques acting on the system is zero, then the total angular momentum of the system is constant in time and conserved. For the special case of a rigid body rotating around a fixed axis, we can write from Eq. (4.42),

$$\boldsymbol{\Gamma} = I d\omega/dt$$

$$= I\alpha \tag{4.52}$$

where $\boldsymbol{\alpha}$ is the angular acceleration.

Further, we realise that, in general, the radius vector $\mathbf{r}_i$ of its mass point with respect to the centre of coordinates, is related to the radius vector with respect to the centre of mass through

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}'_i \tag{4.53a}$$

where $\mathbf{R}$ is the radius vector of the centre of mass. We then write for velocities,

$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{r}}'_i$$

or

$$\mathbf{v}_i = \mathbf{V} + \mathbf{v}'_i \tag{4.53b}$$

One can now write the angular momentum as

$$\begin{aligned}
\mathbf{L} &= \sum_i \mathbf{r}_i \times \mathbf{p}_i \\
&= \sum_i (\mathbf{R} + \dot{\mathbf{r}}'_i) \times m_i \mathbf{v}_i \\
&= \sum_i m_i (\mathbf{R} + \dot{\mathbf{r}}'_i) \times (\dot{\mathbf{R}} + \dot{\mathbf{r}}'_i) \\
&= \sum_i m_i (\mathbf{R} \times \dot{\mathbf{R}} + \mathbf{R} \times \dot{\mathbf{r}}'_i + \mathbf{r}'_i \times \dot{\mathbf{R}} + \mathbf{r}'_i \times \dot{\mathbf{r}}'_i)
\end{aligned} \tag{4.54}$$

Now

$$\begin{aligned}
\sum_i \mathbf{R} \times m_i \dot{\mathbf{r}}'_i &= \mathbf{R} \times \sum_i m_i \dot{\mathbf{r}}'_i \\
&= \mathbf{R} \times \frac{d}{dt} \sum_i m_i \dot{\mathbf{r}}'_i
\end{aligned}$$

But from the definition of the centre of mass Eq. (4.34) $\sum_i m_i \dot{\mathbf{r}}'_i = 0$; therefore, the above term is zero.

Also,

$$\begin{aligned}
&\sum_i m_i \dot{\mathbf{r}}'_i \times \dot{\mathbf{R}} \\
&= - \dot{\mathbf{R}} \times \sum_i m_i \mathbf{r}'_i = 0
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbf{L} &= \mathbf{R} \times \dot{\mathbf{R}} \sum_i m_i + \sum_i \mathbf{r}'_i \times m_i \dot{\mathbf{r}}'_i \\
&= \mathbf{R} \times \dot{\mathbf{R}} M + \sum_i \mathbf{r}'_i \times \mathbf{p}'_i \\
&= \mathbf{R} \times \mathbf{P} + \sum_i \mathbf{L}'_i \\
&= \mathbf{R} \times \mathbf{P} + \mathbf{L}'
\end{aligned} \tag{4.55}$$

where $\mathbf{L}'$ is the total angular momentum around the centre of mass and $\mathbf{R} \times \mathbf{P}$ is the angular momentum of the centre of mass around the reference point. If the reference point itself is taken around the centre of mass, then $\mathbf{R} = 0$ and $\mathbf{L} = \mathbf{L}'$.

(d) Energy of the System of Particles

In a system of particles, the interactions among the particles are always present. These give rise to the potential energy of the system. Therefore, when the configuration of the system is altered, its potential energy is also changed. In addition, if

external forces are acting, then the system will also have potential energy due to the field responsible for the forces. The total work done in changing from configuration (1) to configuration (2) is then given by

$$W_{12} = \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i \tag{4.56}$$

As we have seen before, the ith particle may be under the influence of an external force $\mathbf{F}_i^{\text{ext}}$ and the sum of the internal forces due to interaction with the particles inside the body, i.e. $\sum_j{}' \mathbf{F}_{ij}^{\text{int}}$. Thus

$$\mathbf{F}_i = \mathbf{F}_i^{\text{ext}} + \sum_j{}' \mathbf{F}_{ij}^{\text{int}} \tag{4.57}$$

Hence,

$$W_{12} = \sum_i \int_i^2 \left[\mathbf{F}_i^{\text{ext}} \cdot d\mathbf{r}_i + \sum_j{}' \mathbf{F}_{ij}^{\text{int}} \cdot d\mathbf{r}_i \right] \tag{4.58}$$

If the internal stucture of the system does not change, the effect of the total force $\mathbf{F}_i$ on the particle will be to impart it an acceleration $\frac{d\mathbf{r}_i}{dt} = d\mathbf{v}_i/dt$. Hence

$$\sum_i \int_i^2 \mathbf{F}_i \cdot d\mathbf{r}_i = \sum_i \int_i^2 m_i \frac{d\mathbf{v}_i}{dt} \cdot \frac{d\mathbf{r}_i}{dt}\, dt$$

$$= \sum_i \int_i^2 m_i \mathbf{v}_i \cdot d\mathbf{v}_i = \sum_i \left[\tfrac{1}{2} m_i v_i^2 \right]_1^2$$

$$= \sum_i |K_i|_1^2 = |\sum_i K_i|_2 - |\sum_i K_i|_1$$

$$= K_2 - K_1 \tag{4.59}$$

where K_2 and K_1 are total kinetic energies of configurations (2) and (1) respectively.

The right-hand side of Eq. (4.57) consists of two terms corresponding to external and internal forces. Let us assume that both of them are derivable from position-dependent potentials, i.e.

$$\mathbf{F}_i^{\text{ext}} = -\nabla_i U_i^{\text{ext}} \tag{4.60a}$$

and

$$\mathbf{F}_{ij}^{\text{int}} = -\nabla_i U_i^{\text{int}} \tag{4.60b}$$

where U_i^{ext} is the potential due to external fields, such as gravitational or electrical and U_i^{int} is the potential due to internal forces.

The right-hand side of Eq. (4.58) may, therefore, be written as

$$-\sum_i \int_1^2 (\nabla_i U_i^{\text{ext}}) \cdot d\mathbf{r}_i - \sum_{ij}{}' \int_1^2 \nabla_{ij} U_{ij}^{\text{int}} \cdot d\mathbf{r}_i \tag{4.61}$$

Now let us define

$$U_i = U_i^{\text{ext}} + U_i^{\text{int}}$$

where

$$U_i^{\text{int}} = \sum_i{}' U_{ij}^{\text{int}} \tag{4.62a}$$

and $$U = \sum_i U_i = U^{ext} + U^{int} \tag{4.62b}$$

with $$U^{ext} = \sum_i U_i^{ext} \quad \text{and} \quad U^{int} = \sum_i U_i^{int} = \sum_{ij}'' U_{ij}^{int} \tag{4.62c}$$

The summation $\sum_{ij}''$ corresponds to the situation where summation is carried out so that $i = j$ is excluded and $i < j$. The terms with $i = j$ correspond to U_{ii}, which is zero and $i < j$ ensures that the terms are not counted twice.

The first term of the expression in Eq. (4.61) can, therefore, be written as

$$-\sum_i \int_1^2 \nabla_i U_i^{ext} \cdot d\mathbf{r}_i = -\sum_i \int_1^2 dU_i^{ext}$$

$$= \left| -\sum_i U_i^{ext} \right|_1^2 = U_1^{ext} - U_2^{ext} \tag{4.63}$$

For calculating ihe second part of the expression in Eq. (4.61), we note that:

1. $$\nabla_{ij} U_{ij}^{int} = \nabla_{ij} U_{ji}^{int} \tag{4.64}$$

This can be seen from the fact that the potential energy U_{ij}^{int} between two particles is a scalar quantity which depends on their separation. Therefore, it should be independent of the order in which the particles are mentioned i.e.

$$U_{ij}^{int} = U_{ji}^{int} \tag{4.65a}$$

Hence $$U^{int} = \sum_{ij}'' U_{ij}^{int} = \frac{1}{2} \sum_{ij}' U_{ij}^{int} \tag{4.65b}$$

where in $\sum_{ij}'$ one carries out summations over all i's and j's except $i = j$. The term has been multiplied by $\frac{1}{2}$ to take into account the fact that now we are counting all pairs and hence counting terms twice.

2. $$\sum_{ij}' \mathbf{F}_{ij}^{int} \cdot d\mathbf{r}_i = \sum_{ij}' \mathbf{F}_{ij}^{int} \cdot d\mathbf{r}_j$$

$$= - \sum_{ij}' \mathbf{F}_{ij}^{int} \cdot d\mathbf{r}_j$$

Hence $$\sum_{ij}' \mathbf{F}_{ij}^{int} \cdot d\mathbf{r}_i = \frac{1}{2} [\sum_{ij}' \mathbf{F}_{ij}^{int} \cdot (d\mathbf{r}_i - d\mathbf{r}_j)]$$

$$= \frac{1}{2} \sum_{ij}' \mathbf{F}_{ij}^{int} \cdot d\mathbf{r}_{ij} \tag{4.66}$$

Hence the potential energy due to internal forces is given by Eq. (4.61) as:

$$-\sum_{ij}' \int_1^2 \nabla_{ij} U_{ij}^{int} \cdot d\mathbf{r}_i = \sum_{ij}'' \int_1^2 \mathbf{F}_{ij}^{int} \cdot d\mathbf{r}_i$$

$$= \frac{1}{2} \sum_{ij}' \int_1^2 \mathbf{F}_{ij}^{int} \cdot d\mathbf{r}_{ij}$$

$$= - \frac{1}{2} \sum_{ij}' \int_1^2 \nabla_{ij} U_{ij}^{int} \cdot d\mathbf{r}_{ij}$$

$$= -\frac{1}{2}\sum_{ij}{}' \int_1^2 dU_{ij}^{\text{int}} = \left| -\frac{1}{2}\sum_{ij}{}' U_{ij}^{\text{int}} \right|_1^2$$

$$= -|U^{\text{int}}|_1^2 = U_1^{\text{int}} - U_2^{\text{int}} \tag{4.67}$$

Here we have made use of Eq. (4.65b)

Combining all terms in Eq. (4.56) as given by Eqs (4.59), (4.63) and (4.67), we get,

$$K_2 - K_1 = \left(U_1^{\text{ext}} + U_1^{\text{int}}\right) - \left(U_2^{\text{ext}} + U_2^{\text{int}}\right)$$
$$= U_1 - U_2 \tag{4.68}$$

Therefore, $$U_1 + K_1 = U_2 + K_2 \tag{4.69}$$

Substituting $U_1 + K_1 = E_1$ = Total energy in configuration (1)

and $U_2 + K_2 = E_2$ = Total energy in configuration (2)

we have $$E_1 = E_2 \tag{4.70}$$

This proves that if external and internal forces are derivable from position-dependent potentials, then the total energy of the system of particles is conserved.

(i) *Kinetic energy of a system of particles*: It is easily seen from the previous discussion that the total kinetic energy of a system of particles is given by

$$K = \sum_i K_i = \sum_i \frac{1}{2} mv_1^2 \tag{4.71}$$

There are two types of situations that arise:

1. When the inter-particle distances do not change, i.e. $\mathbf{r}_{ij}$ for each pair remains constant. This will correspond to a rigid body.

2. When the inter-particle distances vary in a random manner, e.g. in the case of a gas.

For a rigid body one can talk of a centre of mass, as defined earlier, and the kinetic energies can be calculated with respect to the centre of mass. The velocity $\mathbf{v}_i$ of the ith paiticle with respect to the origin of the coordinate system is related to its velocity $\mathbf{v}_i'$ with respect to the centre of mass through

$$\mathbf{v}_i = \mathbf{V} + \mathbf{v}_i'$$

where $\mathbf{V}$ is the velocity of the centre of mass with respect to the origin of the coordinate system. (Prove it.)

Hence $$K = \sum_i \frac{1}{2} m_i (\mathbf{V} + \mathbf{v}_i')^2$$

$$= \sum_i \frac{1}{2} m_i (\mathbf{V}\cdot\mathbf{V} + \mathbf{v}_i'\cdot\mathbf{v}_i' + 2\mathbf{V}\cdot\mathbf{v}_i')$$

$$= \frac{1}{2}\sum_i m_i \mathbf{V}^2 + \sum_i \frac{1}{2} m_i \mathbf{v}_i'^2 + \sum_i m_i \mathbf{V}\cdot\mathbf{v}_i') \tag{4.72}$$

Now $$\sum_i m_i \mathbf{V}\cdot\mathbf{v}_i' = \mathbf{V}\cdot\sum_i m_i \dot{\mathbf{r}}_i'$$

$$= \mathbf{v} \cdot \frac{d}{dt} \sum_i m_i \mathbf{r}_i'$$

But $\sum_i m_i \mathbf{r}_i' = 0$, as $\mathbf{r}_i'$ are radius vectors with respect to the centre of mass [Eq. (4.34)]. Therefore,

$$\sum_i m\mathbf{V} \cdot \mathbf{v}_i' = 0$$

Hence
$$K = \sum_i \frac{1}{2} m_i \mathbf{V}^2 + \sum_i \frac{1}{2} m_i \mathbf{v}_i'^2$$

$$= \frac{1}{2} MV^2 + K' = K_0 + K' \tag{4.73}$$

where $K' = \sum_i \frac{1}{2} m_i \mathbf{v}_i'^2$ is the kinetic energy of the system of the particles with respect to the centre of mass and $K_0 = \frac{1}{2} MV^2$ is the kinetic energy of the whole system with respect to the observer and corresponds to the linear motion of the centre of mass. This can be either due to linear motion or even angular motion if the body is rolling with respect to the observer.

The term K' denotes the kinetic energy with respect to the centre of mass. For rigid bodies, this will only correspond to rotation around an axis passing through the centre of mass. On the other hand, in the case of a gas where particles are moving randomly, we can only talk of average values, which can be dealt through the virial theorem as discussed below.

(ii) *Virial theorem*: Let us define a quantity G for a gas given by

$$G \equiv \sum_i \mathbf{p}_i \cdot \mathbf{r}_i \tag{4.74}$$

Differentiating both sides with respect to time, we can write

$$\frac{dG}{dt} = \sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i + \sum_i \mathbf{p}_i \cdot \dot{\mathbf{r}}_i \tag{4.75}$$

But
$$\dot{\mathbf{p}}_i = \mathbf{F}_i$$

The force $\mathbf{F}_i$ includes the external force and any force of constraint due to the inter-particle potential or boundary conditions of the vessel containing gas.

Also,
$$\sum_i \mathbf{p}_i \cdot \dot{\mathbf{r}}_i = \sum_i m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i$$

$$= 2K \tag{4.76}$$

Hence, we can write

$$\frac{dG}{dt} = 2K + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \tag{4.77}$$

Let us integrate the two sides over time τ, which is much larger than the time for one collision and divide by τ. This essentially amounts to finding the average of the quantities. We then get

$$\frac{1}{\tau}\int_0^\tau \frac{dG}{dt}\,dt = \frac{1}{\tau}\int_0^\tau (2K)\,dt + \frac{1}{\tau}\int_0^\tau \sum_i (\mathbf{F}_i \cdot \mathbf{r}_i)\,dt \tag{4.78}$$

or
$$\frac{1}{\tau}\,[G(\tau)-G(0)] = \overline{2K} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} \tag{4.79}$$

The average has been taken over long times, over which the conditions of the gas contained in a bounded vessel can repeat themselves, so that the value of $G(0)$ and $G(\tau)$ become the same. Then from Eq. (4.79),

$$\overline{K} = -\frac{1}{2}\overline{\sum_i \mathbf{F}_i\ \mathbf{r}_i} \tag{4.80}$$

If $\mathbf{F}_i$ is derivable from the potential, then it can be written as

$$\mathbf{F}_i = -\nabla\,\overline{U_i}$$

If
$$\overline{U_i} = ar^{n+1}$$

then for a single particle,

$$\overline{K_i} = \frac{1}{2}\overline{\frac{\partial U_i}{\partial r_i}\hat{\mathbf{r}}\cdot\mathbf{r}_i} = \frac{n+1}{2}\,ar_i^{\,n}\,r_i$$

$$= \frac{n+1}{2}\,ar_i^{\,n+1} = \frac{n+1}{2}\overline{U_i} \tag{4.81}$$

In $n = -2$, i.e. the forces obey the inverse square laws, then

$$\overline{K_i} = -\frac{1}{2}\overline{U_i} \tag{4.82}$$

Equation (4.80) is known as virial theorem and quantity $\overline{\frac{1}{2}\sum_i \mathbf{F}_i\cdot\mathbf{r}_i}$, is called the virial of Clausius, which gives the average kinetic energy of the moving particles in a gas in a vessel.

Equation (4.81) is a realistic equation for a general case, where n will correspond to van der Waals forces and Eq. (4.82) represents a special case.

(iii) *Potential energy of a system of particles*: We have already seen that the total potential energy of a system of particles can be written as

$$U = \sum_i U_i^{\text{ext}} + \frac{1}{2}\sum_i U_{ij}^{\text{int}}$$
$$= U^{\text{ext}} + U^{\text{int}} \tag{4.84}$$

The potential energy U_i^{ext}, of course, can arise from external electric fields, magnetic fields or gravitational fields, etc. The potential energy U_{ij}^{int} may or may not satisfy Eq. (4.60b). The results obtained in the previous section hold only when Eq. (4.60b) is satisfied. For the potentials which depend on velocity, etc., such as those due to frictional forces, energy has to be supplied to the system continuously.

EXAMPLE 4.9

The position vectors of three particles of masses 1, 2 and 4 g respectively are given by

$$\mathbf{r}_1 = t^4\mathbf{i} + 3t^2\mathbf{j} - 4t^3\mathbf{k}$$
$$\mathbf{r}_2 = t^2\mathbf{i} - 3t^2\mathbf{j} + 2t\mathbf{k}$$

$$\mathbf{r}_3 = t^3\mathbf{i} - 5t^2\mathbf{j} - 2t\mathbf{k}$$

where t is time in seconds and distances are in centimetres. Find expressions for (i) centre of mass, (ii) total linear momentum and (iii) total torque with respect to the origin of the coordinate system.

Solution

The masses and position vectors of the three particles are

$$m_1 = 10^{-3}\text{ kg}, \ \mathbf{r}_1 = (t^4\mathbf{i} + 3t^2\mathbf{j} - 4t^3\mathbf{k}) \times 10^{-2}\text{ m}$$
$$m_2 = 2 \times 10^{-3}\text{ kg}, \ \mathbf{r}_2 = (t^2\mathbf{i} - 3t\mathbf{j} + 2t\mathbf{k}) \times 10^{-2}\text{ m}$$
$$m_3 = 4 \times 10^{-3}\text{ kg}, \ \mathbf{r}_3 = (t^3\mathbf{i} - 5t^2\mathbf{j} - 2t\mathbf{k}) \times 10^{-2}\text{ m}$$

The total mass of the system is

$$M = \sum_i m_i = (1 + 2 + 4) \times 10^{-3}\text{ kg} = 7 \times 10^{-3}\text{ kg}$$

The position vector for the centre of mass is given by

$$R = \sum_i \frac{m_i \mathbf{r}_i}{M}$$

$$= \left[\frac{(t^4\mathbf{i} + 3t^2\mathbf{j} - 4t^3\mathbf{k}) + 2(t^2\mathbf{i} - 3t\mathbf{j} + 2t\mathbf{k}) + 4(t^3\mathbf{i} - 5t^2\mathbf{j} - 2t\mathbf{k})}{7}\right] \times 10^{-2}$$

$$= \left[\frac{(t^4 + 4t^3 + 2t^2)\mathbf{i} + (3t^2 - 6t - 20t^2)\mathbf{j} + (-4t^3 + 4t - 8t)\mathbf{k}}{7}\right] \times 10^{-2}$$

$$= 1/7[t^4 + 4t^3 + 2t^2)\,\mathbf{i} - (17t^2 + 6t)\,\mathbf{j} - 4\,(t^3 + t)\,\mathbf{k}] \times 10^{-2}\text{ m}$$

Total linear momentum of the system is

$$P = M\,\frac{d\mathbf{R}}{dt}$$

$$= 7 \times \frac{10^{-3} \times 1}{7}\,[(4t^3 + 12t^2 + 4t)\,\mathbf{i} - (34t + 6) \times \mathbf{j} - 4\,(3t^2 + 1)\,\mathbf{k}] \times 10^{-2}$$

$$= [4(t^3 + 3t^2 + t)\,\mathbf{i} - 2\,(17t + 3)\,\mathbf{j} - 4\,(3t^2 + 1)\,\mathbf{k}] \times 10^{-5}\text{ kg m s}^{-1}$$

Differentiation of the expressions for position vectors with respect to time gives instantaneous values of velocities of the particles. These are

$$\mathbf{v}_1 = \dot{\mathbf{r}}_i = (4t^3\mathbf{i} + 6t\mathbf{j} - 12t^2\mathbf{k}) \times 10^{-2}\text{ m s}^{-1}$$
$$\mathbf{v}_2 = \dot{\mathbf{r}}_2 = (2t\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \times 10^{-2}\text{ m s}^{-1}$$
$$\mathbf{v}_3 = \dot{\mathbf{r}}_3 = (3t^2\mathbf{i} - 10t\mathbf{j} - 2\mathbf{k}) \times 10^{-2}\text{ m s}^{-1}$$

It may be noted that $\sum_{i=1}^{3} m_i\mathbf{v}_i$, comes out to be equal to P, as determined above.

The total kinetic energy of the system will be

$$K = \frac{1}{2}\sum_{i=1}^{3} m_i\,|\mathbf{v}_i|^2$$

$$= \frac{1}{2}\,[1(16t^6 + 36t^2 + 144t^4) + 2\,(4t^2 + 9 + 4)$$
$$+ 4 \times (9t^4 + 100t^2 + 4)] \times 10^{-7}\text{ J}$$

$$= \frac{1}{2}[16t^6 + 180t^4 + 444t^2 + 42] \times 10^{-7}\text{ J}$$
$$= (8t^6 + 90t^4 + 222t^2 + 21) \times 10^{-7}\text{ J}$$

The total angular momentum of the system is given by

$$\mathbf{L} = \sum_{i=1}^{3} m_i (\mathbf{r}_i \times \mathbf{v}_i)$$

Now

$$\mathbf{r}_1 \times \mathbf{v}_1 = 10^{-4}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^4 & 3t^2 & -4t^2 \\ 4t^3 & 6t & -12t^2 \end{vmatrix}$$

$$= [(-36t^4 + 24t^4)\,\mathbf{i} + (-16t^6 + 12t^6)\,\mathbf{j} + (6t^5 - 12t^5)\,\mathbf{k}] \times 10^{-4}$$
$$= [-12t^4\mathbf{i} - 4t^6\mathbf{j} - 6t^5\mathbf{k}] \times 10^{-4}\text{ m}^2\text{ s}^{-1}$$

Also,

$$\mathbf{r}_2 \times \mathbf{v}_2 = 10^{-4}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & -3t & 2t \\ 2t & -3 & 2 \end{vmatrix}$$

$$= [(-6t + 6t)\,\mathbf{i} + (4t^2 - 2t^2)\,\mathbf{j} + (-3t^2 + 6t^2)\,\mathbf{k})] \times 10^{-4}$$
$$= (2t^2\,\mathbf{j} + 3t^2\,\mathbf{k}) \times 10^{-4}\text{ m}^2\text{ s}^{-1}$$

and

$$\mathbf{r}_3 \times \mathbf{v}_3 = 10^{-4}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^3 & -5t^2 & -2t \\ 3t^2 & -10t & -2 \end{vmatrix}$$

$$= [(10t^2 - 20t^2)\,\mathbf{i} + (-6t^3 + 2t^3)\,\mathbf{j} + (-10t^4 + 15t^4)\,\mathbf{k}] \times 10^{-4}$$
$$= [-10t^2\,\mathbf{i} - 4t^3\,\mathbf{j} + 5t^4\,\mathbf{k}] \times 10^{-4}\text{ m}^2\text{ s}^{-1}$$

Hence

$$\mathbf{L} = [(-12t^4\,\mathbf{i} - 4t^6\,\mathbf{j} - 6t^5\,\mathbf{k}) + 2\,(2t^2\,\mathbf{j} + 3t^2\,\mathbf{k}) + 4\,(-10t^2\,\mathbf{i} - 4t^3\,\mathbf{j} + 5t^4\,\mathbf{k})] \times 10^{-7}$$
$$= [(-12t^4 - 40t^2)\,\mathbf{i} + [(-4t^6 + 4t^2 - 16t^3)\,\mathbf{j} + (-6t^5 + 6t^2 + 20t^4)\,\mathbf{k}] \times 10^{-7}$$
$$= -[12t^4 + 40t^2)\,\mathbf{i} + (4t^6 + 16t^3 - 4t^2)\,\mathbf{j} + (6t^5 - 20t^4 - 6t^2)\,\mathbf{k}] \times 10^{-7}\text{ kg m}^2\text{ s}^{-1}$$

Further, total torque is given by

$$\mathbf{\Gamma} = d\mathbf{L}/dt$$
$$= -[(48t^3 + 80t)\,\mathbf{i} + (24t^5 + 48t^2 - 8t)\,\mathbf{j} + (30t^4 - 80t^3 - 12t)\,\mathbf{k}] \times 10^{-7}\text{ N m}$$

4.4 EQUATION OF MOTION OF A ROCKET

So far we confined ourselves to the motion of systems with constant mass. However, there are occasions in nature as well as technology when we come across situations where the mass of the system is variable during its motion. A drop of water falling through clouds will gain in mass as it descends. A rocket will shed its mass

in the form of burning fuel and it is the recoil momentum imparted to the rocket by the exhaust gas that is responsible for the acceleration of the rocket. We obtain the equation of motion of the system of variable mass through the application of the laws of conservation to such systems.

At any time 't' the rocket is burning fuel and exhausting the gases produced, say with velocity $\mathbf{u}$, wrt the rocket. Let the rocket be moving with velocity $\mathbf{v}$ in the opposite direction to fuel as seen by a stationary observer, say, on the earth, (Fig. 4.10). To this observer, the fuel will appear to be moving with a velocity $\mathbf{v} - \mathbf{u}$ in the direction of motion of the rocket. Let the mass of rocket at time t be m. After time Δt, that is, at time $t + \Delta t$, this mass decreases, that is, becomes $(m - \Delta m)$ and the velocity of the rocket increases to $\mathbf{v} + \Delta\mathbf{v}$.

We proceed to calculate the acceleration and the velocity of the rocket at any arbitrary time. As there is no outside force acting on the system of rocket plus fuel, the total linear momentum of the system plus fuel will be considered.

At time t, the rocket plus fuel in the rocket has a linear momentum, $m\mathbf{v}$. At time $t + \Delta t$,the system consists of rocket fuel in the rocket, with a mass $(m - \Delta m)$ moving with velocity $\mathbf{v} - \mathbf{u}$ as seen by the observer from earth. Therefore, the total momentum of the system at time $t + \Delta t$ is

$$(m - \Delta m)(\mathbf{v} + \Delta\mathbf{v}) + \Delta m(\mathbf{v} - \mathbf{u}) \tag{4.85}$$

To the observer, both the parts are moving in the same direction, and hence, these are to be added. Applying the law of conservation of linear momentum to the system, we can replace the vectors by their magnitudes as all the velocities involved are in the same direction (Fig. 4.10).

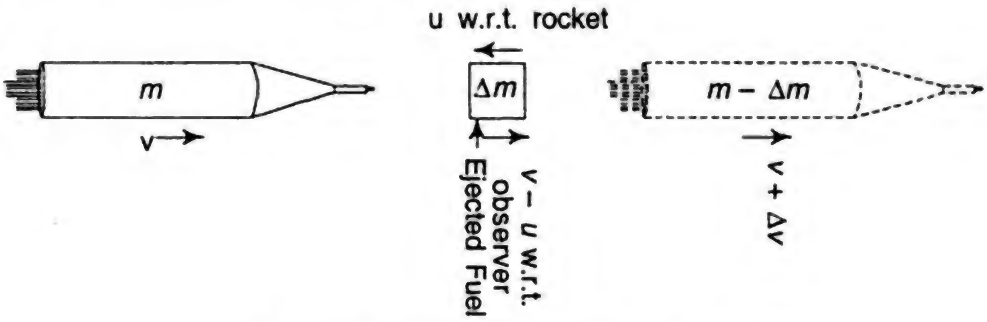


Fig. 4.10 Schematic diagram of a rocket

Thus

$$mv = (m - \Delta m)(v + \Delta v) + \Delta m\,(v - u)$$
$$m\Delta v = \Delta m\,(u + \Delta v)$$

Dividing the both sides by Δt, and taking the limit as $\Delta t \to 0$, we get

$$a = \frac{dv}{dt} = -\frac{dm}{dt}\frac{u}{m} \tag{4.86}$$

Since velocity increases as mass decreases, we added a negative sign on the right-hand side. Rewriting this equation in the form

$$m\frac{dv}{dt} = -u\frac{dm}{dt} \tag{4.87}$$

The left-hand side of Eq. (4.87) is the force exerted on the rocket and the right- hand side is the force exerted by the fuel. This can be seen as follows.

The force on the rocket $F_r = \dfrac{dP_r}{dt}$

Now, $P_r = m_r v_r$ (the subscript r stands for the rocket). As the rocket is accelerated, v_r is changing fast, but m_r is nearly constant as the dead mass of the rocket and fuel is much larger than the loss of mass of fuel. Hence, one can write.

$$\frac{dP_r}{dt} = \frac{d}{dt}(m_r v_r) = m_r\frac{dv_r}{dt} + v_r\frac{dm_r}{dt} \approx m_r\frac{dv_r}{dt}$$

On the other hand, the force exerted by the fuel on the rocket in the opposite direction may be written as $F_f = \dfrac{dP_f}{dt}$

where the subscript f stands for fuel. The fuel is being ejected at a constant velocity but its mass ejected per unit time is given by $\dfrac{dm_f}{dt}$.

Hence, $$\frac{dP_f}{dt} = \frac{d(m_f v_f)}{dt} = v_f\frac{dm_f}{dt}$$

From the third law of motion

$$F_r = -F_f$$

$$m_r\frac{dv_r}{dt} = -v_f\frac{dm_f}{dt}$$

$$= -u\frac{dm}{dt}$$

Thus, we get the same result as in Eq. (4.87).

The velocity v of the rocket at any time t, obtained by integrating Eq. (4.87), is

$$\int_o^t \frac{dv}{dt}dt = -\int_o^t \frac{u}{m}\frac{dm}{dt}dt$$

or $$v = -u\ln m + C \tag{4.88}$$

where C is the constant of integration. If at $t = 0$, $v = v_0$ and $m = m_0$, we get

$$v_o = -u\ln m_o + C$$

$$C = v_o + u\ln m_o$$

Therefore, $$v = v_0 + u\ln\frac{m_o}{m} \tag{4.89}$$

Further, if $\dfrac{dm}{dt} = \alpha$, constant, then

$$m = m_0 - \frac{dm}{dt}t = m_0 - \alpha t \tag{4.90}$$

Therefore, $$v(t) = v_0 + u\ln\left(\frac{m_o}{m_o - \alpha t}\right)$$

$$v(t) - v_0 = u\ln\left(\frac{m_o}{m_o - \alpha t}\right) \tag{4.91}$$

We see from Eq. (4.91) that if $m = \alpha t$, the velocity of the rocket at time t becomes infinite. This is, however, impossible to achieve because this implies that the whole mass of the rocket turns into fuel. However, it has been possible to design rockets such that $\alpha t = 0.9m_o$. This enables the rocket to achieve very high velocities.

It is obvious from Eq. (4.91) that the rocket will attain a higher velocity if the value of u is larger. In principle, it should be possible to have photons as our exhaust gas. Then $u = c$, the velocity of light. However, it is difficult to attain the large values of $\frac{dm}{dt}$ for photons. Hence, α is small. This implies that $\ln\frac{m_o}{m_o - \alpha t}$ will be quite small. Thus, with photons one can attain very large velocities, only if a way is found out to make the value of $\frac{dm}{dt} = \alpha$, large. It may be recalled that photons do not have rest mass but they do possess linear momentum equal to $\frac{h\upsilon}{c}$. The rate of change of momentum of the photons will be given by $\frac{nh\upsilon}{c}$, where n is the number of photons emitted per second from the rocket and υ is the frequency of the photons and h is Planck's constant. A larger value of n corresponds to large number of photons being emitted per second.

So far we neglected the force of gravity exerted by earth on the rocket. The equation of motion of the rocket, assuming that the gravitational pull of earth on it is constant, becomes

$$m\frac{dv}{dt} = -u\frac{dm}{dt} - mg \tag{4.92}$$

Integrating it wrt time, we get

$$\int\frac{dv}{dt}\,dt = -u\int\frac{dm}{m\,dt}\,dt - \int g\,dt \text{ and}$$

$$v = -u\ln m - gt + C_1$$

where C_1 is the constant of integration. According to initial conditions, at $t = 0$, $v = v_0$ and $m = m_o$, so we get $C_1 = v_o + u\ln m_o$

Therefore, $$v = v_o + u\ln\frac{m_o}{m} - gt \tag{4.93}$$

Let $\beta = \frac{\alpha}{m_o}$, the rate of change of mass in terms of initial mass, then

$$m = m_o(1 - \beta t) \tag{4.94}$$

Then, $$v = v_o - u\,ln(1 - \beta t) - gt \tag{4.95}$$

This gives the velocity of the rocket at any time t. The distance travelled by the rocket can be obtained by integrating it wrt time.

EXAMPLE 4.10

A rocket starts from rest with the exhaust velocity of gases u km/s. Calculate the velocity attained by the rocket when the mass of the rocket reduces to 1/50th of the initial mass due to burning of the fuel. The gravitational attraction may be neglected.

Solution

Let the initial mass of rocket be m_0.

The instantaneous velocity of the rocket is given by

$$\mathbf{v} = \mathbf{u} \ln \frac{m_o}{m}$$
$$= \mathbf{u} \ln 50$$
$$= \mathbf{u} (\ln 10 + \ln 5)$$
$$= \mathbf{u} (2.3 + 1.609) = 3.909\, \mathbf{u}$$

EXAMPLE 4.11

The stages of a two-stage rocket separately weigh 100 kg and 10 kg and contain 800 kg and 90 kg of fuel, respectively. Calculate the final velocity of the rocket that can be achieved with an exhaust velocity of 2 km/s. The gravitational attraction may be neglected.

Solution

The initial velocity of the rocket $v_0 = 0$

The rocket velocity after the exhaustion of the first stage is

$$\mathbf{v} = \mathbf{u} \ln \frac{m_o}{m}$$

Here, $m_o = 100 + 10 + 800 + 90 = 1000$ kg

and $m = 100 + 10 + 90 = 200$ kg

Thus,
$$|\mathbf{v}| = 2 \ln 5$$
$$= 2 \times 1.609$$
$$= 3.218 \text{ km/s}$$

The second stage becomes operative when the first is detached from it.

Thus,
$$u_o = 3.218 \text{ km/s}$$
$$m_o = 10 + 90 = 100 \text{ kg}$$
$$m = 10 \text{ kg}$$

Thus,
$$|\mathbf{v}| = 3.22 + 2 \ln \left(\frac{100}{10}\right)$$
$$= 3.22 + 2 \times (2.3)$$
$$= 7.82 \text{ km/s}$$

EXAMPLE 4.12

The final velocity of the last stage of a multistage rocket is much greater than the final velocity of a single stage rocket of the same total weight and fuel supply. Why is this so?

Solution

Let us consider a two- stage rocket for the sake of simplicity. It consists of a rocket of mass M_1, with fuel of mass m_1, and it carries with it a second rocket of mass M_2 with fuel mass m_2.

The total initial mass of the total system

$$(M_i)_1 = (M_1 + m_1) + (M_2 + m_2)$$

The rocket starts from rest and after the rocket has used up all the fuel (m_1), the final velocity attained is

$$v_1 = u_o \ln \frac{(M_i)_I}{(M_f)_I} \tag{1}$$

The second rocket has the initial velocity of v_1 and initial mass

$$(M_i)_{II} = M_2 + m_2$$

The final velocity v_2 when all the fuel is used up in both the stages

$$v_2 = v_1 + u_o \ln \frac{(M_i)_{II}}{(M_f)_{II}} \tag{2}$$

where the exhaust velocity u_0 is the same in both the stages.
Substituting for v_1 in (2) from (1), we get

$$v_2 = u_o \ln \frac{(M_i)_I}{(M_f)_I} + u_o \ln \frac{(M_i)_{II}}{(M_f)_{II}}$$

$$= u_o \ln \left[\frac{(M_i)_I}{(M_f)_I} \cdot \frac{(M_i)_{II}}{(M_f)_{II}} \right] \tag{3}$$

Assuming that $M_1 = M_2 = m_1 = m_2 = M$, say, we get

$$(M_i)_I = 4M;\ (M_f)_I = 3M;\ (M_i)_{II} = 2M;\ (M_f)_{II} = M$$

The final velocity

$$v_2 = u_o \ln\left(\frac{4M}{3M} \frac{2M}{M}\right)$$

$$= u_o \ln\left(\frac{8}{3}\right) = 0.982\, u_o$$

For single-stage rocket, the final velocity

$$v_1 = u_o \ln\left(\frac{2M}{M}\right)$$

$$= 0.6931\, u_o$$

Thus, for the same amount of fuel, the final velocity attained by the rocket will be greater if it is fired in two stages rather than one.

QUESTIONS

4.1 Comment on the need of space, time and mass as fundamental quantities of mechanics. Discuss the possibility of using force as a fundamental quantity in place of mass.

4.2 List out the properties of space that are taken as assumptions in classical mechanics.

4.3 'In Newtonian mechanics, space is taken to be three-dimensional and not four.' Discuss.

4.4 'Time flows uniformly from the present to the future.' Discuss.

4.5 Define mass and bring out its difference from the term weight.

4.6 'Definitions of mass and force are interlinked.' Comment.

4.7 State and discuss Newton's first law of motion.

4.8 Starting from the statement of Newton's second law of motion, show that it provides a means of measuring force.

4.9 'Newton's second law of motion is the most fundamental law of mechanics.' Elaborate this statement.

4.10 Give the range of values of length, time, mass and velocity over which the Newton's laws of motion are valid.

4.11 Define momentum and bring out its physical significance.

4.12 Bring out the meaning of the term 'impulse'.

4.13 Define angular momentum and justify the term 'moment of momentum' for it.

4.14 What is torque? How is it related to angular momentum?

4.15 Bring out the meaning of the term 'work'. When is it taken positive and when negative? How is it that it is assigned a sign but still taken as a scalar.

4.16 Work and torque have the same dimensions, but these are scalar and vector quantities respectively. Justify this statement.

4.17 Define energy and discuss the term kinetic energy.

4.18 The potential energy of a system is referred to the reference point. Discuss this aspect taking gravitational and electrostatic potential energies as examples.

4.19 Find an expression for the potential energy of a stretched spring.

4.20 What are conservative forces? Show lhat $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for these forces.

4.21 Define centres of mass and gravity and bring out the advantage of introducing these concepts.

4.22 Prove that the centre of mass of a system of particles moves as if the total mass and applied force were located at this point.

4.23 Write an expression for the angular momentum of a system of particles and use it to obtain an expression for the torque acting on the system.

4.24 When we consider a system of particles, the inter-particle forces are also present. Do these contribute to the expression for torque on such a system? Justify your answer.

4.25 Show that the angular momentum of a system of particles with respect to the origin of the coordinate system is equal to the vector sum of the angular momentum of the centre of mass with respect to the origin and the angular momentum of the system with respect to the centre of mass. Comment on lhe case when the centre of mass is taken as the origin of the coordinate system.

4.26 Obtain an expression for the energy of a system of particles and show that the conservation of energy holds good for these also.

4.27 Prove that the kinetic energy of a system of particles with respect to the origin of a coordinate system is the sum of the kinetic energy of the system with respect to the centre of mass and that of the centre of mass with respect to the origin of the coordinate system.

4.28 Define a virial and establish the virial theorem.

4.29 Define a central force and show that it is conservative in nature.

PROBLEMS

4.1 A particle of mass 0.004 kg moves in such a way that its position vector in metres is given by

$$\mathbf{r} = 5t^2\mathbf{i} + (3t^3 - 2t^2 + 4)\,\mathbf{j} + (t^2 - 8t)\,\mathbf{k}$$

Determine the force acting, the angular momentum about the origin of the coordinate system and the torque at $t = 2$s. *Ans.* $\mathbf{F} = 0.008\,(5\mathbf{i} + 16\mathbf{j} + \mathbf{k})$ N

$\mathbf{L} = 0.128\,(8\mathbf{i} - 5\mathbf{j} + 5\mathbf{k})$ N m

$\mathbf{\Gamma} = 0.032\,(53\mathbf{i} - 20\mathbf{j} + 55\mathbf{k})$ N m

4.2 A steel ball of 0.020 kg moves under the influence of a force field such that its position vector at time t is given by

$$\mathbf{r} = [(2t - 3)\,\mathbf{i} + (t^2 + 2)\,\mathbf{j} - 2t^3\,\mathbf{k}]\ \mathrm{m}$$

Determine the angular momentum of the ball about the origin of the coordinate system and the torque acting on it at $t = 2$s. *Ans.* $\mathbf{L} = -0.160[10\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}]$ kg m^2 s^{-1}

$\mathbf{\Gamma} = 0.040\ (-56\mathbf{i} + 12\mathbf{j} + \mathbf{k})$ N m

4.3 The motion of a particle of mass m is described by the position vector $\mathbf{r} = at^3\mathbf{i} + bt^2\mathbf{j} + ct\,\mathbf{k}$. Find expressions for linear momentum $\mathbf{p}$ and force $\mathbf{F}$ at any time t. Use these to determine angular momentum $\mathbf{L}$ and torque $\mathbf{\Gamma}$ acting on the particle. Hence show that

$$\mathbf{\Gamma} = d\mathbf{L}/dt$$

Ans. $\mathbf{p} = 3\ mat^2\,\mathbf{i} + 2\ mbt\,\mathbf{j} + mc\,\mathbf{k}$

$\mathbf{F} = 6\ mat\,\mathbf{i} + 2\ mb\,\mathbf{j}$

$\mathbf{L} = -\ mbct^2\,\mathbf{i} + 2\ mact^3\,\mathbf{j} - mab\,t^4\,\mathbf{k}$

$\mathbf{\Gamma} = -2\ mbct\,\mathbf{i} + 6\ mact^2\,\mathbf{j} - 4\ mabt^3\,\mathbf{k}$

4.4 Find the work done in moving a particle from (0, 0, 0) to (2, 3, 4) along a straight line path by force $\mathbf{F} = 4\mathbf{i} + 6\mathbf{i} + 8\mathbf{k}$. *Ans.* $W = 58$ units

4.5 A particle is under the influence of a force $\mathbf{F}$ and has instantaneous velocity $\mathbf{v}$. Find the rate at which its kinetic energy is changing. *Ans.* $dK/dt = \mathbf{F}\cdot\mathbf{v}$

4.6 A particle confined to move along the z-direction, is under the influence of force $\mathbf{F} = Ate^{-Bt}$, where A and B are positive constants. Find the change in momentum during the interval in which force increases from 0 to its maximum value. Also, determine the work done by the force during this time if the particle were at rest to begin with.

Hint: Force is maximum when $d\mathbf{F}/dt = 0$. Further, use expression for instantaneous acceleration to find velocity and hence work done through Eqs (4.9) and (4.10b).]

Ans. $\Delta p = -2A/B^2e$, $W = [2A^2/mB^4e^2]$

4.7 A particle of mass 0.020 kg has $\mathbf{p}_1 = (12\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}) \times 10^{-2}$ kg ms^{-1} at $\mathbf{r}_1 = (5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$m. Find its kinetic energy at $\mathbf{r}_2 = (8\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$ m if the force acting is $\mathbf{F} = (4\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})$ N and the particle moves from $\mathbf{r}_1$ to $\mathbf{r}_2$ along a straight path.

Ans. $K_2 = 14.7$J

4.8 Show that the force acting on a particle of mass m confined to move in a plane such that

$$\mathbf{r} = A \sin \omega t\,\mathbf{i} + B \cos \omega t\mathbf{j}$$

is conservative. Also, find expressions for its potential energy, angular momentum about the origin and the torque for rotation around an axis through the origin of the coordinate system.

Ans. $V(r) = \frac{1}{2}\,m\omega^2 r^2$

$\mathbf{L} = -\,m\omega AB\mathbf{k}$

$\mathbf{\Gamma} = 0$

4.9 The force acting on a particle of mass m moving along the x-axis is given to be

$$F(x) = Ax^2 - Bx$$

Find expressions for its acceleration and potential energy.

Ans. $a = (A/m)\,x^2 - (B/m)\,x$, $V(x) = (x^2/6)\,(2\,Ax - 3B)$

4.10 Classify the following forces as conservative and non-conservative. If possible, also determine the potential energy.

(a) $\mathbf{F} = (x^2y + z^3)\,\mathbf{i} + (3xyz - xz^2)\,\mathbf{j} + (x^2y + yz^2)\,\mathbf{k}$

(b) $\mathbf{F} = (3\ abyz^3 - 10bx^3y^2)\,\mathbf{i} + (3abxz^3 - 5bx^4y)\,\mathbf{j} + 9abxyz^2\,\mathbf{k}$

Hint: For evaluation of $\int_{0,0,0}^{x,y,z} \mathbf{F}\cdot d\mathbf{r}$, consider paths $(0, 0, 0) \to (x, 0, 0)$, $(x, 0, 0) \to (x, y, 0)$ and $(x, y, 0) \to (x, y, z)$.]

Ans. (a) Non-conservative; (b) $V = (5/2)\, bx^4y^2 - 3abxyz^3$.

4.11 Show that the gravitational force between two masses is conservative.

4.12 The mass of moon is 0.0123 times that of the earth and the distance between their centres is 384400 km. Determine the location of the centre of mass of the earth-moon system. *Ans.* $r_0 = 4668.5$ km from the centre of the earth

4.13 Show that if the total momentum of a system of particles is constant, then its centre of mass is either at rest or is moving with constant velocity.

4.14 Find the coordinates of the centre of mass of the system consisting of four particles of mass 0.005 kg at (3, 0 ,3), 0.008 kg at (–3, 2, 2); 0.01 kg at (3, – 2, 4) and 0.002 kg at (2, 2, 2). *Ans.* $x = 1$, $y = 0$, $z = 3$

4.15 The instantaneous values of position coordinates (in cm) of three particles of mass 2, 3 and 5g, respectively are given by

$$\mathbf{r}_1 = 2t^2\mathbf{i} - 4\mathbf{j}$$
$$\mathbf{r}_2 = 3t^3\mathbf{i} + 4t\mathbf{k}$$
$$\mathbf{r}_3 = 2t\mathbf{j} + 5\mathbf{k}$$

Determine the total torque acting on the system at

$$t = 0 \text{ s} \quad \text{and} \quad t = 2 \text{ s}.$$

Ans. $\boldsymbol{\Gamma}$ (0 s) = 32×10^{-7} **k** N m; $\boldsymbol{\Gamma}$ (2 s) = 8.88×10^{-6} N m

4.16 Four particles of mass 1, 2, 3 and 4 g, respectively, move under the influence of a force field so that their position vectors (in cm) are defined by

$$\mathbf{r}_1 = 2t^2\mathbf{i} + 4t\mathbf{j} + 5t\mathbf{k}$$
$$\mathbf{r}_2 = 3t\mathbf{j} - 2t^2\mathbf{k}$$
$$\mathbf{r}_3 = t^2\mathbf{i} + 2t\mathbf{k}$$
$$\mathbf{r}_4 = 4t\mathbf{i} - 2t\mathbf{j}$$

Find expressions for (i) centre of mass, (ii) lotal linear momentum, (iii) total angular momentum, (iv) total torque acting and (v) total kinetic energy with respect to the origin of the coordinate system.

Ans. $\mathbf{R} = [(5t^2 + 16t)\,\mathbf{i} + 2t\mathbf{j} - (4t^2 - 11t)\,\mathbf{k}] \times 10^{-3}$ m

$\mathbf{P} = [(10t + 16)\,\mathbf{i} + 2\mathbf{j} - (8t - 11)\,\mathbf{k}] \times 10^{-5}$ kg m s^{-1}

$\mathbf{L} = -4 \times 10^{-7}\, t^2\, (3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$ kg m^2 s^{-1}

$\boldsymbol{\Gamma} = -8 \times 10^{-7} t\, (3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$ kg m^2 s^{-2}

$K = 5\,(60t^2 + 151) \times 10^{-3}$ J

4.17 Suppose that a rocket that starts from rest, falls in a constant gravitational field. At the instant it starts to fall, it ejects gas at the constant rate α in the direction of the gravitational field and at speed u_o wrt rocket. Find its speed after time t. How far does the rocket travel in time t?

$$\left[gt - v_o \ln\left(\frac{m_o}{m_o - \alpha t}\right), \frac{1}{2}gt^2 - v_o\left\{t + \left(\frac{m_o - \alpha t}{\alpha}\right)\ln\left(\frac{m_o - \alpha t}{m_o}\right)\right\}\right]$$

5

Conservation Laws and Properties of Space and Time

5.1 INTRODUCTION

We have discussed in Chapters 3 and 4 the properties of space and time, and also the laws, of conservation of linear momentum, angular momentum and energy. However, a relationship between conservation laws and the properties of space and time, was not brought out explicitly. The properties of space and time were given as assumptions made by Newton, while the laws of conservation arose out of the definition of force, work, potential energy and kinetic energy, torque and linear as well as angular momenta. The law of conservation were obtained in a self-consistent manner from these definitions without involving the properties of space and time explicitly. Of course, the assumptions regarding the properties of space and time are built implicitly into the definitions ot various physical quantities.

We will now discuss the relationship of the properties of space and time with the laws of mechanics, especially the conservation laws in an explicit manner. To do so, we start from the three laws of motion.

1(a) According to the first law, a body continues at rest if no external force acts on it. This means that the free space in itself does not create any forces with time to move the body. Consequently, the free space (i.e. the space without any external forces) continues to be without any forces for all time to come. In other words, the properties of free space do not change with time. This is true for every point in space. Hence, the properties of every point in free space, where no external forces or fields are acting, are invariant with time.

(b) Further, the first law states that if a body is moving with a constant velocity, it continues in that direction with the same velocity, if no external force is acting. In other words,

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \text{const} \tag{5.1}$$

at different points lying on the trajectory of the particle. It is evident from Eq. (5.1) that this is possible only if:

(i) $\Delta \mathbf{r}$ has the same meaning at different points along the line, and

(ii) Δt has the same meaning at different times, as the particle progresses in its motion.

These two implicit assumptions mean that (a) the space has the same properties along any line, i.e. space is homogeneous (b) the time interval has the same properties at different times.

(iii) For a free space, the invariance of the properties of space and time are expected to continue till infinity in both space and time. This requires that the space is flat throughout, as we assume in classical mechanics, and time has the same meaning from the infinite past to the infinite future.

2. From the second law of motion, we know that

$$\mathbf{F} = m\mathbf{a}$$

$$= md^2\, \mathbf{r}/dt^2 \tag{5.2}$$

(a) If we change t to $-t$ in Eq. (5.2), it is easy to see that this equation remains unchanged. What is the significance of this? It means that laws of mechanics which are based on Newton's second law of motion are true not only for the forward advance of time from the present to the future, but also for the backward motion of time from the present to the past. Though this cannot be physically verified, because we cannot go from the present to the past, this is built in the assumptions of classical mechanics.

(b) Again, if we replace $\mathbf{r}$ by $-\mathbf{r}$ in Eq. (5.2), it remains unchanged, i.e. the equation is invariant under reflection. As the second law of motion determines the dynamics of any body, we conclude that the dynamics of any body remains unchanged, if we reflect the radius vector in the origin, i.e. change $\mathbf{r}$ to $-\mathbf{r}$ or in other words the space coordinates from x, y, z to $-x$, $-y$ and $-z$. This is possible only if space has the same property on reflection.

Summarising the above conclusions, we see that Newton's laws of motion assume that:

1. The interval has the same meaning for all times, i.e. time flows uniformly.
2. The dynamics of a system does not change, if we change from $+t$ to $-t$. In other words, time is isotropic.
3. The properties of free space are invariant with time.
4. The free space has the same properties along all straight lines, or in other words, it is homogeneous.
5. Free space has the same properties on reflection.

It will be shown that these symmetry properties of space and time lead to three very useful laws of conservation in classical mechanics. These conservation laws are stated as follows:

1. Conservation of linear momentum: According to this law, the linear momentum of an isolated body or system is conserved if no external force is acting on it.
2. Conservation of angular momentum: The angular momentum of an isolated body or system is conserved if no external torque is acting on it.
3. Conservation of total energy: The sum of potential and kinetic energies of an isolated body or system is conserved if no dissipative forces are present, i.e. for conservative forces.

Experimentally, these three conservation laws have always been found to hold good. We will now show that they are related very intimately to the properties of space and time, discussed above.

5.2 LINEAR UNIFORMITY OF SPACE AND CONSERVATION OF LINEAR MOMENTUM

Consider two particles which interact with each other. According to Newton's third law of motion,

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

where $\mathbf{F}_{21}$ is the force acting on the first body and $\mathbf{F}_{12}$ the force on the second body. Now from the second law of motion,

$$\mathbf{F}_{12} = m_2\Delta\mathbf{v}_2/\Delta t \qquad \Delta t \to 0$$

and

$$\mathbf{F}_{21} = m_1\Delta\mathbf{v}_1/\Delta t \qquad \Delta t \to 0 \tag{5.3}$$

Hence

$$(\mathbf{F}_{12} + \mathbf{F}_{21})\, \Delta t = (m_2\Delta\mathbf{v}_2 + m_1\Delta\mathbf{v}_1) \qquad \Delta t \to 0 \tag{5.4}$$

The left side of Eq. (5.4) is zero (since $\mathbf{F}_{12} = -\mathbf{F}_{21}$).

$$m_2\Delta\mathbf{v}_2 + m_1\Delta\mathbf{v}_1 = 0$$

which for $\Delta t \to 0$, becomes

$$m_2 d\mathbf{v}_2 + m_1 d\mathbf{v}_1 = 0 \tag{5.5}$$

On integration, we have

$$m_2\mathbf{v}_2 + m_1\mathbf{v}_1 = \text{const} \tag{5.6}$$

This is the law of conservation of linear momentum for a system of two particles. Hence, we conclude that if the second and third laws of motion hold good, then the linear momentum is conserved for a system of two particles. These arguments can be further extended to establish the validity of the law of conservation of linear momentum for a larger number of particles. The law of conservation of linear momentum is, in fact, a basic law and is true even in the domains of atomic and nuclear physics as well as in relativistic mechanics with a modified concept of mass.

We will now formally prove that if the space is linearly uniform, it leads to the third law of motion and hence to the conservation of linear momentum, as shown above. For the sake of simplicity, we will deal with motion in one dimension and will take the x-axis as the direction of motion. As the motion is along the x-axis; we will not use vector notation. All the motion and forces will be considered along the x-axis.

We take a system of two particles P_1 and P_2 lying along the x-axis. The force F_x acting on any particle can be expressed in the case of conservative forces as

$$F_x = -\frac{\partial U}{\partial x} \tag{5.7}$$

where U is the potential energy of the particle and x is the coordinate on the x-axis of the particle in question. The potential energy U should obey the following conditions:

1. Its form should be such that the force derived from it, is independent of the inertial frame of reference*, which are displaced linearly from each other. This is required if the linear uniformity of space is assumed. Let us take the two frames of reference such that their y- and z-axes are parallel to each other and x-axes are along the same line, but the origins are displaced by a distance b as shown in Fig. 5.1. Then

$$x' = x + b \tag{5.8}$$

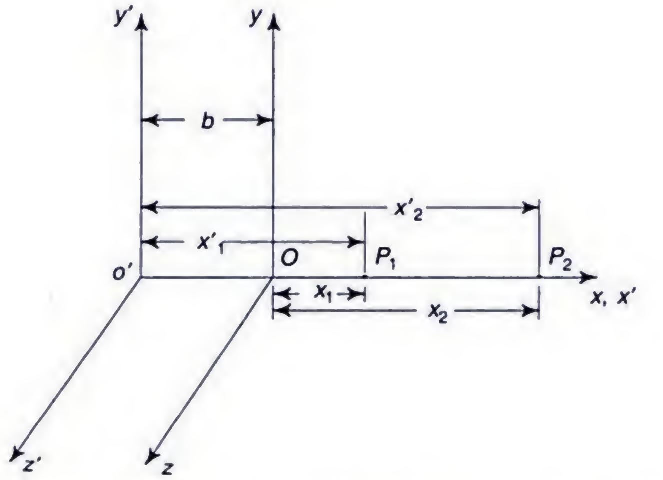


Fig. 5.1 Coordinate systems xyz and $x'y'z'$ with their origins displaced by b

where x and x' represent the position of a particle in the two frames of reference. Now linear uniformity of space demands that the force is independent of the displacement of the frames of reference. Hence

$$F_x = -\frac{\partial U}{\partial x} = -\frac{\partial U'}{\partial x'} \tag{5.9}$$

where U and U' are the potential energies in the two systems.

2. The potential energy due to interaction between two particles has to depend on their coordinates, i.e.

$$U = U(x_1, x_2) \tag{5.10}$$

where x_1 and x_2 are the x-coordinates of particles 1 and 2 respectively. Obviously, the form of $U(x_1, x_2)$ should be such that Eq. (5.9) holds good. Since U is a scalar quantity, one such form is

$$U(x_1, x_2) = (x_1 - x_2)^2 \tag{5.11}$$

Similarly, in the primed frame of reference, the potential energy is given by

$$\begin{aligned} U'(x'_1, x'_2) &= (x'_1 - x'_2)^2 \\ &= [(x_1 + b) - (x_2 + b)]^2 \\ &= (x_1 - x_2)^2 = U(x_1, x_2) \end{aligned}$$

This shows that the potential energy is independent of the frame of reference. Hence, forces on the two particles in the two frames of reference are given by

---

*Inertial frames are discussed in Chapter 10 of this book.

$$(F_x)_{21} = -\partial U/\partial x_1 = -\partial U'/\partial x'_1$$

and
$$(F_x)_{12} = -\partial U/\partial x_2 = -\partial U'/\partial x'_2 \tag{5.13}$$

Consequently, if

$$U(x_1, x_2) = (x_1 - x_2)^2$$

and
$$U'(x'_1, x'_2) = (x'_1 - x'_2)^2$$

then the forces are independent of the frames of reference.

If we define,

$$x_1 - x_2 \equiv u \tag{5.14}$$

we can write
$$(F_x)_{21} = -\partial U/\partial x_1 = -(\partial U/\partial u)(\partial u/\partial x_1)$$
$$= -\partial U/\partial u \tag{5.15}$$

because $\partial u/\partial x_1 = 1$. Similarly,

$$(F_x)_{12} = -\partial U/\partial x_2 = -(\partial U/\partial u)(\partial u/\partial x_2)$$
$$= \partial U/\partial u \tag{5.16}$$

as $\partial u/\partial x_2 = -1$. Hence

$$(F_x)_{21} = -(F_x)_{12}$$

which is the third law of motion.

We have thus proved that if we assume the property of linear uniformity of free space, and hence make the forces and potentials independent of the linear displacement of the frames of reference, then Newton's third law of motion is obtained. We had earlier proved that from the third law follows the conservation of linear momentum in the absence of external forces. Hence linear uniformity of space leads to the conservation of linear momentum via the third law of motion.

What is the implication of the equation: $U(x_1, x_2) = (x_1 - x_2)^2$? This assumes that U is independent of the absolute values of x_1 and x_2 as long as $(x_1 - x_2)^2$ is the same i.e. the value of U is the same for all pairs of points on the line as long as $|x_1 - x_2|$ is the same. This means that the properties of space are independent of the positions x_1 and x_2 or the free space is linearly uniform. It may be noted that the form of U chosen is one such possible form. It can also have some other forms, such as

$$U = \text{const} \times |x_1 - x_2| \tag{5.17a}$$
$$U = \text{const}/|x_1 - x_2| \tag{5.17b}$$

and so on.

The only condition is that U should be scalar and independent of the absolute values of x_1 and x_2. These arguments can be extended for three-dimensional space and then x_1 and x_2 will be replaced by position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$.

EXAMPLE 5.1

The potential energy of interaction between two particles at x_1 and x_2 on the x-axis is given by

$$U = A(x_2 - x_1)^2 + B/(x_2 - x_1)^2$$

Show that this potential is in accord with the requirements for linear uniformity of space and that Newton's third law is valid for forces acting on the two particles.

Solution

The expression for the given potential energy is

$$U = A(x_2 - x_1)^2 + B/(x_2 - x_1)^2$$

Since the value of U depends on $(x_2 - x_1)^2$ and $(x_2 - x_1)^{-2}$, it is scalar. Furthermore, it is independent of the absolute values of x_1 and x_2 and hence also the frames of reference. Consequently, this potential energy pertains to linear uniformity of space.

Now the force acting on particle 1 will be given by

$$(F_x)_{21} = -\,\partial U/\partial x_1 = -\,(\partial/\partial x_1)\,[A\,(x_2 - x_1)^2 + B(x_2 - x_1)^{-2}]$$
$$-2A\,(x_2 - x_1)\,)\,(-1)\,-B\,(-2)\,(x_2 - x_1)^{-3}\,(-1)$$
$$= 2A\,(x_2 - x_1) - 2B/(x_2 - x_1)^3$$

The force on the second particle will be

$$(F_x)_{12} = -\partial U/\partial x_2 = -\,(\partial/\partial x_2)\,[A(x_2 - x_1)^2 + B(x_2 - x_1)^{-2}]$$
$$= -2A\,(x_2 - x_1) - B\,(-2)\,(x_2 - x_1)^{-2}$$
$$= -\,2A\,(x_2 - x_1) + 2B/(x_2 - x_1)^3$$
$$= -\,(F_x)_{21}$$

This means that the potential energy is such that Newton's third law of motion and hence law of conservation of linear momentum holds good.

5.3 ROTATIONAL INVARIANCE OF SPACE AND LAW OF CONSERVATION OF ANGULAR MOMENTUM

The rotational invariance of space means that the potential energy of interaction between two particles does not change if the position coordinate of both of them are rotated through some angle about an arbitrary axis, i.e.

$$U\,(\mathbf{r}_1, \mathbf{r}_2) = U(\Omega\mathbf{r}_1, \Omega\mathbf{r}_2) \tag{5.18}$$

where Ω represents the rotation of $\mathbf{r}_1$ and $\mathbf{r}_2$ through a finite angle about the given axis. The form of U should be selected in such a manner that Eq. (5.11) for linear uniformity of space as well as Eq. (5.18) for rotational invariance hold good.

It can be shown that the form

$$U(\mathbf{r}_1, \mathbf{r}_2) = U(|\mathbf{r}_1 - \mathbf{r}_2|) = U(r) \tag{5.19}$$

where $r = |\mathbf{r}| = |\mathbf{r}_1 - \mathbf{r}_2|$ satisfies these conditions. The potential with this form depends on the magnitude of the separation of the two points but is independent of the coordinates of the points and the direction.

From such a potential, we can write the expression for forces as

$$F_{21} = -(\partial U/\partial r_1)\,\hat{\mathbf{r}} = -(\partial U/\partial r)\,(\partial r/\partial r_1)\,\hat{\mathbf{r}}$$

but $\quad r = |\mathbf{r}_1 - \mathbf{r}_2|$ when $|\mathbf{r}_1| > |\mathbf{r}_2|$ $\quad$ (5.20)

and $\quad r = |\mathbf{r}_2 - \mathbf{r}_1|$ when $|\mathbf{r}_1| < |\mathbf{r}_2|$

Therefore $\quad \partial r/\partial r_1 = 1 \quad$ if $r_1 > r_2$

$\quad = -1 \quad$ if $r_1 < r_2$ $\quad$ (5.21)

Hence $\quad F_{21} = \pm\,F(r)\hat{\mathbf{r}} \quad$ + for $r_1 > r_2$

$\quad$ – for $r_1 < r_2$ $\quad$ (5.22)

where $\quad F(r) \equiv -\partial U/\partial r$ $\quad$ (5.23)

Furthermore $\quad F_{12} = -\,(\partial U/\partial r_2)\hat{\mathbf{r}} = -\,(\partial U/\partial r)\,(\partial r/\partial r_2)\hat{\mathbf{r}}$

Now $\quad \partial r/\partial r_2 = -1 \quad$ for $r_1 > r_2$

$\quad = +1 \quad$ for $r_1 < r_2$ $\quad$ (5.24)

and $\quad F_{12} = \mp F(r)\,\hat{\mathbf{r}} \quad -\text{ for } r_1 > r_2, + \text{ for } r_1 < r_2 \qquad (5.25)$

Hence $\quad F_{12} = -F_{21}$ (both for $r_1 > r_2$ and $r_1 < r_2$) $\qquad (5.26)$

In other words, the potential given in Eq. (5.19) leads to the third law of motion and to the fact that these forces can be written as

$$\mathbf{F}(r) = F(r)\,\hat{\mathbf{r}} \tag{5.27}$$

The force as given in Eq. (5.27) depends only on r, i.e. the distance between the centres of the two particles and is called the central force.

The torque $\boldsymbol{\Gamma}$ for such a force about the centre of force is given by

$$\begin{aligned}\boldsymbol{\Gamma} &= \mathbf{r} \times F(r)\,\hat{\mathbf{r}} \\ &= F(r)\,\mathbf{r} \times \hat{\mathbf{r}} \\ &= F(r)\,r\,[\hat{\mathbf{r}} \times \hat{\mathbf{r}}] = 0\end{aligned} \tag{5.28}$$

We also know that torque $\boldsymbol{\Gamma}$ and angular momentum $\mathbf{L}$ are related by

$$\boldsymbol{\Gamma} = d\mathbf{L}/dt$$

Hence if $\boldsymbol{\Gamma} = 0$, $\mathbf{L}$ is constant.

This means that for central forces, not only does the magnitude of angular momentum remain constant, but its direction is also constant.

From this we conclude that if space has proprieties of invariance of rotation as indicated by the expression of U in Eq. (5.19), this leads to the third law of motion and to the central forces, and hence to the constancy of angular momentum. We can, therefore, say that the property of rotational invariance leads to the conservation of angular momentum.

EXAMPLE 5.2

The interaction energy between two nucleons (neutrons and protons) at $\mathbf{r}_1$ and $\mathbf{r}_2$ may be expressed in terms of Yukawa potential as

$$U(r) = -C\frac{\exp(-\alpha\,|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

where C and α are positive constants and $r = |\mathbf{r}_1 - \mathbf{r}_2|$. Show that this satisfies the requirements of rotational invariance of space and hence is in accord with Newton's third law of motion.

Solution

The interaction energy of two nucleons at $\mathbf{r}_1$ and $\mathbf{r}_2$ is

$$U = -C\,\frac{\exp(-\alpha\,|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

The potential depends only on $|\mathbf{r}_1 - \mathbf{r}_2|$, which is the separation of the two nucleons and is independent of the position vectors of individual nucleons. Furthermore, the force acting on nucleon 1 will be

$$\mathbf{F}_{21} = -\frac{\partial U}{\partial r_1}\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is unit vector along the vector $\mathbf{r}_1 - \mathbf{r}_2$. Substituting for U, we get

$$\mathbf{F}_{21} = +C\,\frac{\partial}{\partial \mathbf{r}_1}\left[\frac{\exp(-\alpha\,|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|}\right]\hat{\mathbf{r}}$$

$$= C\left[\frac{-\alpha \exp(-\alpha|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|} - \frac{\exp(-\alpha|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\right] \times (\pm 1)\,\hat{\mathbf{r}}$$

$$= \pm C\,\frac{\exp(-\alpha|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\,[-\alpha|\mathbf{r}_1 - \mathbf{r}_2| - 1]\,\hat{\mathbf{r}}$$

$$= \mp C\,\frac{\exp(-\alpha|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\,[1 + \alpha|\mathbf{r}_1 - \mathbf{r}_2|]\,\hat{\mathbf{r}}$$

The factor $\pm$ 1 in the second and subsequent expressions above comes from the fact that $|\mathbf{r}_1 - \mathbf{r}_2| = r_1 - r_2$ when $r_1 > r_2$ and $|\mathbf{r}_1 - \mathbf{r}_2| = r_2 - r_1$ for $r_1 < r_2$, and derivative of $|\mathbf{r}_1 - \mathbf{r}_2|$ with respect to r_1 will be + 1 or −1 depending on whether $r_1 > r_2$ or $r_1 < r_2$.

Similarly, the force acting on the second nucleon will be

$$\mathbf{F}_{12} = -\frac{\partial U}{\partial r_2}\hat{\mathbf{r}}$$

$$= C\frac{\partial}{\partial r_2}\left[\frac{\exp(-\alpha|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|}\right]\hat{\mathbf{r}}$$

$$= C\left[\frac{-\alpha \exp(-\alpha|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|} - \frac{\exp(-\alpha|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\right] \times (\mp 1)\,\hat{\mathbf{r}}$$

$$= \pm C\,\frac{\exp(-\alpha|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\,[1 + \alpha|\mathbf{r}_1 - \mathbf{r}_2|]\,\hat{\mathbf{r}}$$

Once again the factor $\mp$ 1 originates from the differentiation of $|\mathbf{r}_1 - \mathbf{r}_2|$ with respect to $\mathbf{r}_2$, the upper sign corresponding to $r_1 > r_2$ and the lower to the situation $r_1 < r_2$.

Comparing the expressions for $\mathbf{F}_{21}$ and $\mathbf{F}_{12}$, we note that

$$\mathbf{F}_{21} = -\mathbf{F}_{12}$$

Thus the given potential satisfies the conditions of rotational symmetry and the force involved is central; $\mathbf{F}(r) = F(|\mathbf{r}_1 - \mathbf{r}_2|)\hat{\mathbf{r}}$. The torque for such a force will be zero and hence the angular momentum will be conserved.

5.4 HOMOGENEITY OF FLOW OF TIME AND CONSERVATION OF ENERGY

Conservation of total energy is applicable for a closed system, if the forces are nondissipative or conservative, i.e. when force is related to potential energy through.

$$\mathbf{F} = -(\partial U/\partial r)\,\hat{\mathbf{r}}$$

Now the total energy E of a system is given by

$$E = U + T \tag{5.30}$$

where U is the potential energy and T is the kinetic energy. If the time flows uniformly, then it implies that the intrinsic properties of space do not depend on time explicitly. In other words, if a force does not have any explicit dependence on time, it will not change with time. It may be remarked that the phrase 'explicit dependence' means that the time should occur as such or directly in the expression

for force. However, as we know, the expressions for Coulomb or Gravitational forces are given by:

$$\mathbf{F}_e = (q_1 q_2/r^2)\hat{\mathbf{r}}$$

and $$\mathbf{F}_g = (Gm_1 m_2/r^2)\hat{\mathbf{r}} \qquad (5.31)$$

so that time does not occur explicitly. Hence U will also not be explicitly dependent on time. Nevertheless, U may depend on time implicitly, i.e. indirectly through the coordinates r depending on time. Consequently, we have

$$\partial U/\partial t = 0 \qquad (5.32)$$

Keeping this in mind, we can correlate the law of conservation of energy with the homogeneous flow of time as follows:

From Eq. (5.30), if

$$E = E\,(r, t)$$

then $$dE = (\partial E/\partial r)\,dr + (\partial E/\partial t)\,dt$$

$$= (\partial/\partial r)\,(U + T)\,dr + (\partial/\partial t)\,(U + T)\,dt \qquad (5.33)$$

But from Eq. (5.32) $\partial U/\partial t = 0$ and $\partial T/\partial t$ is also zero because there is no explicit dependence of the expression for kinetic energy ($T = (1/2)\,mv^2$] on time. Hence we can write

$$dE = (\partial U/\partial r)\,dr + (\partial T/\partial r)\,dr$$

or $$dE/dt = [(\partial U/\partial r) + (\partial T/\partial r)]\,(dr/dt) \qquad (5.34)$$

Now $$\partial U/\partial r = -F$$

and $$\partial T/\partial r = \partial/\partial r\,[(1/2)\,mv^2)] = mv\,(\partial v/\partial r)$$

Since v is not explicitly dependent on time, $\partial v/\partial r = dv/dr$.

Therefore $$\partial T/\partial r = mv\,(dv/dr)$$

$$= m\,(dr/dt)\,(dv/dr) = m(dv/dt)$$

$$= ma \qquad (5.35)$$

Hence $$dE/dt = (-F + ma)\,(dr/dt) \qquad (5.36)$$

But from the second law of motion, the expression in parentheses on the right-hand side is zero, Hence,

$$dE/dt = 0 \qquad (5.37)$$

or the total energy E is constant with time. Thus it may be concluded that if U is not an explicit function of time, or in other words, if the time flows uniformly, the total energy of the system where nondissipative forces operate, remains constant with time. It may further be pointed out that the above derivation assumes the validity of the second law of motion.

The interaction potentials used in examples 5.1 and 5.2 are such that these do not possess any explicit dependence on time. Consequently, for both of these $\partial U/\partial t = 0$ and hence the total energy E will be constant in time. Thus energy conservation will hold good for problems involving such interactions.

The discussion presented in this chapter shows that the assumption of linear uniformity of space leads to the conservation of liner momentum; the assumption regarding rotational invariance of space corresponds to the conservation of angular momentum; and the assumption of homogeneity of flow of time leads to the conservation of energy. Of course, these conclusions are arrived at by keeping in mind Newton's three laws of motion.

QUESTIONS

5.1 State Newton's first law of motion and show that it assumes that the properties of free space along a straight line are the same as well as invariant with time.
5.2 State Newton's second law of motion and show that it is based on the assumptions that time is homogeneous in nature and space has the same property on reflection.
5.3 List out the assumptions that are implicit in Newton's three laws of motion.
5.4 State the law of conservation of linear momentum and prove that it is a consequence of Newton's second and third laws of motion.
5.5 Comment on the statement: 'Law of conservation of linear momentum is a basic law of physics'.
5.6 What is conservative force? How is it related to potential energy?
5.7 Show that the function $U = |(\mathbf{r}_1 - \mathbf{r}_2)|^2$ for potential energy is in accord with linear uniformity of space.
5.8 Show that the choice of potential energy being in accord with the uniformity of space leads to Newton's third law of motion, which, in turn, forms the basis of conservation of linear momentum.
5.9 What does the term 'rotational invariance' imply?
5.10 What is a central force? Give two examples of such forces.
5.11 Write down the form of the potential energy function corresponding to a central force and show that this leads to the third law of motion.
5.12 Show that angular momentum is constant for motion under a central force.
5.13 Show that rotational invariance of space requires motion under a central force and leads to the conservation of angular momentum.
5.14 Prove that angular momentum is not necessarily conserved about an origin not coinciding with the position of the source of the central field.
[*Hint*: See Eq. (5.28).]
5.15 What are explicitly and implicitly dependent functions? Give two examples of each?
5.16 What does conservation of energy mean? Show that this law follows from homogeneity of time and Newton's second law of motion.
5.17 Show that $\partial T/\partial r = ma$, where T is kinetic energy of particle of mass m having acceleration a.
5.18 'If potential energy U does not depend explicitly on time then, the total mechanical energy E is constant in time. Discus.

PROBLEMS

5.1 The potential energy of interaction between two particles at x_1 and x_2 is given by

$$U = \frac{A}{1 + |x_1 - x_2|}$$

where A is some constant. Prove that this potential satisfies the requirements of linear uniformity and is in accord with Newton's third law of motion.
5.2 Two particles constrained to move along the x-axis are known to repel each other in such a way that their interaction potential is given by

$$U = \frac{C}{(x_1 - x_2)^2}$$

Show that the space is linearly uniform for this interaction and also that action and reaction are equal and opposite.

5.3 The long range interactions between the atoms of a linear polymer are expressed as

$$U = -\frac{C}{(x_i - x_j)^m}$$

where x_i and x_j are positions of atoms forming a pair and m is a positive integer. Show that this potential is in accord with the linear uniformity of space. Also, derive expressions for forces acting on the two atoms.

5.4 The potential energy of interaction between two particles at x_1 and x_2 on the x-axis is found to be given by

$$U(x) = A(x_1 - x_2)^2 \exp[-(x_1 - x_2)^2]$$

Argue to show that this potential satisfies the requirements of linear uniformity of space. Also, prove that the forces exerted by the two particles on each other are equal and opposite.

5.5 The interaction between two atoms with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is generally expressed in terms of the Lennard-Jones formula

$$U = -\frac{A}{|\mathbf{r}_1 - \mathbf{r}_2|^2} + \frac{B}{|\mathbf{r}_1 - \mathbf{r}_2|^{12}}$$

where A and B are positive constants. Show that this interaction corresponds to rotational invariance of space and hence Newton's third law of motion is satisfied.

5.6 A crystal lattice is defined to be a periodic three-dimensional arrangement of ions or atoms, in which there are some long-range interactions in addition to the interactions with the immediate neighbours. The potential energy of an ion in a lattice of anions and cations arranged alternatively is given by

$$U = -\frac{C_1 e^2}{|\mathbf{r}_1 - \mathbf{r}_2|} + \frac{C_2}{|\mathbf{r}_1 - \mathbf{r}_2|^n}$$

where n is an integer of the order of 10. Show that the interaction corresponds to the irrotational invariance of space. Also, check whether action and reaction on the two interacting ions are equal and opposite.

5.7 One of the important forces in physics is the spring force which comes into play when two masses attached at the end of a spring are disturbed from the equilibrium position. An extention of this force is made use of in explaining the behaviour of diatomic molecules. For two atoms (or masses) at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ with equilibrium separation r_0, the potential energy is given by

$$U = C[|\mathbf{r}_1 - \mathbf{r}_2| - r_0]^2$$

where C is the force constant. Show that this interaction is in agreement with rotational invariance of space and also with Newton's third law of motion.

5.8 Show that for all the interactions listed in the above problems, the conservation of energy will hold good.

6

Inverse Square Law Force

One of the most important problems of classical mechanics is to understand the motion of a particle or body moving under the influence of a field and hence obtain an equation for its description. The first such problem to be studied in detail by physicists, both experimentally and theoretically, was the motion of planets around the sun. In fact, it was the fascination of the planetary motion that provided the impetus to the development of mechanics. The motion of electrons around the nucleus in an atom is another example of such a motion. Also, the motion of loose nucleons around the central core in a nucleus is similar to the motion of electrons in an atom, except that the forces involved are stronger. Although, the last two problems require quantum-mechanical formulation for correct description, these are intimately related to the classical situation. It is, therefore, important to understand theoretically the method of obtaining the trajectory of a particle in a given field.

6.1 FORCES IN THE UNIVERSE

In nature, we have basically four kind of forces in the following order of increasing interaction strength:

1. Gravitational,
2. Weak,
3. Electromagnetic, and
4. Nuclear or strong.

The nature of these forces and their consequences still continue to be topics of investigation, but their spatial dependence and relative strengths are reasonably known today. Some of their significant features are listed below.

1. The gravitational potential energy between two masses is due to the gravitational attraction of two masses and can be written as

$$U_g = -G(m_1 m_2)/r \tag{6.1}$$

where G is the gravitational constant, and m_1 and m_2 are the interacting masses separated by distance r. The gravitational forces play a prominent role in the dynamics of various planets in the solar system and of the galaxies in the whole universe.

2. The weak interaction operates in beta decay or any process in which the decay products of a nuclear process are leptons, i.e. electrons, positrons, neutrons, μ-mesons, etc. These are weakly interacting particles and the forces that are responsible for their interaction in the decay process are called weak forces. It may be mentioned that the electric charge of electrons, positrons, etc. can give rise to the electromagnetic part of the interaction, but that should be taken into account separately. The intrinsic interaction responsible for the decay and emission of electrons and neutrinos is weak interaction.

Without going into details, it may be mentioned that the interaction energy of such an interaction can be written as

$$U_{\beta i} = g_{\beta i}^2 \, \delta(\mathbf{r}_i - \mathbf{r}_L) \, F_i \tag{6.2}$$

where $g_{\beta i}$ is the β-decay constant and i denotes the ith term in the interaction. There are many such terms in the full expression. The vector $\mathbf{r}_i$ gives the radial position of the source from where the lepton is being emitted and $\mathbf{r}_L$ is the radial position of the emitted lepton so that $\mathbf{r}_i - \mathbf{r}_L$ is the distance between the emitting nucleon and lepton. The function $\delta(\mathbf{r}_i - \mathbf{r}_L)$ is called the δ-function and is zero when $\mathbf{r}_i - \mathbf{r}_L \neq 0$ and unity when $\mathbf{r}_i - \mathbf{r}_L = 0$. In physical terms, this simply means that $\delta(\mathbf{r}_i - \mathbf{r}_L) = 1$ when the lepton is just at the site of the nucleon and vanishes as soon as the lepton gets away from the emitting nucleon. The function F_i is a complicated function depending on the spin orientations of the emitted particles. The type of potential represented by Eq. (6.2) is called the contact potential as it contains the $\delta(\mathbf{r}_i - \mathbf{r}_L)$ term, which makes the interaction nonzero only when $\mathbf{r}_i = \mathbf{r}_L$. Such an interaction takes place only when two particles are in contact or exactly overlap each other.

3. Electromagnetic interaction has electric and magnetic parts. Without going into full details of the complete electromagnetic interaction, we write below the expression for electrostatic or coulomb potential energy U_C between two electric charges

$$U_C = \frac{q_1 q_2}{kr} \tag{6.3}$$

where k is a constant called the dielectric constant, and q_1 and q_2 are the charges. The behaviour and properties of condensed matter such as solids and liquids depend very much on the electromagnetic interaction. Consequently, the whole chemistry, biology and even human life depend on the interplay of electromagnetic interactions.

4. The nuclear or strong interaction operates between nucleons (neutrons and protons) in a nucleus and is the strongest of the interactions. Its spatial dependence can be written as

$$U_s = U_0 \frac{\exp(-r/r_0)}{r/r_0} + f_T S_{12} \tag{6.4}$$

Here r is the distance between the nucleons and r_0 is a constant having the value 2×10^{-12} cm, U_0 is a constant nearly equal to 40 MeV, f_T is a constant function and S_{12} is a function given by

$$S_{12} = \frac{3(\mathbf{S}_1 \cdot \mathbf{r})(\mathbf{S}_2 \cdot \mathbf{r})}{r^2} - \mathbf{S}_1 \cdot \mathbf{S}_2 \tag{6.5}$$

Here $\mathbf{S}_1$ and $\mathbf{S}_2$ are the spins of nucleons and $|\mathbf{r}|$ is the distance between them (Fig. 6.1). The first term in Eq. (6.5) gives the dependence of potential on r only, and therefore, corresponds to the central force part, whereas the second term depending on the orientation of the spins, pertains to the non-central force.

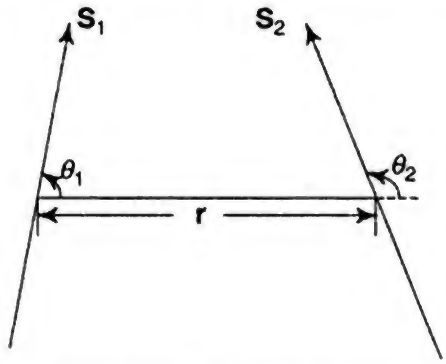


Fig. 6.1 The interaction between two nuclear spins

There are two features of the forces discussed above which require detailed discussions:

1. their spatial dependence, and
2. the relative strengths.

6.1.1 Spatial Dependence

Of the four types of the basic forces, two of them, namely gravitational and electromagnetic forces obey the inverse square law. In other words, forces are inversely proportional to the square of the distance between the two bodies. These are long range forces as they exist even when the bodies are at very large distances from each other.

Further, these forces are central, i.e. the force between two point charges or masses depends only on the distance between the centres of the two bodies and not on the orientation of the bodies, involving any angles, etc.

On the other hand, the weak forces due to, say beta decay, and strong forces (or nuclear forces) between nucleons (protons and neutrons) are short range forces. As may be seen in Eq. (6.2), weak forces exist only when particles are in contact with each other and are zero outside that range. The spatial dependence of nuclear potential is, on the other hand, given by

$$U_s \propto \exp(-r/r_0)/(r/r_0)$$

We compare in Fig. 6.2 the spatial dependence of the electrostatic or coulomb and the nuclear potentials. It can be easily seen that nuclear potential becomes nearly zero at a certain small value of r, depending on the value of r_0 while the coulomb potential approaches zero only at infinity. For example, the values of the nuclear potentials at $r_1 = 10^{-14}$ cm, $r_2 = 10^{-13}$ cm and $r_3 = 10^{-12}$ cm have a ratio of $U_{1'} : U_{2'} : U_{3'} = 1 : 0.3 : 10^{-3}$, if $r_0 = 2 \times 10^{-13}$ cm.

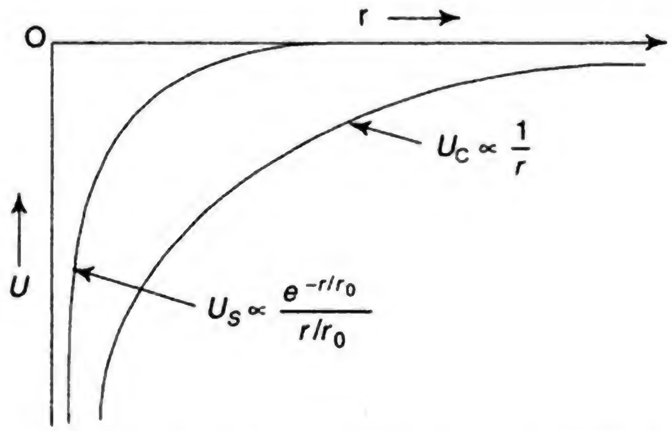


Fig. 6.2 Dependence of long and short range potentials on distance

On the other hand, for coulomb potential, it can be seen that $U_{1C} : U_{2C} : U_{3C} = 1 : 0.1 : 0.01$. It is clear from here, that the expression given in Eq. (6.4) represents the short range force while the coulomb potential (and similarly the gravitational potential) is long-ranged.

6.1.2 Relative Strength of Various Interactions

The intrinsic strengths of various interactions can be expressed in terms of constants, in each case. A few examples are given below.

1. In gravitational interaction, the strength is determined by Gm_1m_2, where G is the gravitational constant, whose experimentally determined value is 6.67×10^{-11} in S.I. units.
2. In electromagnetic interaction, it is determined by q_1q_2/k, where $k = 1$ in vacuum, and q_1 and q_2 are the electric charges.
3. In weak interaction the value of g_β determines the strength. Its value has been experimentally measured as $g_\beta = 1.4 \times 10^{-49}$ cgs units (erg-cm^3).
4. In strong or nuclear interaction the strength is determined by U_0, whose value has been found to be 40 MeV. Though these constants determine the strengths of various interactions, it is not easy to compare them with each other as they have different dimensions. It is customary to express the relative strengths of various interactions in terms of dimensionless coupling constants.

The arguments for representing them in proper dimensionless form arise from the microscopic theories of relativistic quantum mechanics. We cannot go into that here. We will however, mention that universal constants, $\hbar c$ (which has the dimensions of erg-cm) and $\hbar / mc$ (having dimensions of length and called the Compton wavelength) are used for getting the dimensionless forms as shown below:

1. In gravitational interaction, one uses the dimensionless gravitational coupling constant $Gm^2/\hbar c = 2 \times 10^{-45}$ representing the strength of the interaction; here m is the mass of the electron.
2. For weak interaction, the β-decay coupling constant is $g_\beta = 1.4 \times 10^{-49}$ erg-cm^3. The corresponding dimensionless quantity is obtained by dividing g_β by $\hbar c\, (\hbar / m_\pi c)^2$, which yields

$$\frac{g_\beta}{\hbar c\,(\hbar / m_\pi c)^2} = 2.5 \times 10^{-7}$$

However, it is the square of the β-decay constant that enters in the physically meaningful quantities. Therefore, the dimensionless coupling constant of our interest is

$$\left[\frac{g_\beta}{\hbar c\,(\hbar / m_\pi c)^2}\right]^2 \simeq 10^{-13}$$

Here m_π is the mass of the π-meson which is taken to be the particle involved in exchange-coupling of nucleons in the nucleus.

3. In the electrostatic case, the coupling constant is given by

$$\frac{e^2}{\hbar c} \simeq \frac{1}{137}$$

where e is the electronic charge.

4. In the case of strong interaction, one uses $g_N^2 = r_0 U_0$ as the constant factor and the dimensionless coupling constant as $g_N^2/\hbar c$. Putting the value $U_0 = 40$ MeV and $r_0 = 2 \times 10^{-13}$ cm, one obtains

$$g_N^2/\hbar c \simeq 0.4$$

In this way, it is possible to compare the intrinsic strengths of various interactions as follows:

Interaction	*Coupling constant (strength)*
Gravitational	$\dfrac{Gm^2}{\hbar c} \simeq 10^{-45}$
Weak	$\dfrac{g_\beta^2}{(\hbar^3/m_\pi^2 c)^2} \simeq 10^{-13}$
Electromagnetic	$\dfrac{e^2}{\hbar c} \simeq 10^{-2}$
Strong (nuclear)	$\dfrac{g_N^2}{\hbar c} \simeq 1$

Obviously nuclear interaction is the strongest, while gravitational is the weakest. Electromagnetic interaction is weaker than nuclear but stronger than the "weak".

6.2 GRAVITATIONAL FIELD AND POTENTIAL

Gravitational field is said to exist at a point if a gravitational force is exerted on a material particle at that point. The physical concept of the field in the general theory of relativity as well as electrodynamics differs from that in classical mechanics due to the finite velocity of propagation of interactions and the relative corrections introduced by it. In classical-mechanics, the finite velocity of propagation of gravitational force introduces a negligible correction to the result since the macroscopic

bodies that are responsible for gravitational force move slowly when compared to the velocity of propagation of interactions. However, the concept of field is useful in dealing with a so-called action at a distance gravitational force.

The intensity of gravitational field **E** at a point is defined as the gravitational force per unit mass on a test mass at that point, that is

$$\mathbf{E} = \frac{\mathbf{F}_g}{m_o} \tag{6.6}$$

where **E** is the gravitational intensity, $\mathbf{F}_g$ is the gravitational force being exerted on the test particle of mass m_0. **E** is a vector field and its direction is that of gravitational field.

According to Newton's law of universal gravitation, the gravitational force between mass particles m and m_0, separated by distance r is given by

$$\mathbf{F}_g = -\frac{Gmm_o}{r^2}\hat{\mathbf{r}} \tag{6.7}$$

where G is a universal constant, called gravitation constant, and $\hat{\mathbf{r}}$ is a unit vector along the vector **r**.

Thus,
$$\mathbf{E} = -\frac{Gmm_o}{m_o r^2}\hat{\mathbf{r}}$$

$$= -\frac{Gm}{r^2}\hat{\mathbf{r}} \tag{6.8}$$

The intensity of the field is directed towards the particle opposite to **r**. The gravitational field units are Newton/kg in MKS system and dyne/gm in the CGS system.

In case a number of material particles are present, the resultant gravitational field is the vector sum of the fields due to all particles. Thus,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \ldots . \tag{6.9}$$

E, the intensity of gravitational field is a vector function of space coordinates and is also defined as a space rate of change of a scalar function, called gravitational potential, V. The gravitational potential is defined as the potential energy per unit of a test mass m_o as

$$V = \frac{U}{m_o} \tag{6.10}$$

where U is the potential energy of the test mass.

We take the reference point at infinity since the gravitational force and potential are zero there. The gravitational potential V at a point located at a distance r from a body of mass m is equal to the amount of work done in moving a unit mass from infinity to that point. Thus,

$$V = -\int_{\infty}^{r} \mathbf{E}\cdot d\mathbf{r} = \int_{\infty}^{r} \frac{Gmdr}{r^2} = -\frac{Gm}{r} \tag{6.11}$$

This is the potential energy of unit mass at the point r units distant from the body of mass m. It may be remarked that the gravitational potential V, and the potential

energy U, are always negative. This is a consequence of the fact that the reference point is chosen arbitrarily at infinity to have zero potential energy.
The units of gravitational system are in MKS system Joule/kg and in CGS system, erg/gm.
The resultant gravitational potential due to a number of material masses m_1, m_2, m_3, . . . at distances r_1, r_2, r_3, . . ., respectively, from the point under consideration is given by the sum of the potentials. Thus,

$$V = V_1 + V_2 + V_3 + \ldots$$

$$= -G\left(\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3} + \ldots\right) \quad (6.12)$$

The principle of superposition holds for gravitational field and potential. This principle of linear superposition is familiar to us from various kinds of wave phenomenon. This principle is also well established in electromagnetism when the charges are located in vacuum (i.e. not inside a material medium) and are separated by distances involving classical (i.e. non-quantum mechanical) length scales.

6.2.1 Equipotential Surfaces

An equipotential surface is a surface, for which at all points, the gravitational potential has the same value. We will see in Section 6.4 that at all points of a spherical shell, the gravitational potential is the same, that is, $-MG/R$ where M is its mass and R its radius. Thus, its surface is an equipotential surface.

The potential difference between any two points on such a surface being zero, no work is done in moving any mass along it. This implies that the gravitational field has no component along the surface and is perpendicular to it.

One can easily prove that at every point on an equipotential surface, the gravitational field is perpendicular to the surface at that point.

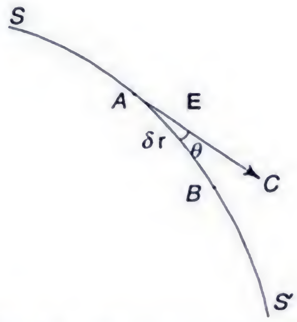


Fig. 6.3 Equipotential surface

Consider two points A and B, which are δr apart on an equipotential surface SS' (Fig. 6.3). If E is the intensity of the gravitational field at A, directed along C, the component of the field along the surface is $E \cos \theta$.
The work done is moving a unit mass from A to $B = E \cos \theta \, \delta r$
As A and B lie on the equipotential surface, the work done is zero, which is possible only if $\theta = 90°$, since neither $\mathbf{E}$ nor δr are zero.

Having defined the gravitational field and gravitational potential earlier, we proceed to show how these are related to each other.

Let a mass particle m lie at the origin of the coordinate system. The gravitational field at a distance r from the origin is given by

$$\mathbf{E} = -\frac{Gm}{|\mathbf{r}|^2}\hat{\mathbf{r}} \quad (6.8)$$

Also,
$$\text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = -\frac{1}{|\mathbf{r}|^2}\hat{\mathbf{r}}$$

Therefore, $$\mathbf{E} = Gm\nabla\left(\frac{1}{r}\right) = -\nabla\left(\frac{-Gm}{r}\right)$$

$$= -\nabla V \tag{6.13}$$

where $$V = -\frac{Gm}{r} \tag{6.11}$$

Thus, once the potential distribution is known, the field at any point can be found from Eq. (6.13). It is relatively easy to work with scalar potential than the field which is a vector quantity.

6.2.2 Experimental Determination of Constant of Gravitation G. Boys' Method

The first accurate experiment for finding the value of gravitation constant G, was performed by Cavendish in 1798, using a torsion balance. The mean value of G as a result of 29 different sets of experiments was 6.56×10^{-8} cgs units. Boys eliminated the difficulties and errors of the Cavendish method in the apparatus that was designed by him in 1895 but the principle of the method remained the same.

The apparatus consists of a mirror strip PQ, which is suspended from a torsion head by a quartz fibre (Fig. 6.4(a)).

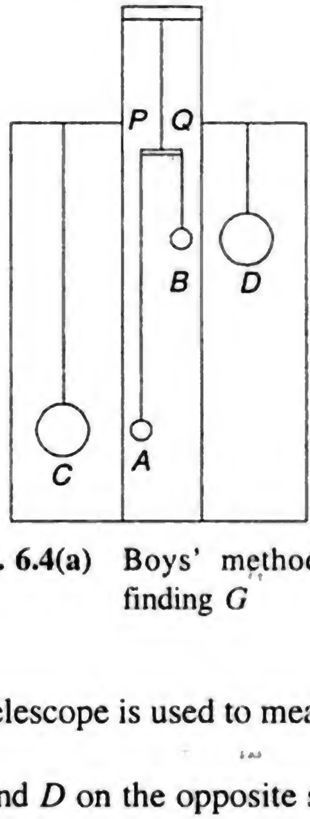


Fig. 6.4(a) Boys' method for finding G

From the ends of the mirror strip are suspended two small gold spheres, A and B, by quartz fibres of unequal length. The mirror strip and the balls are enclosed in a narrow tube, thus eliminating the air draughts altogether. The quartz fibres are perfectly elastic, stronger than steel, and the twisting torque required to twist through a unit angle is very small. Two identical lead spheres C and D of mass M each are suspended outside the tube from the revolving lid of an outer coaxial tube such that the centers of A and C are at the same horizontal level and the centers of B and D are at the same horizontal level. C is in front of A, and D is behind B. The distance between the centers of A and C is equal to the distance between the centers of B and D. A telescope is used to measure the deflection using a lamp and scale arrangement.

The lid is rotated so as to put the large lead balls C and D on the opposite sides of the small gold spheres A and B, but not in line with the mirror strip, till the deflection is maximum. Next, the lid is rotated so as to set the large balls in a similar position on the other sides of the corresponding small gold balls till there results maximum deflection.

A, B, C, and D are the four balls in the position of maximum deflection θ, radian, Fig. 6.4(b). O is the middle point of the mirror strip whose length is equal to $2\ l$. Calling $OC = a$, $\angle BOC = \alpha$; OE is drawn perpendicular to CB produced.

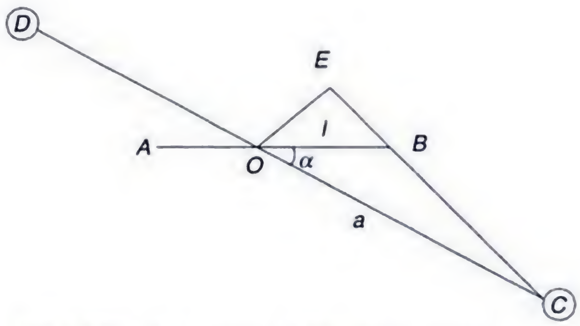


Fig. 6.4(b) Calculation of deflection couple

Now, according to the law of cosines we get

$$BC = (a^2 + l^2 - 2al\cos\alpha)^{\frac{1}{2}}$$

According to the law of sines of a triangle, we have

$$\frac{\sin\alpha}{\sin\angle BCO} = \frac{BC}{l}$$

This gives $$\sin\angle BCO = \frac{l\sin\alpha}{BC}$$

Also, $$OE = a\sin\angle BCO = \frac{al\sin\alpha}{BC}$$

C attracts A with a force $\frac{GMm}{BC^2}$ and D attracts B with the same force in the opposite direction. The two equal and opposite forces on A and B constitute a couple of moment $\frac{GMm}{BC^2} \times 2OE$. This couple produces a deflection in the mirror PQ about the suspension fibre as axis, which in turn gets twisted. As a result of the twist, an elastic restoring couple is set up in the fibre and the mirror comes to rest when the deflecting and the restoring couples balance each other.

$$\text{The deflecting couple} = \frac{GMm}{BC^2} \times 2OE$$

$$= \frac{2GMmal\sin\alpha}{(a^2 + l^2 - 2al\cos\alpha)^{\frac{3}{2}}}$$

$$\text{The restoring couple} = c\theta$$

or $$G = \frac{c\theta(a^2 + l^2 - 2al\cos\alpha)^{\frac{3}{2}}}{2Mmal\sin\alpha}$$

Boys obtained the value of $G = 6.6576 \times 10^{-8}$ cgs units.

Advantages of Boys' Over Cavendish's Method

1 The apparatus size was greatly reduced and enclosed in a vessel, thus avoiding the disturbances caused by air currents.
2 The apparatus being small in extent, the temperature difference over the small size will be small, and thus, minimize the convection currents.
3 The spheres A and B are at different levels and due to the large difference in their heights, the cross attractions between A and B as well as between C and D are made negligible.
4 A quartz fibre used as suspension is both fine and strong. It is perfectly elastic and requires small couple per unit twist with the result that the deflection of the mirror strip is large and proportional to the applied torque.
5. The use of the lamp and scale arrangement facilitates an accurate measurement of deflection.

6.3 ELECTRIC FIELD AND POTENTIAL

As both the gravitational and electrostatic fields are inverse square fields, we can treat the case of the electric field and potential in a manner analogous to that of the gravitational field at a point.

The electric field, **E**, is defined as the force acting on a unit positive charge at that point.

Thus,
$$\mathbf{E} = \frac{\mathbf{F}}{q_o} \tag{6.14}$$

where q_o is the test charge. The electric field for a point charge q at a distance r from it is given by

$$\mathbf{E} = k\frac{q}{r^2}\hat{\mathbf{r}} = k\frac{q}{r^3}\mathbf{r} \tag{6.15}$$

where k is the constant of proportionality and depends upon the system of units. The value of k is determined from the unit of charge. In the electrostatic system of units (esu), a unit charge is a charge that is repelled by a force of one dyne when kept at a distance of 1 cm from an equal charge in vacuum. Thus, $k = 1$ with esu units and the unit of charge is called the stat-coulomb. In the MKS system the unit of force is Newton and that of charge Coulomb, $k = 1/4\pi\varepsilon_0 = 9 \times 10^{10}$ metre/farad and ε_0 is the permittivity of free space.

Electric potential at a point in an electric field is defined as the potential energy of a unit positive test charge at that point. If the electric potential is taken as zero at an infinite distance, then it is the amount of work done against the electric force per unit charge as the positive test charge is taken from infinity to that point.

Thus,
$$V = -\int_{0}^{\infty} \mathbf{F}\cdot ds \tag{6.16}$$

It is related to the electric field through the relation

$$\mathbf{E} = -\nabla V \tag{6.13}$$

The unit of electric potential in esu system is statvolt and of electric field is dyne/statcoulomb. However, in the MKS system of units, the electric potential has the unit volt and the electric field Newton/coulomb (also volt/m).
As 1 volt = 1/300 statvolt, one has

1 volt/cm = 100 volt/m = 1/300 statvolt/cm.

6.4 GRAVITATIONAL POTENTIAL AND FIELD DUE TO A THIN SPHERICAL SHELL

Consider a thin spherical shell of mass M, radius R and surface mass density σ. P is the point where the potential is to be found. P is at a distance r from 0, the centre of the shell (Fig. 6.5). BC and DE are two parallel planes, perpendicular to the axis OAP, such that $\angle AOB = \theta$ and $\angle BOD = d\theta$.

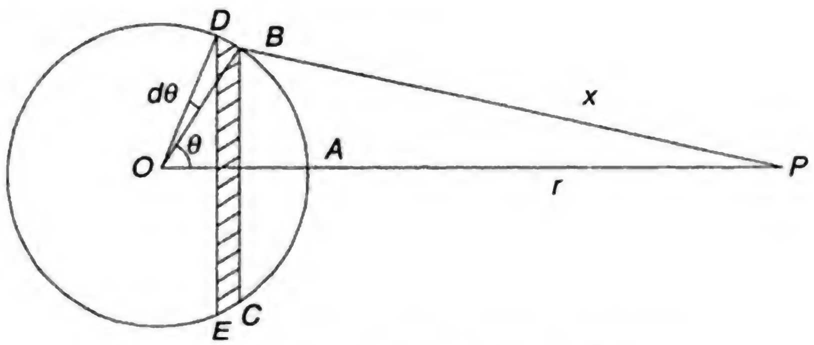


Fig. 6.5 Potential due to a thin spherical shell

The planes cut a ring $DBCE$ from the shell of radius $R \sin\theta$ and width $BD = Rd\theta$.
Mass of the ring $DBCE$, $dM = 2\pi R \sin \theta \times Rd\theta \times \sigma$

$$= 2\pi R^2 \sin \theta \, d\theta \frac{M}{4\pi R^2}$$

$$= \frac{M \sin \theta \, d\theta}{2} \tag{6.17}$$

All the points on the ring are at a distance x from P; the gravitational potential dV at P

$$= -\frac{GdM}{x} = -\frac{GM \sin \theta \, d\theta}{2x} \tag{6.18}$$

We evaluate the term $\sin \theta \, d\theta$ from the geometry of the arrangement. From the law of cosines applied to the triangle OBP, we get

$$x^2 = R^2 + r^2 - 2Rr \cos \theta$$

Differentiating both sides, we get

$$2xdx = 2Rr \sin \theta \, d\theta$$

or

$$\sin \theta \, d\theta = \frac{xdx}{Rr} \tag{6.19}$$

In view of Eq. (6.19) and Eq. (6.18) becomes

$$dV = -\frac{GMxdx}{Rr2x}$$

$$= -\frac{GMdx}{2Rr} \tag{6.20}$$

The potential V due to the entire shell is obtained by integrating the Eq. (6.20). Three cases arise.

Case I: Point P is outside the shell

Under this condition, x can have the range from $r - R$ to $r + R$.

Thus, gravitational potential V at P

$$= \int_{r-R}^{r+R} -\frac{GM}{2Rr}\,dx$$

$$= -\frac{GM}{2Rr}\,x\Big|_{r-R}^{r+R} = -\frac{GM}{r} \tag{6.21}$$

Now, gravitational field $\mathbf{E} = -\dfrac{dV}{dr}$

$$= -\frac{GM}{r^2}\hat{\mathbf{r}} \tag{6.22}$$

Case II: P lies on the surface of the shell

Thus $V = \displaystyle\int_{o}^{2R} -\frac{GM}{2R^2}\,dx$

$$= \int_{o}^{2R} -\frac{GM}{2R^2}\,dx$$

$$= -\frac{GM}{2R^2}\,x\Big|_{o}^{2R}$$

$$= -\frac{GM}{R} \tag{6.23}$$

The gravitational field intensity at a point on its surface

$$\mathbf{E} = -\frac{dV}{dr} = 0 \tag{6.24}$$

Case III: P lies inside the shell

Here x will vary between $R - r$ and $R + r$

Thus, $V = \displaystyle\int_{R-r}^{R+r} -\frac{GM}{2Rr}\,dx$

$$= -\frac{GM}{2Rr}\,x\Big|_{R-r}^{R+r}$$

$$= -\frac{GM}{R} \tag{6.25}$$

The gravitational intensity at a point inside the shell,

$$\mathbf{E} = -\frac{dV}{dr} = 0 \tag{6.26}$$

6.4.1 Electrostatic Potential and Field due to a Charged Spherical Shell

By going through a similar calculation, it is a straight forward matter to obtain the expressions for the potential and field at a point for a charged spherical shell having a surface charge density, $\sigma = \frac{q}{4\pi R^2}$, where q is the charge on the shell of radius R.

The expressions for this charged shell can be obtained from the corresponding expressions for the thin spherical shell by the following prescription.

Replace the term $-GM$ by the expression q/k, where k is the dieletric constant of the intervening medium. The expressions for the potential and field for a charged spherical shell are as follows:

1. *At points outside the shell*

Electrostatic potential, $$V = \frac{q}{kr}$$

Electrostatic field, $$\mathbf{E} = \frac{q}{kr^2}\hat{\mathbf{r}}$$

2. *At points inside the shell*

Electrostatic potential, $$V = \frac{q}{kr}$$

This is the potential at the surface.

Electrostatic field, $$\mathbf{E} = 0$$

The variations of potential $\mathbf{E} = \frac{q}{kr^2}\hat{\mathbf{r}}$ and field as a function of distance from the centre of the shell are displayed in the Fig. 6.6(a) and (b) for the gravitational and electorstatic cases, respectively.

6.5 GRAVITATIONAL FIELD AND FIELD DUE TO A SOLID SPHERE

Consider a solid sphere of mass M and radius R, divided into a number of spherical shells whose radii vary from 0 to R. Consider one shell of radius x and thickness dx (Fig. 6.7(a)). We will take the different cases separately, depending upon the location of the observation point P.

Case I: When P is located outside the sphere

Mass of the spherical shell of radius x and thickness $dx = 4\pi x^2 dx\rho$

Putting $$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

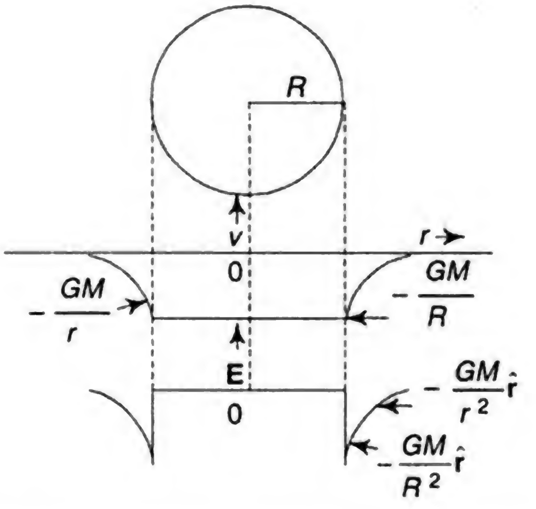


Fig. 6.6(a) Gravitational potential and field as a function of r, the distance from the centre of the shell

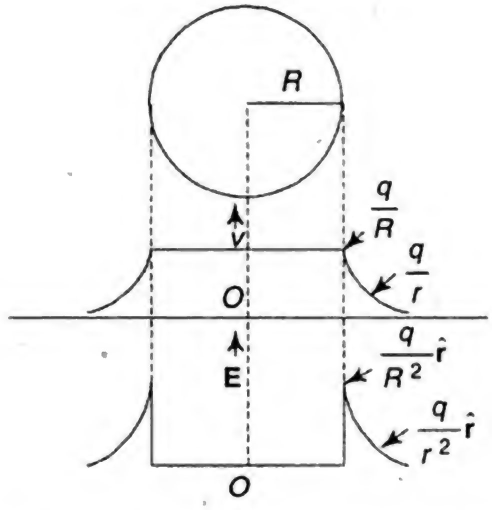


Fig. 6.6(b) Electrostatic potential and field as a function of r, the distance from the centre of the shell

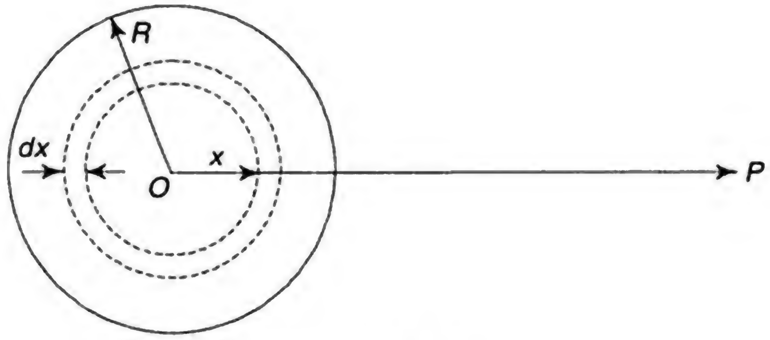


Fig. 6.7(a) A spherical shell

Potential at P due to this shell $\quad dV = -G\dfrac{4\pi x^2 dx}{r}\dfrac{M}{\frac{4}{3}\pi R^3}$

$$= -\frac{3GM}{rR^3}x^2\,dx \tag{6.27}$$

The potential due to the solid sphere is obtained by integrating the Eq. (6.27) as

$$V = -\frac{3GM}{rR^3}\int_0^R x^2\,dx$$

$$= -\frac{GM}{r} \tag{6.28}$$

This expression for potential is the same as if the entire mass of the sphere is located at the centre of the sphere.

Gravitational field $\mathbf{E} = -\nabla V$

$$= -\frac{GM}{r^2}\hat{\mathbf{r}} \tag{6.29}$$

Case II: P lies inside the sphere

The point of observation P lies at a distance r from 0, the centre of the sphere (Fig. 6.7(b)). Imagine a spherical surface of radius r, concentric with the sphere and passing through the point P.

Let the potential at P due to the sphere of radius r be denoted by V_1, and the potential at P due to thick spherical shell of internal radius r and external radius R by V_2.

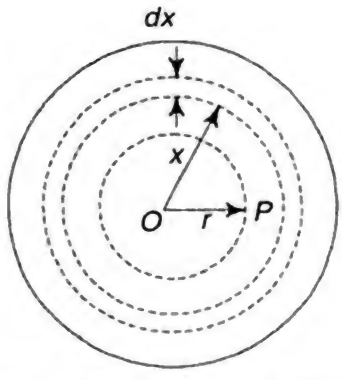


Fig. 6.7(b) Point P lies inside the sphere

Thus, potential at P due to the sphere,

$$V = V_1 + V_2 \tag{6.30}$$

Obviously,

$$V_1 = -G\frac{\frac{4}{3}\pi r^3\rho}{r}$$

$$= -\frac{4}{3}\pi G r^2 \rho \tag{6.31}$$

For evaluating V_2, the potential due to the thick spherical shell, we consider a thin spherical shell of radius x and thickness dx concentric with the sphere.

Then,

$$V_2 = \int_r^R -G\frac{\frac{4}{3}\pi x^2\,dx\rho}{x}$$

$$= -4\pi G\rho\int_r^R x\,dx$$

$$= -2\pi G\rho(R^2 - r^2) \tag{6.32}$$

Combining Eqs (6.31) and (6.32), we get

$$V = -4\pi G\rho\left(\frac{R^2}{2} - \frac{r^2}{6}\right)$$

$$= \frac{-3GM}{R^3}\left(\frac{R^2}{2} - \frac{r^2}{6}\right) \tag{6.33}$$

Therefore, the gravitational intensity at P

$$\mathbf{E} = -\nabla V$$

$$= -\frac{3GM}{R^3}\frac{2r}{6}\hat{\mathbf{r}}$$

$$= -\frac{GM}{R^3}\mathbf{r} \tag{6.34}$$

The gravitational field intensity at a point inside the sphere is directly proportional to its distance from the centre of the sphere.The potential V and the field $\mathbf{E}$ due to a solid sphere are plotted as a function of the distance from the centre of the sphere.

From Eq. (6.34), it is obvious that the gravitational field $\mathbf{E}$ is zero at the centre and maximum at the surface whereas the gravitational potential Eq. (6.33) is maximum at the centre. The Fig. 6.8 gives the graphical plots of gravitational V and $\mathbf{E}$ for a solid sphere as a function of distance from the centre of the sphere.

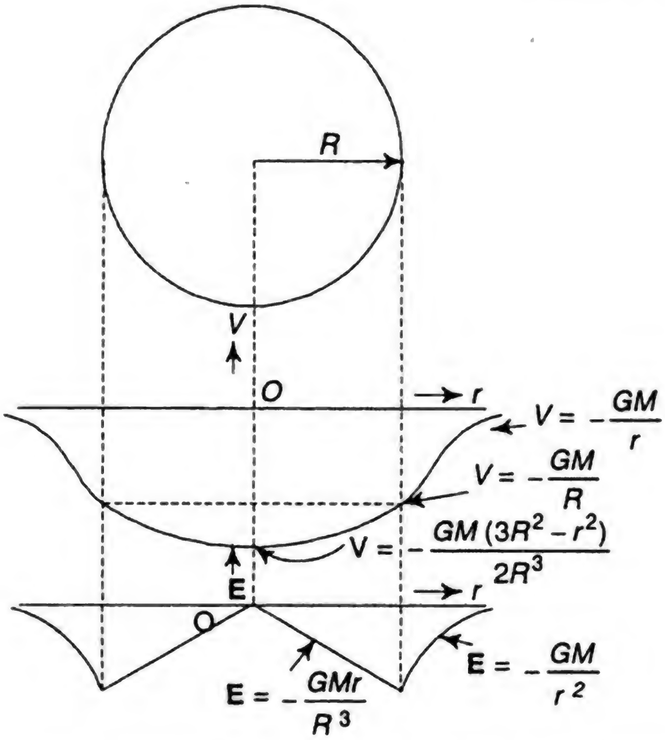


Fig. 6.8 Graphical plots of gravitational V and field $\mathbf{E}$ for a solid sphere as a function of its distance from the centre of the sphere

6.5.1 Electrostatic Potential and Field Due to a Uniformly Charged Sphere

The corresponding expressions for the electric case are easily derivable from those for the gravitational case by replacing $-GM$ by q/k. Thus, we get the following expressions for the different cases.

Case I: When the point of observation P lies outside the sphere

Electrostatic potential $$V = \frac{q}{kr} \tag{6.35}$$

Electrostatic field $$\mathbf{E} = \frac{q}{kR^2}\hat{\mathbf{r}} \tag{6.36}$$

q, the total charge $= 4/3\ \pi R^3 \rho$

where ρ is the volume charge density.

Case II: When the point of observation lies inside the sphere

Electrostatic potential, $$V = \frac{q(3R^2 - r^2)}{2kR^3} \tag{6.37}$$

Electrostatic field, $$\mathbf{E} = \frac{qr}{kR^3}\hat{\mathbf{r}} \tag{6.38}$$

The expression for the electrostatic potential and field are plotted as a function of distance from the centre of the sphere in Fig. 6.9.

The charge on the sphere will get distributed uniformly over a dielectric sphere of homogeneous material. However, if the sphere is made of a conducting material, then the charge will reside on the surface and the expressions for the potential and field will be those pertaining to those of the spherical shell.

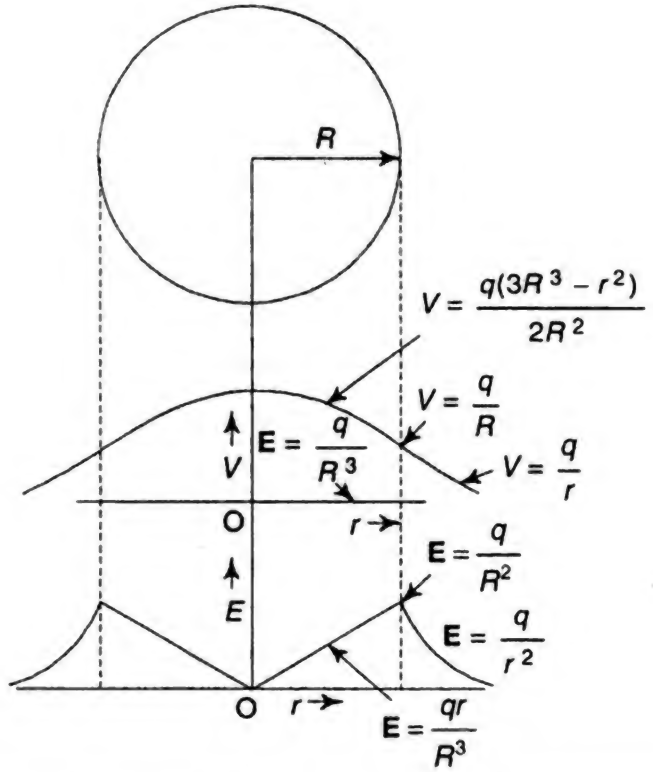


Fig. 6.9 Plots of electrostatic potential and electrostatic field for a uniformly charged sphere as a function of distance from the centre of the sphere

6.6 EARTH'S GRAVITATIONAL FIELD, ESCAPE AND ORBITING VELOCITIES

Escape velocity is defined as the velocity that an object requires to escape from the surface of a heavenly body, say, earth, moon, mars, and so on.

Let us consider the case of escape from earth. If an object is thrown away from the earth, with initial velocity v, it will possess at any given point in its travel, a potential energy due to its position and a kinetic energy due to its motion.

The potential energy of the body at a distance r from the centre of earth is given by the amount of work required to be done on the body to move it from that point to infinity, and is given by

$$E_p = -\int_r^\infty \mathbf{E} \cdot d\mathbf{r}$$

$$= -\int_r^\infty \frac{GMm}{r^2}\, dr = -\frac{GMm}{r} \tag{6.39}$$

The negative sign means that work is done on the body, and therefore,

$$E_p = \frac{GMm}{r} \tag{6.40}$$

If this point is taken on the surface of earth, then $r = R$, and

$$E_p = \frac{GMm}{R}$$

In other words, this much energy should be provided to the body to move it from the surface of earth to infinity. This energy can be given to the body by imparting it the kinetic energy, E_K, which is given by

$$E_K = \frac{1}{2} m v_e^2$$

Thus, for the body to escape to infinity, we should have

$$\frac{1}{2} m v_e^2 = \frac{GMm}{R}$$

or
$$v_e = \sqrt{\frac{2GM}{R}} \tag{6.41}$$

However, at the surface of earth, $g = \dfrac{GM}{R^2}$

or
$$GM = gR^2$$

Therefore,
$$v_e = \sqrt{2gR} \tag{6.42}$$

The value of escape velocity for earth

$$v_e = \sqrt{2 \times 981 \times 6.4 \times 10^8}$$
$$= 11.2 \text{ km/s} = 7 \text{ miles/s} \tag{6.43}$$

As the escape velocity depends on the values of g and R, it will be different for different planets.

Next, we consider orbiting velocity. The orbiting velocity of a body is the velocity with which it is to be projected so that it orbits around the earth. The centrifugal force is just counter-balanced by the earth's attraction.

Consider a satellite of mass m, moving in stable orbit at distance h from earth's centre.

Then,
$$\frac{m v_o^2}{(R_E + h)} = \frac{GmM}{(R_E + h)^2}$$

where v_o is the orbiting velocity, m is the mass of the satellite, M is the mass of earth, and R_E is the earths' radius. Therefore,

$$v_o^2 = \frac{GM}{R_E + h}$$

Now, near earth's surface $h << R_E$ and the orbiting velocity becomes the launching velocity v_1 so that

$$v_1 = \sqrt{\frac{GM}{R_E}} = \sqrt{gR} \tag{6.44}$$

Comparing Eqs (6.41) and (6.44), we get

$$v_e = \sqrt{2} v_1$$

Thus, the launching velocity for a satellite from earth

$$= \frac{11.2}{\sqrt{2}} \text{ km/s} = 8 \text{ km/s} \tag{6.45}$$

6.7 EXISTENCE OF ATMOSPHERE AROUND A PLANET

If a particle on the surface of a planet has velocity equal to its escape velocity $v_e = \sqrt{2gR}$ on that planet, then the particle escapes permanently from the planet.

Thus, the important consequence of v_e for a planet is that it can help us to determine the probable nature of the atmosphere there.

Consider a molecule of mass m in the atmosphere on the planet. If the mean temperature of the atmosphere is T°K, the mean kinetic energy of the molecule is 3/2 kT, where k is Boltzmann constant and its mean speed is given by

$$1/2\ mv^2 = 3/2\ kT$$

or
$$v = \sqrt{\frac{3kT}{m}}$$

If $v >> v_e$, the particular molecule will leave the atmosphere and over a length of time it will be completely depleted from there. However, there is Maxwellian distribution of the velocities of the molecules and there are always molecules with speeds greater than v_e. Thus, even if all those molecules with $v > v_e$ leave, the remaining molecules will help establish the statistical equilibrium with the consequence that there will result more molecules with $v > v_e$. All the molecules will eventually escape in the course of time. The required time interval will be greater if the difference $(v - v_e)$ is greater. According to J.H. Jeans, the mean speed of a molecule should be equal to or greater than $v_e/5$ if the gas is to remain on the planet for a period of 1 billion years. It implies that all those molecules with mean speed $\bar{v} >> v_e$ will escape completely over a period of one billion years and only those gases remain on the planet whose $\bar{v} < v_e$ for that planet.

The mean molecular speed of a gas at temperature T is given by

$$\bar{v} = \sqrt{\frac{8kT}{\pi m}}$$

Thus, the condition to be satisfied by those molecules that can remain in the atmosphere after about 1 billion years, becomes

$$\bar{v} = \sqrt{\frac{8kT}{\pi m}} \le \frac{v_e}{5} \tag{6.46}$$

Since $v_e = \sqrt{2gR}$, we have

$$\frac{8kT}{\pi m} \le \frac{2gR}{25} \tag{6.47}$$

The minimum molecular mass that can still remain

$$m = \frac{25}{v_e^2}\frac{8kT}{\pi}$$

For earth, v_e = 11.2 km/s = 11.2 × 10^5 cm/s

$$T = 300°\text{K}$$

we get,
$$m = \frac{25 \times 8 \times 1.38 \times 10^{-16} \times 300}{(11.2 \times 10^5)^2 \times 3.14}$$

The unit of mass is 1.66 × 10^{-24} g, so the minimum molecular weight of mass m

$$= \frac{25 \times 8 \times 1.38 \times 10^{-16} \times 300}{(11.2 \times 10^5)^2 \times 3.14 \times 1.66 \times 10^{-24}}$$

$$= 1.34$$

Thus, there is no possibility for atomic hydrogen ($m \approx 1$) to exist. However, in the upper regions of the atmosphere, the temperature is very low, and therefore, atomic hydrogen can exist.

For moon, $v_e = 2.4$ km/s $= 2.4 \times 10^5$ cm/s

$T = 400$ °K

So, minimum molecular weight

$$= \left(\frac{11.2 \times 10^5}{2.4 \times 10^5}\right)^2 \times \left(\frac{400}{300}\right) \times 1.34 = 39$$

The gases nitrogen (28) and oxygen (32), which are the predominant constituents of earths' atmosphere, cannot be present on the moon. Therefore, we conclude that for a planet, higher is the escape velocity, denser is the atmosphere around it.

6.8 GRAVITATIONAL SELF-ENERGY

The gravitational self-energy of any material body is defined as its potential energy or the amount of work done in assembling the body from its constituent particles, which initially are placed at infinite distance from each other. If N particles constitute the body, then like the potential energy of a system of masses, Eq. (4.65 b), the self-energy is

$$U_s = -\frac{1}{2} G \sum_{i=1}^{n} \sum_{\substack{j=1 \\ \neq i}}^{n} \frac{m_i\, m_j}{r_{ij}} \tag{6.48}$$

The negative sign indicates that in the process of putting together these particles in the form of the material body, this much energy is converted into the kinetic energy of the particles, and eventually, radiated away.

EXAMPLE 6.1

Show that the gravitational self-energy of a system of n particles, each of mass m, at an average distance r from each other is given by $U_s = -\frac{1}{2} Gn\,(n-1)\,\frac{m^2}{r}$.

Solution

There are n masses in the system, the total potential energy or the gravitational self-energy of the n-particle system is

$$U_s = -G \sum_{\substack{\text{all pairs} \\ i \neq j}} \frac{m_i m_j}{r_{ij}} \tag{1}$$

where the summation extends over all pairs of masses i and j. The particular case $i = j$ is excluded since it does not constitute a pair by itself, which however, does not contribute to the potential energy of the system.

Alternatively, ensuring that each pair of masses i and j is to be counted only once, we may express the above result as

$$U_s = -G \sum_{i>j}^{n} \sum_{j=1}^{n} \frac{m_i m_j}{r_{ij}} \tag{2}$$

However, one may remove this restriction by counting all possible pairs, which clearly implies counting each pair twice and then putting1/2 before the expression. Thus,

$$U_s = -\frac{1}{2} G \sum_{i=1}^{n} \sum_{\substack{j=1 \\ \neq i}}^{n} \frac{m_i m_j}{r_{ij}}$$

Here, $m_i = m_j = m$ and there are n equal terms in the summation $\sum_{i=1}^{n}$ and $(n - 1)$ terms in $\sum_{\substack{i=1 \\ \neq i}}^{n}$. Therefore,

$$U_s = -\frac{1}{2} Gn (n - 1) \frac{m^2}{r}$$

6.8.1 Gravitational Self-Energy of Uniform Solid Sphere

The gravitational self-energy of a uniform solid sphere is equal to the work done in assembling together its constituent particles, which initially lie at infinite distance from each other (Eq. (6.34)). This will be both a tedious affair as it will necessitate the conversion of the summation signs into integrals and then doing the multiple integrations.

However, the spherical symmetry of the sphere enables us to adopt an alternative approach, which considers the sphere to be formed by the deposition of successive spherical shells around the inner core of radius r till the sphere acquires radius R.

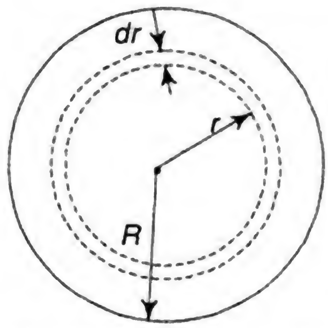


Fig. 6.10 Gravitational self-energy of a uniform solid sphere

Consider a sphere of radius R and mass M distributed uniformly (Fig. 6.10). Let ρ be the material density.

When the spherical shell has radius r, its mass content is $\frac{4}{3}\pi r^3\rho$. By depositing the material so as to increase the radius of the shell to $r + dr$, the mass content of the thin shell of thickness dr is given by

$$dm = 4\pi r^2 dr\rho \tag{6.49}$$

The energy released in increasing the mass by dm of the shell

$$dU_s = \frac{-G(\frac{4}{3}\pi r^3\rho)(4\pi r^2\rho dr)}{r}$$

$$= -\frac{16}{3}\pi^2\rho^2 G r^4 dr \tag{6.50}$$

The total energy released in the formation of sphere of radius R

$$= -\int_0^R \frac{16}{3}\pi^2\rho^2 G r^4 dr$$

$$= -\frac{16}{15}\pi^2\rho^2 G R^5$$

$$= -\frac{16}{15}\pi^2 G R^5\left(\frac{M}{\frac{4\pi}{3}R^3}\right)^2$$

$$= -\frac{3}{5}\frac{GM^2}{R} \tag{6.51}$$

Let us apply Eq. (6.51) to calculate the self-energy of earth.

Putting $G = 6.67 \times 10^{-8}$ dynes cm^2 gm^{-2}

$M_e = 5.98 \times 10^{27}$ gm

$R_e = 6.4 \times 10^8$ cm

$$U_s = -\frac{3}{5}\times\frac{6.67\times10^{10}\times(5.98\times10^{27})^2}{6.4\times10^8}$$

$$= -2.3 \times 10^{38} \text{ ergs}$$

EXAMPLE 6.2

Calculate the gravitational self-energy of

(a) the sun

(b) the earth

(c) the earth-sun system. Given that the mass of the sun = 2×10^{30} kg and its radius = 7×10^8 metres, mass of earth = 6×10^{24} kg and its radius = 6.4×10^8 cm, the mean earth-sun distance = 1.5×10^8 km, and $G = 7 \times 10^{-11}$ N m^2/kg^2.

(d) The amount of heat that would have been evolved at the time of the formation of earth.

Solution

(a) Self-energy of sun $= -\frac{3}{5}\frac{M_s^2 G}{R_s}$

$$= -\frac{3}{5}\times\frac{(2\times10^{30})^2\times7\times10^{-11}}{7\times10^8}\text{J}$$

$$= -2.4 \times 10^{41} \text{ J}$$

(b) Self-energy of earth $= -\frac{3}{5}\frac{M_e^2 G}{R_e}$

$$= -\frac{3}{5}\times\frac{(6\times10^{24})^2\times7\times10^{-11}\times10^2}{6.4\times10^8}$$

$$= -23.62 \times 10^{31} \text{ J}$$

(c) Self-energy of earth-sun system $= -\frac{M_s M_e G}{r_{es}}$

$$= -\frac{2 \times 10^{30} \times 6 \times 10^{24}}{1.5 \times 10^{11}} \times 7 \times 10^{-11}$$
$$= -5.6 \times 10^{33} \text{ J}$$

(d) The amount of heat that would have evolved at the time of the formation of earth is equivalent to its self-energy. Thus, it is equal to

$$= \frac{23.62 \times 10^{31}}{4.2} \text{ cal}$$
$$= 5.6 \times 10^{31} \text{ cal.}$$

EXAMPLE 6.3

If a 500 kg meteor falls on the earth, how much does the self-energy of the earth increase or decrease? What is the potential energy lost by the meteor? If the meteor started from rest, with what velocity does it strike the earth? Given that radius of earth = 6.37×10^6 metre, $g = 9.80$ m/sec^2.

Solution

The potential energy of the earth-meteor system will decrease by $\frac{G M_e M_m}{R}$ when the meteor falls to the surface of earth. M_e is the mass of earth, M_m the mass of the meteor and R the radius of earth.

Now, $\frac{M_e G}{R^2} = g$, therefore, $\frac{M_e M_m G}{R} = M_m g R$

Therefore, loss in the potential energy of the earth-meteor system

$$= 500 \times 9.8 \times 6.37 \times 10^6$$
$$= 3.12 \times 10^{10} \text{ J}$$

The loss in potential energy of the system is the gain of the kinetic energy of the meteor.

Since the meteor starts from the position of rest, the gain in its kinetic energy = $\frac{1}{2} M_m v^2$, where v is the velocity with which it strikes the earth.

Thus,
$$v = \left(\frac{2 \times 3.12 \times 10^{10}}{M_m}\right)^{\frac{1}{2}}$$

$$= \left(\frac{2 \times 3.12 \times 10^{10}}{500}\right)^{\frac{1}{2}} = 11.2 \text{ km/s}$$

EXAMPLE 6.4

Gravitational energy of a galaxy;

Estimate the gravitational energy of a galaxy consisting of 1.6×10^{11} stars, each equal to the mass of the sun and with an average distance $r = 10^{21}$ metres between each pair of stars. Given, mass of sun = 2×10^{30} kg and G = 7×10^{-11} N-M/kg^2.

Solution

The gravitational potential energy or self-energy of n stars is

$$U_s = -G \sum_{\substack{\text{all pairs} \\ i \neq j}} \frac{M_i\, M_j}{r_{ij}} = -\frac{1}{2} G \sum_{i=1}^{n} \sum_{\substack{j=1 \\ \neq i}}^{n} \frac{M_i M_j}{r_{ij}}$$

where M_i and M_j are the individual masses and r_{ij} is the distance apart of those individual masses. The case $i = j$ is omitted because this is not a pair at all. The self-energy of individual masses is also ignored since only the mutual interactions of the masses are considered.

Here $M_i = M_j = M$ and there are n equal terms in $\sum_{i=1}^{n}$ and $(n - 1)$ terms in $\sum_{\substack{j=1 \\ \neq i}}^{n}$ and in so doing we count each pair twice.

$$U_s = -\frac{1}{2}\, Gn\,(n-1)\, \frac{M^2}{r}$$

$$= -\frac{1}{2}\, \frac{7 \times 10^{-11} \times 1.6 \times 10^{-11} \times (1.6 \times 10^{11} - 1) \times (2 \times 10^{30})^2}{10^{21}}$$

$$= -4 \times 10^{51}\ \text{J}$$

6.9 ELECTROSTATIC SELF-ENERGY

Electrostatic self-energy of a charged body is the potential energy due to its own charge that is, the work done in charging the body to its present state by bringing infinitesimal fractions of charge from infinity
If there are n charges, then the electrostatic self-energy is expressed as

$$U_s = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ \neq i}}^{n} \frac{q_i\, q_j}{k r_{ij}} \tag{6.52}$$

This can be expressed by carrying out the summation unrestricted and taking $\frac{1}{2}$ of it as

$$U_s = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{q_i\, q_j}{k r_{ij}} \tag{6.53}$$

The self-energy terms corresponding to $i = j$ are not included in the sum; k is the dielectric constant of the intervening medium, and for air, $k = 1$.

The self-energy calculation in case of crystals of metals and dielectrics plays an important role in solid-state physics.

EXAMPLE 6.5

Calculate the electrostatic potential energy of three point charges , $+q$, $+q$, and $-q$ at the corners of an equilateral triangle of side a.

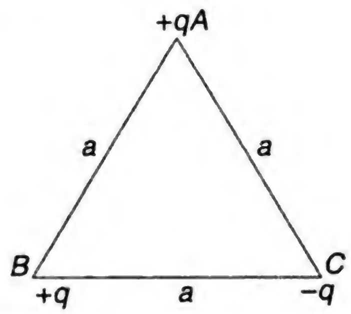


Fig. E6.5 Electrostatic potential energy

Solution

The electrostatic potential energy of the three point charges (Fig. E6.5)

$$= U_{AB} + U_{AC} + U_{BC}$$

$$= \frac{q^2}{4\pi\varepsilon_o a} - \frac{q^2}{4\pi\varepsilon_o a} - \frac{q^2}{4\pi\varepsilon_o a} = \frac{-q^2}{4\pi\varepsilon_o a}.$$

EXAMPLE 6.6

Four charges $+q$, $-q$, $+q$, $-q$ are placed in the same order on the four consecutive corners of a square of side a.

(a) Calculate the energy W of the system

(b) Calculate the work done (ΔW) in interchanging the positions of any two neighbouring charges of opposite signs.

Solution

(a) The energy of the system (Fig. E6.6(a)) is given by

$$W = \sum_{\substack{i,j=1 \\ i \neq j}}^{4} \frac{q_i q_j}{r_{ij}}$$

$$= -\frac{q^2}{4\pi\varepsilon_o a} + \frac{q^2}{4\pi\varepsilon_o \sqrt{2}a} - \frac{q^2}{4\pi\varepsilon_o a} - \frac{q^2}{4\pi\varepsilon_o a} + \frac{q^2}{4\pi\varepsilon_o \sqrt{2}a} - \frac{q^2}{4\pi\varepsilon_o a}.$$

$$= \frac{q^2}{4\pi\varepsilon_o a}\left[\frac{2}{\sqrt{2}} - 4\right]$$

$$= \frac{q^2}{4\pi\varepsilon_o a}\sqrt{2}\left(1 - 2\sqrt{2}\right)$$

(b) The energy of the system created by interchanging the charges at B and C (Fig. E6.6(b)) is

$$W' = \sum_{\substack{i,j=1 \\ 1 \neq j}}^{4} \frac{q_i q_j}{r_{ij}}$$

$$= \frac{q^2}{4\pi\varepsilon_o a} - \frac{q^2}{4\pi\varepsilon_o \sqrt{2}a} - \frac{q^2}{4\pi\varepsilon_o a} - \frac{q^2}{4\pi\varepsilon_o \sqrt{2}a} + \frac{q^2}{4\pi\varepsilon_o a}$$

$$= -\frac{q^2\sqrt{2}}{4\pi\varepsilon_o a}$$

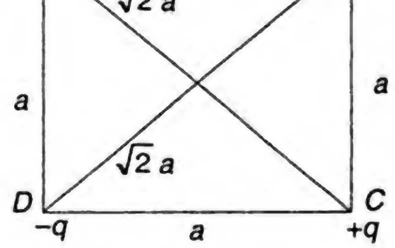


Fig. E6.6(a)

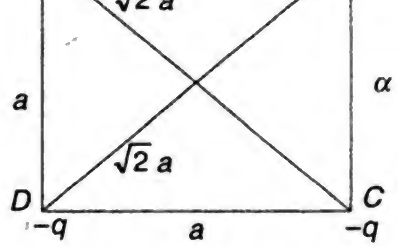


Fig. E6.6(b)

The change in energy of the configuration

$$\Delta W = W' - W = \frac{q^2}{4\pi\varepsilon_o a}\left[-\sqrt{2}-\sqrt{2}+4\right] = \frac{q^2}{4\pi\varepsilon_o a}\left(4-2\sqrt{2}\right)$$

EXAMPLE 6.7

The electrostatic potential due to certain charge distribution is given by the expression $\phi\ (x, y, z) = \dfrac{-V_o}{a^4}\ (x^2yz + xy^2z + xyz^2)V$

where V_0 and a are constants. Calculate the electric field at the points $A\ (o, o\ , a)$, $B\ (o, a, a)$, and $C\ (a, a, a)$. What is the magnitude of the field at C? Also, find the charge density at points, A, B, and C.

Solution

Electric field at a point $\mathbf{E} = -\ \nabla\phi$

$$= -\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\mathbf{k}\frac{\partial}{\partial z}\right)\phi$$

We get

$$-\mathbf{i}\frac{\partial\phi}{\partial x} = \mathbf{i}\frac{V_o}{a^4}\ (2xyz + y^2z + yz^2) \qquad (1)$$

$$-\mathbf{j}\frac{\partial\phi}{\partial y} = \mathbf{j}\frac{V_o}{a^4}\ (x^2z + 2xyz + xz^2) \qquad (2)$$

$$-\mathbf{k}\frac{\partial\phi}{\partial z} = \mathbf{k}\frac{V_o}{a^4}\ (x^2y + xy^2 + 2xyz) \qquad (3)$$

Combining (1), (2), and (3), we get

$$\mathbf{E} = \frac{V_o}{a^4}[\mathbf{i}(2xyz + y^2z + yz^2) + \mathbf{j}(x^2z + 2xyz + xz^2) + \mathbf{k}(x^2y + xy^2 + 2xyz)] \qquad (4)$$

Let us evaluate the electric field at different points.

$$\mathbf{E}(0, 0, \mathrm{a}) = 0$$

$$\mathbf{E}(0, \mathrm{a}, \mathrm{a}) = \frac{V_o}{a^4}\ [\mathbf{i}(a^3 + a^3) + \mathbf{j}(0) + \mathbf{k}(0)]$$

$$= \frac{2V_o}{a}\ \mathbf{i}$$

$$\mathbf{E}(a, a, a) = \frac{V_o}{a^4}\ [\mathbf{i}(2a^3 + a^3 + a^3) + \mathbf{j}(a^3 + 2a^3 + a^3) + \mathbf{k}(a^3 + a^3 + 2a^3)]$$

$$= \frac{4V_o}{a}\ (\mathbf{i} + \mathbf{j} + \mathbf{k})$$

The magnitude of electric field at $C = \dfrac{4V_o}{a}\sqrt{(1)^2 + (1)^2 + (1)^2}$

$$= \frac{4\sqrt{3}\,V_o}{a}$$

To find the charge density at different points, we make use of Poisson equation.

$$\nabla^2\phi = -\rho/\varepsilon_o$$

or

$$\nabla \cdot (\nabla\phi) = -\rho/\varepsilon_o$$

$\therefore$

$$\rho = \varepsilon_o \nabla \cdot \mathbf{E} \tag{5}$$

Rewriting Eq. (4),

$$\mathbf{E} = \frac{V_o}{a^4}\,[\mathbf{i}(2xyz + y^2z + yz^2) + \mathbf{j}(x^2z + 2xzy + xz^2) + \mathbf{k}(x^2y + xy^2 + 2xzy)]$$

$$\nabla \cdot \mathbf{E} = \frac{V_o}{a^4}\,[2yz + 2xz + 2xy]$$

$$\therefore \quad \rho = \frac{V_o\,\varepsilon_o}{a^4}\,[2yz + 2xz + 2xy] \tag{6}$$

Thus,

$$\rho(A) = 0$$

$$\rho(B) = \frac{V_o\,\varepsilon_o}{a^4}\,[2a^2] = \frac{2V_o\,\varepsilon_o}{a^2}$$

$$\rho(C) = \frac{V_o\,\varepsilon_o}{a^4}\,[2a^2 + 2a^2 + 2a^2] = \frac{6V_o\,\varepsilon_o}{a^2}$$

EXAMPLE 6.8

Find the spherical surface of zero potential due to charges $+2q$ and $-3q$ fixed at (4, 0, 0) and (9, 0, 0), respectively.

Solution

Let the point (x, y, z) be the locus of zero potential surface. Then,

$$\frac{2q}{\sqrt{(x-4)^2 + y^2 + z^2}} - \frac{3q}{\sqrt{(x-9)^2 + y^2 + z^2}} = 0$$

$$9[x^2 + 16 - 8x + y^2 + z^2] = 4[x^2 + 81 - 18x + y^2 + z^2]$$

Simplifying we get, $x^2 + y^2 + z^2 = 36$

This is the equation of a circle of radius 6 units, with its centre at (0, 0, 0).

6.9.1 Electrostatic Self-Energy of a Charged Sphere

The sphere can be of a conducting material like a metal or of a dielectric. We will consider these cases separately.

Case I:

Conducting Sphere

In this case the charges on the body will reside on the surface and spread uniformly on it. Imagine that initially all the charges are at infinity and are being brought bit by bit to charge the body. In this process, imagine increasing the charge of the body.

Then the work done against the repulsive force is $\frac{q}{C}dq$, where $\frac{q}{C}$ is the potential of the body, q being the charge and C its capacity. Thus, the total work done is given by

$$U_s = \int_o^q \frac{q}{C} dq = \frac{q^2}{2C}$$

For a charged sphere, $C = R$ in esu. Thus,

$$U_s = \frac{q^2}{2R} \tag{6.54}$$

Case II:

Dielectric ***Sphere***

In this case, the charge is distributed uniformly throughout its volume. Analogous to the gravitational sphere, the self-energy of charge q distributed uniformly within the sphere of radius R is given by

$$U_s = \frac{3q^2}{5R} \tag{6.55}$$

6.9.2 Classical Radius of Electron

As we do not know exactly the distribution of charge inside an electron, we attribute to it an electron radius in the sense that a charge distribution totaling an electronic charge must have a radius equal to it if its electrostatic self-energy is to equal the rest-energy of the electron. Thus,

$$\frac{e^2}{r_o} = mc^2 \tag{6.56}$$

$$r_o = \frac{e^2}{mc^2} \tag{6.57}$$

where e, the electronic charge = 4.8×10^{-14} esu.
m, the mass of the electron = 9.1×10^{-28} gm
and c, velocity of light in vacuum = 3×10^{10} cm/s
Putting these values, the classical radius of the electron

$$r_o = \frac{(4.8 \times 10^{-10})^2}{9.1 \times 10^{-28} \times (3 \times 10^{10})^2} = 2.81 \times 10^{-13} \text{ cm} \tag{6.58}$$

6.10 MOTION UNDER FORCE OBEYING INVERSE SQUARE LAW

As discussed in the previous section, of the four types of force existing in nature, two, viz., electromagnetic and gravitational have an inverse square dependence, and the other two, viz., strong and weak interactions are short-ranged. Whereas long range forces, obeying the inverse square law are mainly observed in the macroscopic world, short range forces manifest themselves in microscopic especially nuclear processes. As the nuclei are very small in size (10^{-13} cm) and as the short range forces have the same range as the nuclear size, they can only be dealt with through the methods of quantum mechanics.

Since the forces obeying the inverse square law are effective over large distances, the distances involved in these interactions are also large. Furthermore, in the

case of gravitational problems, generally heavenly bodies are involved which are very large in sizes. However, the problems of electromagnetic interaction generally pertain to charged particles, such as electrons, protons, etc. which have small dimensions. Therefore, while in gravitational problems classical mechanics is directly applicable, in electromagnetic problems the concepts of quantum mechanics also have to be invoked.

The aim of the following discussion is to develop an equation of motion of a body moving under the influence of the force obeying the inverse square law, exemplified by the case of motion of a planet under the influence of gravitational attraction of the sun. We will consider not only the motion of the planet around the sun, but also the motion of the sun itself. In other words, it is a two-body problem. In practice, one solves such a problem by first reducing it to a one-body problem as described in the next section.

6.11 EQUIVALENT ONE BODY PROBLEM

(a) Centre of Mass System

As already discussed in Sec. 4.3, if there is a system of many mass points, its centre of mass is given by

$$MR = \sum_i m_i \mathbf{r}_i \tag{6.59}$$

where $\mathbf{r}_i$ is the vector radial position of the ith mass point with mass m_i; $m_i\mathbf{r}_i$ is its moment around the origin, $\mathbf{R}$ is the vector radial position of the centre of mass and M is the total mass of the system of mass points.

If the centre of the coordinate system is chosen to be at the centre of mass, then $\mathbf{R} = 0$ and Eq. (6.59) reduces to

$$\sum_i m_i \mathbf{r}_i = 0 \tag{6.60}$$

For two mass points, m_1 and m_2, their distances from CM are given by Eq. (6.60), i.e.

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0$$

or
$$(|\mathbf{r}_1|/|\mathbf{r}_2| = -(m_2/m_1) \tag{6.61}$$

i.e. the CM divides the line joining m_1 and m_2 in the ratio of m_2/m_1.

In general, for any coordinate system, one simply writes for two masses, from Eq. (6.59)

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = (m_1 + m_2)\, \dot{\mathbf{R}} \tag{6.62}$$

Differentiating, we get

$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = (m_1 + m_2)\, \dot{\mathbf{R}}$$

or
$$\dot{\mathbf{R}} = (m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2)/(m_1 + m_2) \tag{6.63}$$

Differentiating again, we obtain

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = (m_1 + m_2)\, \ddot{\mathbf{R}}$$

or
$$\ddot{\mathbf{R}} = (m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2)/(m_1 + m_2) \tag{6.64}$$

In the case of gravitational forces that are central, i.e. dependent only on r and are independent of angle θ and φ, and obey the inverse square law, the equations of motion of the two bodies will be given by

$$m_1 \ddot{\mathbf{r}}_1 = -G \frac{m_1 m_2}{r^2} \hat{\mathbf{r}} \tag{6.65a}$$

and
$$m_2 \ddot{\mathbf{r}}_2 = +G \frac{m_1 m_2}{r^2} \hat{\mathbf{r}} \tag{6.65b}$$

where $\hat{\mathbf{r}}$ is the unit vector along

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

and
$$r^2 \equiv |(\mathbf{r}_1 - \mathbf{r}_2)|^2$$

We have used opposite signs in Eqs (6.65a) and (6.65b) because the forces on m_1 and m_2 are in opposite directions as shown in Fig. 6.11. The force on m_1 is towards m_2 and hence opposite to $\mathbf{r}_1 - \mathbf{r}_2$ and that on m_2 is towards m_1 and hence along $\mathbf{r}_1 - \mathbf{r}_2$. Adding Eqs (6.65a) and (6.65b), we have

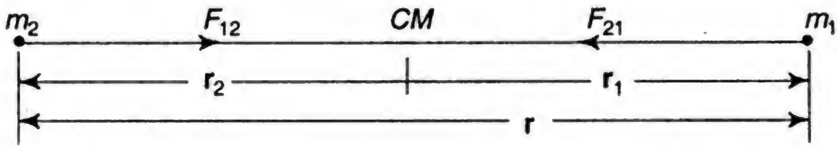


Fig. 6.11 Representation of CM for a two-particle system

$$m_1 \ddot{\mathbf{r}}_1 + m_2 \ddot{\mathbf{r}}_2 = 0 \tag{6.66}$$

which on substitution in Eq. (6.64) leads to

$$\ddot{\mathbf{R}} = 0$$

Integrating Eq. (6.66), we get

$$m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2 = \text{const}$$

Combining this result with Eq. (6.63), we obtain

$$\mathbf{R} = \frac{m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2}{m_1 + m_2} = \text{const} \tag{6.67}$$

Thus, we see that in the case of central forces obeying the inverse square law, the centre of mass has a constant velocity with respect to an arbitrary coordinate system but its acceleration is zero. It may be mentioned that Eq. (6.66) would have been obtained, even when the denominator on the right-hand side of Eq. (6.65) contained other powers of r, because the difference in sign in Eqs (6.65a) and (6.65b) comes from the property of central forces. Consequently the condition of central forces along with Newton's second law of motion is essential for the validity of Eq. (6.67) and not the inverse square law.

(b) Equation of Motion of One-Body Problem

Equations (6.65a) and (6.65) can be written as

$$\ddot{\mathbf{r}}_1 = (-1/m_1)\,[G(m_1 m_2)/r^2]\,\hat{\mathbf{r}} \tag{6.68a}$$

and
$$\ddot{\mathbf{r}}_2 = (1/m_2)\,[G(m_1 m_2)/r^2]\,\hat{\mathbf{r}} \tag{6.68b}$$

Subtracting Eq. (6.68b) from (6.68a), we get

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right) G \frac{m_1 m_2}{r^2} \hat{\mathbf{r}}$$

or
$$\ddot{\mathbf{r}} = -\frac{1}{\mu} G \frac{m_1 m_2}{r^2} \hat{\mathbf{r}} \tag{6.69}$$

where we define
$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \tag{6.70a}$$

and
$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M} \tag{6.70b}$$

is called the reduced mass or effective mass. Equation (6.69) can be rewritten as

$$\mu \ddot{\mathbf{r}} = -G\,\frac{m_1 m_2}{r^2}\,\hat{\mathbf{r}}$$
$$= F(r)\,\hat{\mathbf{r}} = \mathbf{F}(r) \tag{6.71a}$$

It is easy to see that Eq. (6.71a) is an equation of motion of one body with mass equal to reduced mass μ moving under the influence of force given by

$$\mathbf{F}(r) = -\,(Gm_1m_2/r^2)\,\hat{\mathbf{r}} \tag{6.71b}$$

Physically, Eq. (6.71a) represents the motion of a mass μ moving at a distance of $\mathbf{r}_1 - \mathbf{r}_2$. Since $\mathbf{r}_1$ are the vectors in opposite direction, $|\mathbf{r}_1 - \mathbf{r}_2| = |\mathbf{r}_1| + |\mathbf{r}_2|$. Therefore, the motion can be taken as that of the reduced mass concentrated at one of the point masses and revolving around the second mass point.

Equation (6.71a) can also be reduced to

$$\ddot{\mathbf{r}} = -\,G(M/r^2)\,\hat{\mathbf{r}} \tag{6.72}$$

It can be easily seen from Eq. (6.72) that the magnitude of the acceleration of one body with respect to the other body will appear to be the same whether the observer is at mass m_1 or m_2, but in opposite directions.

On the other hand, if the observer is located at the centre of mass, then Eq. (6.68) hold good. Furthermore, if $m_2 \gg m_1$, then $M = m_1 + m_2 \simeq m_2$ and from Eq. (6.61) and definition of $\mathbf{r}$, we get

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_1 + (m_1/m_2)\,\mathbf{r}_1$$
$$= \mathbf{r}_1\,(1 + m_1/m_2) \simeq \mathbf{r}_1$$

Hence, one can see that for $m_2 \gg m_1$,

$$\ddot{\mathbf{r}} \simeq \ddot{\mathbf{r}}_1$$

and
$$\frac{-GM}{r^2}\hat{\mathbf{r}} \simeq \frac{-Gm_2}{r_1^2}\hat{\mathbf{r}} \tag{6.73}$$

Therefore, whether the observer is at the heavy mass or at the centre of mass, the motion of m_1 looks similar. However, to the observer at the centre of mass, the heavy mass m_2 is at a very short distance from the centre of mass because $|\mathbf{r}_2 = -\,(m_1/m_2)\,|\mathbf{r}_1|$ is very much less than $|\mathbf{r}_1|$ and it seems to describe a circle with a small radius. On the other hand, the motion of the light mass, say m_1 will describe a larger circle, if one is sitting at the centre of mass or at the heavy mass (Fig. 6.12).

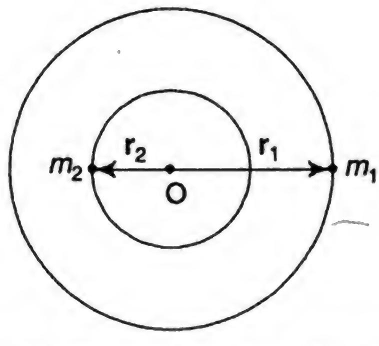


Fig. 6.12 The circles described by the mass m_1 and m_2 as seen by the *CM* at *O*

EXAMPLE 6.9

Two particles of masses m_1 and m_2, having constant velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are moving parallel to each other with separation d. Find expressions for angular momentum

and energy of the equivalent one-body problem assuming that the particles pass each other undeflected.

Solution

It is given that the particles pass each other undeflected. Therefore, the interaction between these, if any, is negligible. Suppose that the position vectors of the particles are $\mathbf{r}_1$ and $\mathbf{r}_2$ when these are at A and B, as shown, then

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

and the relative velocity

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2 = \mathbf{v}_1 - \mathbf{v}_2$$

because $\dot{\mathbf{r}}_2 = \mathbf{v}_2$ (Fig. 6.13). Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are constant, $\mathbf{v}$ will be constant. The effective mass of the equivalent one-body system is

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

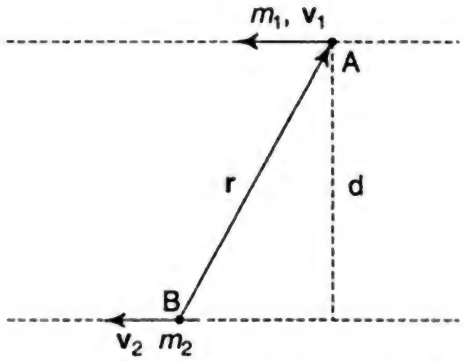


Fig. 6.13 Two particles moving parallel to each other

The angular momentum of one-body system will be (Fig. 6.13):

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = d\mu\,|\mathbf{v}|\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane defined by $\mathbf{r}$ and $\mathbf{v}$. Since there is no interaction energy, the total energy of the system will be kinetic, which is

$$E = T = \frac{1}{2}\mu v^2$$

Combining the expressions for angular momentum and energy, we have

$$E = \frac{1}{2}\frac{L^2}{\mu d^2}$$

Obviously, the angular momentum $\mathbf{L}$ and the energy E are constant for the system.

EXAMPLE 6.10

Find the changes in the values of energy and angular momentum when the problem of a two-body system interacting through gravitational force is reduced to an equivalent one-body case.

Solution

Consider two particles of masses m_1 and m_2 at positions $\mathbf{r}_1$ and $\mathbf{r}_2$. In the case of a two-body problem

Total energy of system = sum of kinetic energies of two particles + potential energy of interaction

Using subscript 2 for this case, the energy becomes

$$E_2 = (1/2)m_1|\dot{\mathbf{r}}_1|^2 + (1/2)m_2|\dot{\mathbf{r}}_2|^2 - G\frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

Angular momentum of the system = sum of the angular momenta of the two particles

$$\therefore \qquad \mathbf{L}_2 = \mathbf{r}_1 \times m_1\dot{\mathbf{r}}_1 + \mathbf{r}_2 \times m_2\dot{\mathbf{r}}_2$$

Here $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ are velocities of the two particles.

When the two-body problem is reduced to the equivalent one-body problem, we have from Eq. (6.70)

Effective mass, $$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

and

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

In this case, the total energy can be written as

Total energy = kinetic energy of effective mass + potential energy of interaction

Using subscript 1 for this case, have the energy to be

$$E_1 = \left(\frac{1}{2}\right)\mu\,|\dot{\mathbf{r}}|^2 - G\frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

$$= \left(\frac{1}{2}\right)\frac{m_1 m_2}{m_1 + m_2}\,[(|\dot{\mathbf{r}}_1|^2 + |\dot{\mathbf{r}}_2|^2 - 2\dot{\mathbf{r}}_1\,\dot{\mathbf{r}}_2)] - G\frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

Therefore, change in energy

$$E_1 - E_2 = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\,[|\dot{\mathbf{r}}_1|^2 + |\dot{\mathbf{r}}_2|^2 - 2\dot{\mathbf{r}}_1\,\dot{\mathbf{r}}_2]$$

$$-\frac{1}{2}\frac{1}{m_1 + m_2}\,[m_1^2\,|\dot{\mathbf{r}}_1|^2 + m_1 m_2\,|\dot{\mathbf{r}}_1|^2 + m_1 m_2\,|\dot{\mathbf{r}}_2|^2 + m_2^2\,|\dot{\mathbf{r}}_2|^2]$$

$$= -\frac{1}{2}\frac{1}{m_1 + m_2}\,(m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2)^2$$

$$= -\frac{1}{2}\,(m_1 + m_2)\,|\dot{\mathbf{R}}|^2$$

where use has been made of Eq. (6.63).

Hence the total energy of the system is decreased by an amount equal to the kinetic energy of the centre of mass.

Furthermore, for angular momentum, we have

$$\mathbf{L}_1 = \mathbf{r} \times \mu\,\dot{\mathbf{r}}$$

$$= \frac{m_1 m_2}{m_1 + m_2}\,(\mathbf{r}_1 - \mathbf{r}_2) \times (\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2)$$

Accordingly,

$$\mathbf{L}_1 - \mathbf{L}_2 = \frac{m_1 m_2}{m_1 + m_2}\,(\mathbf{r}_1 \times \dot{\mathbf{r}}_1 - \mathbf{r}_1 \times \dot{\mathbf{r}}_2 - \mathbf{r}_2 \times \dot{\mathbf{r}}_1 + \mathbf{r}_2 \times \dot{\mathbf{r}}_2)$$

$$-\,(m_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2)$$

$$= \frac{1}{m_1 + m_2} (-m_1 m_2 \mathbf{r}_1 \times \dot{\mathbf{r}}_2 - m_1 m_2 \mathbf{r}_2 \times \dot{\mathbf{r}}_1$$
$$- m_1^2 \mathbf{r}_1 \times \dot{\mathbf{r}}_1 - m_2^2 \mathbf{r}_2 \times \dot{\mathbf{r}}_2)$$
$$= -\frac{1}{m_1 + m_2} (m_1 \mathbf{r}_1 \times m_2 \dot{\mathbf{r}}_2 + m_2 \mathbf{r}_2 \times m_1 \dot{\mathbf{r}}_1$$
$$+ m_1 \mathbf{r}_1 \times m_1 \dot{\mathbf{r}}_1 + m_2 \mathbf{r}_2 \times m_2 \dot{\mathbf{r}}_2)$$
$$= -\frac{1}{m_1 + m_2} (m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2) \times (m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2)$$
$$= -\frac{1}{m_1 + m_2} (m_1 + m_2)\, \mathbf{R} \times (m_1 + m_2)\, \dot{\mathbf{R}}$$
$$= -\mathbf{R} \times (m_1 + m_2)\, \dot{\mathbf{R}}$$
$$= -\mathbf{R} \times M\mathbf{V} = -\mathbf{R} \times \mathbf{P}$$

where we have made use of Eqs (6.63) and (6.64). Obviously, the angular momentum also decreases by an amount equal to the angular momentum of the centre of mass. We, therefore, conclude that in a one-body problem, the motion of the centre of mass should be taken into account before applying the conservation laws in the reference system of the observer.

6.12 MOTION UNDER CENTRAL FORCES

In order to write the final equations of motion, we use the one-body equation for the reduced mass, i.e. Eq. (6.71a), but will replace μ by m, for convenience. The solution of this equation will involve $\mathbf{r}$, the radial vector distance between masses m_1 and m_2. In general, this will have three components r, θ and φ.

Now we know that a central force can be expressed as $\mathbf{F}(r) = F(r)\, \mathbf{r}$ and as seen in Sec. 6.11, the angular momentum $\mathbf{L}$ is constant for these forces. This implies that for the motion under central forces, the magnitude as well as the direction of angular momentum remain the same. Since $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is an axial vector perpendicular to the plane containing $\mathbf{r}$ and $\mathbf{p}$, the situation of no change in its direction refers to the fact that the plane of motion is unchanged.

It is convenient to choose $\mathbf{L}$ along the z-axis. In that case $\mathbf{r}$ and $\mathbf{p}$ both lie in the xy-plane and the motion is confined to this plane only. Accordingly, in terms of the spherical polar coordinates $\theta = \pi/2$ and the motion is defined by r and φ. However, maintaining the terms of description of motion in a plane (Chapter 3), we use θ instead of φ. Now from Eq. (3.36), we have

$$m\mathbf{a}_r = (m\ddot{r} - mr\,\dot{\theta}^2)\,\hat{\mathbf{r}} = F(r)\,\hat{\mathbf{r}} \tag{6.74a}$$
$$m\mathbf{a}_\theta = (mr\ddot{\theta} + 2m\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}} = F(\theta)\,\hat{\boldsymbol{\theta}} \tag{6.74b}$$

where m is the reduced mass. In Eq. (6.74a), $F(r)\,\hat{\mathbf{r}}$ represents the radial force along the direction of increasing $\mathbf{r}$ and in Eq. (6.74b), $F(\theta)\,\hat{\boldsymbol{\theta}}$ is the force along the direction of increasing θ. However, in the case of central force, there is no component of force showing angular dependence, which implies that $F(\theta)$ ought to be zero. This aspect also becomes clear if we consider

$$\frac{d|\mathbf{L}|}{dt} = \frac{d}{dt}(mr^2\dot{\theta}) = 2\,mr\,\dot{r}\dot{\theta} + mr^2\ddot{\theta}$$
$$= r(mr\ddot{\theta} + 2\,m\dot{r}\dot{\theta}) = rF(\theta) = 0$$

because $|\mathbf{L}|$ is constant. Hence $F(\theta) = 0$.

Since $L = mr^2\dot{\theta}$, Eq. (6.74a) may be written as

$$F(r)\hat{\mathbf{r}} = [m\ddot{r} - mr\dot{\theta}^2]\,\hat{\mathbf{r}}$$

$$= \left[m\ddot{r} - mr\left(\frac{L}{mr^2}\right)^2\right]\hat{\mathbf{r}}$$

or
$$m\ddot{r} - \frac{L^2}{mr^3} = F(r) = -\frac{\partial U}{\partial r} \qquad (6.75)$$

where we have assumed that

$$F(r) = -\frac{\partial U}{\partial r}$$

implying that $F(r)$ is not only central, but also conservative. From here on we will not use vector notation for the sake of convenience. Now

$$\frac{\partial}{\partial r}\left[\left(\frac{L^2}{2mr^2}\right)\right] = -\frac{L^2}{mr^3}$$

so that Eq. (6.75) becomes

$$m\ddot{r} = -\left[\frac{\partial}{\partial r}\left(\frac{L^2}{2mr^2}\right)\right] - \frac{\partial U}{\partial r} \qquad (6.76)$$

We can replace $\partial/\partial\dot{r}$ by d/dr because the only variable in Eq. (6.75) is r. We, therefore, can write Eq. (6.76) as

$$m\ddot{r} = -\frac{d}{dr}\left(U + \frac{L^2}{2mr^2}\right)$$

Multiplying both sides by $\dot{r}$, we get (operationally),

$$m\dot{r}\ddot{r} = -\frac{d}{dt}\frac{d}{dr}\left(U + \frac{L^2}{2mr^2}\right)$$

$$= -\frac{d}{dt}\left(U + \frac{L^2}{2mr^2}\right)$$

Integrating, we have

$$\frac{1}{2}m\dot{r}^2 = -\left(U + \frac{L^2}{2mr^2}\right) + \text{const}$$

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + U = \text{const} = E(\text{say}) \qquad (6.77a)$$

Physically, Eq. (6.77a) can be understood as an equation representing the conservation of total energy. The term (1/2) $m\dot{r}^2$ gives the kinetic energy, $L^2/2mr^2$ represents the energy due to rotational motion of the system as can be seen from the expression

$$\frac{L^2}{2mr^2} = \frac{(mr^2\dot{\theta})^2}{2mr^2} = \frac{1}{2}mr^2\dot{\theta}^2 = \frac{1}{2}\,I\omega^2 \qquad (6.77b)$$

and is called the centripetal energy and U is the potential energy.

Equation (6.77a) may also be rewritten as

$$\dot{r} = (dr/dt) = [2/m\,(E - U - L^2/2mr^2)]^{1/2}$$

or
$$dt = \frac{dr}{[(2/m\,(E - U - (L^2/2mr^2))]^{1/2}} \tag{6.78}$$

We also know that

$$\begin{aligned} L &= \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} \\ &= m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= m[\mathbf{r} \cdot \mathbf{r})\boldsymbol{\omega} - (\mathbf{r} \cdot \boldsymbol{\omega})\mathbf{r}] \\ &= mr^2\omega = mr^2\,\dot{\theta} \end{aligned}$$

$$L = mr^2\dot{\theta} = mr^2\frac{d\theta}{dt}$$

or
$$d\theta = \frac{L}{mr^2}\,dt$$

so that
$$\theta = \int_0^t \frac{L}{mr^2}\,dt \tag{6.79}$$

Equations (6.78) and (6.79) can, in principle, be used to obtain the relationship of r with t as well as of θ with t. We should realise that in Eq. (6.79) we should put the expression of r as a function of t, as obtained from Eq. (6.78). In this manner, we can obtain the progress of the motion, i.e. both r and θ of the particle with time.

6.13 SOME PHYSICAL INSIGHTS INTO THE NATURE OF MOTION UNDER CENTRAL FORCES

It is easy to see from Eq. (6.76) that one can write these equations as

$$m\ddot{r} = -\frac{d}{dr}\left(\frac{L^2}{2mr^2} + U\right) = -\frac{dU'}{dr} = F'(r)$$

where
$$U' = U + (L^2/2mr^2) \tag{6.80}$$

Thus
$$F'(r) = -\frac{dU}{dr} - \frac{d}{dr}\left(\frac{L^2}{2mr^2}\right) = -\frac{dU}{dr} + \frac{L^2}{mr^3}$$

$$= F(r) + \frac{L^2}{mr^3} \tag{6.81}$$

The term $L^2/2mr^2$, therefore, behaves like a potential due to centripetal force. We can now write the total energy E as

$$\begin{aligned} E &= (1/2)\,m\dot{r}^2 + U' \\ &= (1/2)\,m\dot{r}^2 + (U + L^2/2mr^2) = \text{const} \end{aligned} \tag{6.82}$$

which is the same as Eq. (6.77). The plots of U, $L^2/2mr^2$ and U' as a function of r for the central inverse square force problem are shown in Fig. 6.14. These are known as energy diagrams. It should be remembered that the kinetic energy (1/2) $m\dot{r}^2$ is always positive. Furthermore, the centripetal energy $L^2/2mr^2$ is also positive, while the potential energy U can be positive or negative. Since $U' = U + L^3/2mr^2$, the effective potential energy U' can also be both positive or negative depending on the relative magnitudes of U and $L^2/2mr^2$, Fig. 6.14. A positive value of U' means

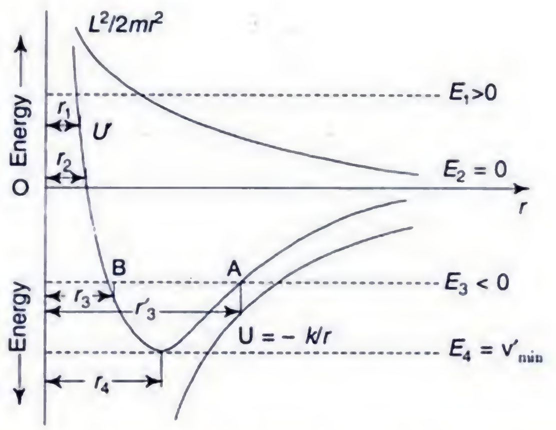


Fig. 6.14 The dependence on distance r of potential energy U corresponding to the inverse square force, the centripetal energy and their sum U'

$\left|\frac{L^2}{2mr^2}\right| > |U|$ and a negative value of U' corresponds to $\left|\frac{L^2}{2mr^2}\right| < |U|$.

It can seen from Fig. 6.14 that depending on the value of total energy E, four different situations can arise. These are:

1. *Total energy* $E = E_1 > 0$

As $E - U' = (1/2)\, m\dot{r}^2$ represents the radial kinetic energy which has to be positive, the body cannot have r less than that corresponding to $E - U' = 0$ or for $E_1 = U'$, as shown in the figure for r_1. Obviously, such a motion will be unbounded, i.e. the particle will come from infinity, go up to r_1 and go back. Detailed calculations show that the path of such a motion is hyperbolic.

2. *Total energy* $E = E_2 = 0$

Then the smallest distance up to which the body can approach is r_2, because as in the previous case, if one goes nearer than this distance, the kinetic energy $E - U'$ is less than zero, which is physically not possible. The path in this case is a parabola.

3. *Total energy* $E = E_3 < 0$

Since $E_3 = U' + \frac{1}{2}m\dot{r}^2 < 0$ and $\frac{1}{2}m\dot{r}^2$ is always positive, this case corresponds to the situation that U' is not only negative but also $|U'| > \left|\frac{1}{2}m\dot{r}^2\right|$. As is clear from Fig. 6.14, in this case the total energy line for E_3 cuts the U' curve at two points A and B. Hence, the distance can have any value between r_3 and r_3'. Thus the motion is between two values of distance, *viz.* r_3 and r_3'. Such a path turns out to be elliptical.

4. *Total energy* $E = E_4 = U'$ minimum

In this case there is only one unique value of $r = r_4$ for which the radial kinetic energy is just zero. The motion here is in a circle because of a single value of r. As this corresponds to the minimum value of U', it is easy to see that

$$dU'/dr = 0$$

or
$$dU/dr = -(d/dr)\,(L^2/2mr^2) = L^2/mr^3 = mr\dot{\theta}^2 \tag{6.83}$$

Hence $F(r) = -mr\dot{\theta}^2$ or the force acting on the body is just equal to the centripetal force for circular motion.

Obviously, the motion under the inverse square force is unbounded for positive and zero values of energy, the orbit being hyperbolic for the former and parabolic for the latter case. On the other hand, if energy is negative, the two particles form a bound system—the path is either elliptical or circular depending on whether energy is larger than or equal to the minimum effective potential energy. Further, if $L = 0$, there will be no centripetal barrier between the particles and these move in a straight path.

EXAMPLE 6.11

Consider a cloud of point particles interacting through gravitational forces and having a distribution of kinetic energy. Discuss the conditions under which this cloud will contract or expand.

Solution

The particles in the cloud are interacting through gravitational force, and therefore, their potential energy U will always be negative, and depend on the interparticle separation. The kinetic energy T of the particles is always positive and has some distribution. In view of the randomness of motion of the particles, we can assume that there is no rotational motion and as such the centripetal energy is zero. Let the average values of potential and kinetic energies be $\overline{U}$ and $\overline{T}$. Then total energy $\overline{E}$ is given by

$$\overline{E} = \overline{U} + \overline{T}$$

It will negative if $|\overline{U}| > \overline{T}$ and be positive if $|\overline{U}| < \overline{T}$.

Now if $\overline{E} < 0$ for any pair of particles, then the motion is bounded and thus these particles continue to hold together. Extending these arguments to the case of a large number of particles, we can say that those particles will hold together for which the magnitude of potential energy is more than their kinetic energy. If the cloud is dominated by such particles (which can happen if the density of cloud is very large), then it will contract.

On the other hand, if a pair is such that $\overline{E} > 0$, then the motion will be unbounded and the particles can fly apart. Obviously, a particle cloud with a larger number of such particles for which the magnitude of potential energy is less than their kinetic energy will expand.

Note: It is believed that condensation of highly dense particle clouds under their own gravitation led to the formation of stars.

6.14 TRAJECTORY OF A PARTICLE AND TURNING POINTS

While Eqs (6.78) and (6.79) connect r and θ separately to t, it is more usefull to connect r and θ directly, to get the trajectory or orbit $r(\theta)$ of the particle. For this purpose, we introduce a new symbol u such that

$$r \equiv 1/u \tag{6.84}$$

Then

$$\begin{aligned}\dot{r} &= -(1/u^2)(du/d\theta)(d\theta/dt)\\ &= -(1/u^2)(du/d\theta)\,\dot{\theta}\\ &= -(L/m)(du/d\theta)\end{aligned} \tag{6.85}$$

where we have made use of the fact that

$$\dot{\theta} = L/mr^2 = Lu^2/m \tag{6.86}$$

Further

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = -\frac{L}{m}\frac{d}{dt}\left(\frac{du}{d\theta}\right)$$

$$= -\frac{L}{m}\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -\frac{L}{m}\frac{d^2u}{d\theta^2}\dot{\theta}$$

$$= -\frac{L^2u^2}{m^2}\frac{d^2u}{d\theta^2} \tag{6.87}$$

using Eq. (6.86). Combining Eqs (6.75), (6.84) and (6.87) and substituting the values of r and $\dot{r}$ from these equations, we get

$$\frac{d^2u}{d\theta^2} = -u - \frac{m}{L^2u^2}F(1/u) \tag{6.88}$$

From Newton's law of gravitation, we have

$$\mathbf{F}(r) = \mathbf{F}(1/u) = -(Gm_1m_2/r^2)\,\hat{\mathbf{r}} = (-k/r^2)\,\hat{\mathbf{r}}$$

and $\qquad U(r) = -k/r$

where $k = Gm_1m_2$. Hence Eq. (6.88) reduces to

$$\frac{d^2u}{d\theta^2} + u = \frac{mk}{L^2} \tag{6.89}$$

It is a second-order differential equation, whose homogeneous part in the operator notation ($D = d/d\theta$) is

$$(D^2 + 1)\,u = 0$$

This gives $D = \pm i$ which implies that the complimentary function for the solution of the above differential equation is

$$u_c = C_1e^{i\theta} + C_2e^{-i\theta}$$

$$= (C_1 + C_2)\cos\theta + i(C_1 - C_2)\sin\theta$$

$$= A\cos\theta_0\cos\theta + A\sin\theta_0\sin\theta$$

$$= A\cos(\theta - \theta_0)$$

where A and θ_0 are constants to be specified. Since the right-hand side of Eq. (6.89) is constant, the particular integral will be the constant itself, ie.

$$u_p = mk/L^2$$

Hence the general solution of Eq. (6.89) is

$$u = \frac{mk}{L^2} + A\cos(\theta - \theta_0) \tag{6.90}$$

or

$$\frac{1}{r} = \frac{mk}{L^2}\left[1 + \frac{AL^2}{mk}\cos(\theta - \theta_0)\right]$$

$$= \frac{mk}{L^2}\,[1 + \epsilon\cos(\theta - \theta_0)] \tag{6.91}$$

where $$\in = \frac{AL^2}{mk}$$

Equation (6.91) represents the equation of a general conic, the exact shape of which will depend on the value of $\in$, the eccentricity. Obviously, $\in$ depends on L and k and hence the shape of the trajectory is determined by the angular momentum L and the interaction (and hence the total) energy E. From the coordinate geometry it is known that

1. for $\in > 1$, the conic is a hyperbola;
2. for $\in = 1$, it is a parabola;
3. for $0 < \in < 1$, it is an ellipse; and
4. for $\in = 0$, it is a circle. (6.92)

Turning Points

From the equation of a trajectory, i.e. Eq. (6.91), it is clear that r will be maximum when $(\theta - \theta_0)$ is equal to π and is minimum when $(\theta - \theta_0)$ is equal to 0. These positions on a conic define the turning points where particle changes its direction.

$$\text{For, } \theta - \theta_0 = 0; \qquad r_{\min} = (mk/L^2 + A)^{-1}$$

and
$$\text{For, } \theta - \theta_0 = \pi; \qquad r_{\max} = (mk/L^2 - A)^{-1} \tag{6.93}$$

Since at the turning point r is either minimum or maximum, $\dot{r} = 0$ so that from Eq. (6.77), we have

$$E = (L^2/2mr^2) + U(r)$$
$$= (L^2/2mr^2) - k/r$$

or
$$(L^2/2m)\,(1/r^2) - k(1/r) - E = 0 \tag{6.94}$$

It is a quadratic equation, whose solutions are given by

$$1/r_1 = mk/L^2 + [(mk/L^2)^2 + (2mE/L^2)]^{1/2} \tag{6.95a}$$

and
$$1/r_2 = mk/L^2 - [(mk/L^2)^2 + (2mE/L^2)]^{1/2} \tag{6.95b}$$

Obviously these values of r (i.e. r_1 and r_2) define the two turning points. Comparing Eq. (6.95) with Eq. (6.93), we see that

$$A^2 = \frac{m^2k^2}{L^4} + \frac{2mE}{L^2} \tag{6.96}$$

Hence
$$\in^2 = \frac{A^2L^4}{m^2k^2} = \frac{L^4}{m^2k^2}\left(\frac{m^2k^2}{L^4} + \frac{2mE}{L^2}\right)$$
$$= 1 + \frac{2L^2E}{mk^2} \tag{6.97}$$

Clearly, the eccentricity and hence the shape of the trajectory depends on the relationship between E and L. Making use of the results listed in Eq. (6.92), Eq. (6.97) shows that:

1. the trajectory is hyperbolic if $E > 0$;
2. it is a parabola if $E = 0$;
3. it is an ellipse for $E < 0$; and
4. it is a circle if $E = -(mk^2/2L^2)$, (6.98)

These findings have already been discussed qualitatively in Sec. 6.13.

EXAMPLE 6.12

An asteroid is seen to be moving towards the earth. It was first observed at a distance of 1.2×10^9 m and was travelling with a velocity of 10^4 m s^{-1} in a direction, which would pass at a distance of 2.5×10^7 m from the centre of the earth. Assuming that the centre of the earth is at the origin of an inertial coordinate system and the sun as well as other planets have negligible effect on the motion of the asteroid, determine the minimum distance at which it will pass the surface of the earth. When will the asteroid come again near the earth? Given the radius of the earth as 6.4×10^6 m, mass of the earth 6×10^{24} kg and gravitational constant 6.67×10^{-11} N m^2/kg^2.

Solution

Taking masses of the earth and the asteroid as m_e and m_a, the effective mass of the system will be

$$\mu = m = \frac{m_e m_a}{m_e + m_a}$$

since the asteroids have masses much smaller than that of the earth $m_e + m_a \simeq m_e$ and $\mu = m \simeq m_a$.

Therefore, the kinetic and centripetal energies of the system can be found by using m in place of the effective mass.

When the asteroid is at a distance $r = 1.2 \times 10^9$ m from the centre of the earth, their potential energy due to gravitational interaction will be

$$U = -\frac{Gm_a m_e}{r}$$

$$= -\frac{6.67 \times 10^{-11} \times m_a \times 6 \times 10^{24}}{1.2 \times 10^9} \text{ J}$$

$$= -3.3 \times 10^5\, m_a \text{J}$$

Since velocity v of the asteroid at this position is 10^4 m s^{-1}, the kinetic energy will be

$$T = \frac{1}{2} m_a v^2$$

$$= \frac{1}{2} \times m_a \times 10^8 \text{ J}$$

Furthermore, when the asteroid is first seen, its perpendicular distance from the centre of the earth is $b = 2.5 \times 10^7$ m, and therefore, its angular momentum will be

$$L = m_a v b$$

$$= 10^4 \times 2.5 \times 10^7\, m_a \text{kg m}^2 \text{ s}^{-1}$$

$$= 2.5 \times 10^{11}\, m_a \text{kg m}^2 \text{ s}^{-1}$$

Accordingly, the centripetal energy of the asteroid at distance $r = 1.2 \times 10^9$ m is

$$\frac{L^2}{2m_a r^2} = \frac{(2.5 \times 10^{11}\, m_a)^2}{2 \times m_a \times (1.2 \times 10^9)^2} \text{ J}$$

$$= 2.2 \times 10^4\, m_a \text{ J}$$

Hence, $$\text{Total energy} = E = T + U + \frac{L^2}{2m_a r^2}$$
$$= (500 - 3.3 + 0.2) \times 10^5\, m_a \text{ J}$$

Since the total energy is positive, it is clear from Eq. (6.97) that the eccentricity of the orbit will be greater than unity and hence the path will be hyperbolic. Consequently, the asteroid will not approach the earth for the second time.

The minimum distance between the centre of the earth and the asteroid will be given by the turning point defined by Eq. (6.95a), where m stands for the effective mass.

$$\frac{1}{r_{\min}} = \frac{Gm_a^2 m_e^2}{(m_a + m_e)L^2} + \left\{\left[\frac{Gm_a^2 m_e^2}{(m_a + m_e)L^2}\right]^2 + \frac{2m_a m_e E}{(m_a + m_e)L^2}\right\}^{1/2}$$

Here we have used the expression for reduced mass and put $k = Gm_a m_e$. Furthermore, in a motion under a central force, angular momentum is conserved. Therefore,

$$L = 2.5 \times 10^{11}\, m_a \text{ kg m}^2 \text{ s}^{-1}$$

Substituting in the expression for $r_{\min}$, we get

$$\frac{1}{r_{\min}} = \frac{6.67 \times 10^{-11} \times 6 \times 10^{24}\, m_a^2}{(2.5 \times 10^{11}\, m_a)^2}$$
$$+ \left[\left(\frac{6.67 \times 6 \times 10^{13}\, m_a^2}{6.25 \times 10^{22}\, m_a^2}\right)^2 + \frac{2m_a \times 496.9 \times 10^5\, m_a}{6.25 \times 10^{22}\, m_a^2}\right]^{1/2}$$

$$= \frac{4 \times 10^{14}}{6.25 \times 10^{22}} + \left[\left(\frac{4 \times 10^{14}}{6.25 \times 10^{22}}\right)^2 + \frac{9.933 \times 10^7}{6.25 \times 10^{22}}\right]^{1/2}$$

$$= [6.4 \times 10^{-9} + (40.96 \times 10^{-18} + 1.59 \times 10^{-15})^{1/2}]\text{ m}^{-1}$$
$$= (6.4 \times 10^{-9} + 40.4 \times 10^{-9}) = 47 \times 10^{-9}\text{ m}^{-1}$$
$$r_{\min} = 2.1 \times 10^7 \text{ m}$$

Minimum distance from the surface of the earth,
$$= (2.1 \times 10^7 - 6.4 \times 10^6)\text{ m}$$
$$= 1.5 \times 10^7 \text{ m}$$

EXAMPLE 6.13

A particle of mass m moves in a central force field. Prove that

(i) its path must be a plane curve
(ii) its angular momentum is conserved
(iii) its equations of motion are

$$m(\ddot{r} - r\dot{\theta}^2) = F(r)$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$

(iv) the principle of conservation of energy is expressed as

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \int F(r)dr = E$$

where E is its total energy.

Solution

(i) Let $\mathbf{F}(r) = F(r)\hat{\mathbf{r}}$ be the central field, where $\hat{\mathbf{r}}$ is a unit vector in the direction of the position vector $\mathbf{r}$. Then,

$$\mathbf{r} \times \mathbf{F} = F(r)\, \mathbf{r} \times \hat{\mathbf{r}} = 0 \tag{1}$$

Putting $\mathbf{F} = m\dfrac{d\mathbf{v}}{dt}$, Eq. (1) becomes

$$\mathbf{r} \times \frac{d\mathbf{v}}{dt} = 0$$

or

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = 0 \tag{2}$$

Integrating one gets $\mathbf{r} \times \mathbf{v} = \mathbf{C}$ (3)

where $\mathbf{C}$ is a constant vector. Multiplying both sides of Eq. (3) by $\mathbf{r}$, we get

$$\mathbf{r}\cdot(\mathbf{r} \times \mathbf{v}) = \mathbf{r}\cdot\mathbf{C}$$

which leads to $\mathbf{r}\cdot\mathbf{C} = 0$ (4)

Therefore, $\mathbf{r}$ is perpendicular to the constant vector $\mathbf{C}$, implying, thereby, that the motion takes place in a plane.

(ii) Rewriting Eq. (3)

$$\mathbf{r} \times \mathbf{v} = \mathbf{C}$$

Multiplying both sides by m, the mass of the particle, we get

$$m(\mathbf{r} \times \mathbf{v}) = m\mathbf{C} \tag{6}$$

The left hand side of Eq. (5) is the angular momentum, and thus, Eq. (5) shows that the angular momentum is conserved, being always constant in magnitude and direction.

(iii) From Eq. (1), it is clear that the motion of a particle takes place in a plane. We choose the plane to be the xy–plane and the position of the particle at any time t can be represented by the polar coordinates (r, θ).

According to Newton's second law of motion

$$m\{(\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}}\} = F(r)\,\hat{\mathbf{r}}$$

Equating the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ parts on both sides, we get

$$m(\ddot{r} - r\dot{\theta}^2) = F(r) \tag{6}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{7}$$

Multiplying Eq. (6) by $\dot{r}$ and Eq. (7) by $r\dot{\theta}$ and adding, we get

$$m(\dot{r}\ddot{r} + r^2\dot{\theta}\ddot{\theta} + r\dot{r}\dot{\theta}^2) = F(r)\,\dot{r} \tag{8}$$

Rewriting Eq. (8) as

$$\frac{1}{2}m\frac{d}{dt}(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{d}{dt}\int F(r)dr \tag{9}$$

Integrating both sides, we get

$$\frac{1}{2} m \frac{d}{dt} (\dot{r}^2 + r^2\dot{\theta}^2) - \int F(r)dr = \frac{1}{2}mv^2 + U = E \tag{10}$$

where U is the potential energy.
This is the expression of the law of conservation of energy.

EXAMPLE 6.14

Show by means of the substitution, $r = \frac{1}{u}$ that the differential equation for the path of a particle in a central field is

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2} F\left(\frac{1}{u}\right)$$

There is a central force acting at the point 0 and under its influence, a particle moves in a circular orbit passing through 0 (Fig. Ex 6.14). Find the law of force.

Solution

The differential equation for the trajectory of a particle in a central field is given by Eq. (6.22)

$$m\ddot{r} - \frac{L^2}{mr^3} = F(r) = -\frac{\partial U}{\partial r} \tag{1}$$

Introducing the symbol u, through the transformation $u = \frac{1}{u}$ (2)

Then,
$$\dot{r} = -\left(\frac{1}{u^2}\right)\left(\frac{du}{d\theta}\right)\left(\frac{d\theta}{dt}\right)$$

$$= -\left(\frac{1}{u^2}\right)\left(\frac{du}{d\theta}\right)\dot{\theta}$$

$$= -\left(\frac{L}{m}\right)\left(\frac{du}{d\theta}\right) \tag{3}$$

where we have made use of the fact that $\dot{\theta} = \frac{L}{mr^2} = \frac{Lu^2}{m}$ (4)

Also,
$$\ddot{r} = \frac{d}{dt}(\dot{r}) = -\frac{L}{m}\frac{d}{dt}\left(\frac{du}{d\theta}\right)$$

$$= -\frac{L}{m}\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -\frac{L}{m}\frac{d^2u}{d\theta^2}\dot{\theta}$$

$$= -\frac{L^2u^2}{m^2}\frac{d^2u}{d\theta^2} \tag{5}$$

using Eq. (4). Combining Eqs (1), (2), and (5), we get

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2} F\left(\frac{1}{u}\right) \tag{6}$$

which is the required equation.

The equation of the circle of radius a passing through 0 (Fig. Ex 6.14) in polar coordinates is given by the relation

$$r = 2a \cos \theta \tag{7}$$

Putting $u = \frac{1}{r} = \frac{\sec\theta}{2a}$

$$\frac{du}{d\theta} = \frac{\sec\theta \tan\theta}{2a}$$

$$\frac{d^2u}{d\theta^2} = \frac{\sec\theta\,(\sec^2\theta) + (\sec\theta\tan\theta)\tan\theta}{2a}$$

$$= \frac{\sec^3\theta + \sec\theta\tan\theta}{2a}$$

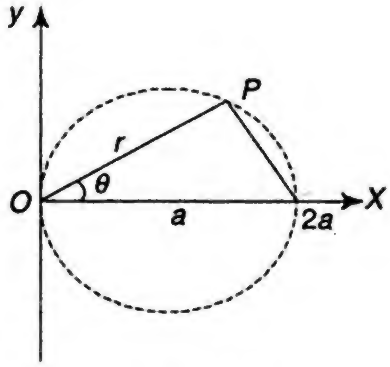


Fig. Ex 6.14

Then, the law of force

$$F\left(\frac{1}{u}\right) = -\frac{L^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$= -\frac{L^2u^2}{m}\left(\frac{\sec^3\theta + \sec\theta\tan^2\theta + \sec\theta}{2a}\right)$$

$$= -\frac{L^2u^2}{2ma} 2\sec^3\theta$$

$$= -\frac{8}{m} L^2a^2u^5$$

or

$$F\left(\frac{1}{r}\right) = -\frac{8L^2a^2}{mr^5} \tag{8}$$

The law of force is of attraction, varying inversely as the fifth power of distance from the origin.

6.15 KEPLER'S LAWS

These laws were formulated by Kepler on the basis of the systematisation of observations of Tycho Brahe on the motion of planets. We can, however, derive these from the discussion presented in this chapter.

In the discussion below, the perturbations caused by other planets on the one under consideration, are ignored for the ease of treatment.

1. The first law states that the planets move in elliptical orbits having the sun at the focus.

This is borne out by Eq. (6.97). It should be remembered that this equation is derived with one of the bodies at the centre of the coordinate system. Therefore, the sun will be at one focus which can be taken as the centre of the coordinate system for the ellipse.

2. Referring to Fig. 6.15, if a planet moves from P to P', in time Δt the area which the radial vector r sweeps is given by $\Delta A = (1/2)\ r(r\Delta\theta)$. Here, we have assumed that $SP'P$ is a triangle, which is true if $\Delta\theta$ is small. This in turn, means that ΔA and Δt are small. The rate at which the area is swept is given by

$$\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt}$$

$$= (1/2) r^2 \frac{d\theta}{dt} = (1/2) r^2 \dot{\theta}$$

But as mentioned earlier,

$$\frac{1}{2} r^2 \dot{\theta} = L/2m = \text{const}$$

Hence $$\frac{dA}{dt} = L/2m = \text{const} \qquad (6.99)$$

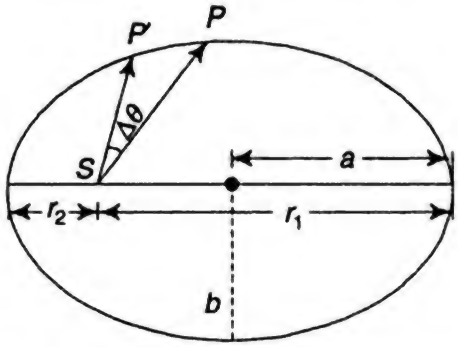


Fig. 6.15 Elliptical trajectory of a planet

This is the statement of Ke]pler's second law, according to which the line joining the sun to a given planet sweeps out equal areas in equal times.

3. Kelper's third law can be obtained by integrating Eq. (6.99) with respect to time over the time period T for one full rotation.

Thus $$\int_0^T \frac{dA}{dt} dt = \int_0^T \frac{L}{2m} dt$$

or $$\int_0^T dA = \frac{LT}{2m} \qquad (6.100)$$

or A = area of the ellipse = $LT/2m$. In terms of the semi-major axis a and semi-minor axis b, the area A is given by

$$A = \pi ab \qquad (6.101)$$

Hence $$T = \frac{2\pi m \, ab}{L}$$

Now the two semi-axes are related through the equation

$$b = a(1 - \epsilon^2)^{1/2} \qquad (6.102)$$

Since the origin is considered at the focus, we have

$$2a = r_{\max} + r_{\min}$$

where, from Eq. (6.91)

$$\frac{1}{r_{\max}} = \frac{mk}{L^2}(1 - \epsilon)$$

and $$\frac{1}{r_{\min}} = \frac{mk}{L^2}(1 + \epsilon) \qquad (6.103)$$

Accordingly, we have

$$2a = r_{\max} + r_{\min} = \frac{L^2}{mk}\left(\frac{1}{1-\epsilon} + \frac{1}{1+\epsilon}\right) = \frac{2L^2}{mk}\left(\frac{1}{1-\epsilon^2}\right) \qquad (6.104)$$

Hence $$T^2 = \frac{(2\pi mab)^2}{L^2} = \frac{(2\pi mab)^2}{mka(1-\epsilon^2)} = \frac{4\pi^2 m^2 a^2 b^2}{mka(1-\epsilon^2)}$$

$$= \frac{4\pi^2 m^2 a^3}{mk} \times \frac{b^2}{a^2(1-\epsilon^2)} = \frac{4\pi^2 m a^3}{k} \qquad (6.105)$$

The last simplification has been obtained through Eq. (6.102). Equation (6.105) is the statement of Kelper's third law according to which the square of the planet's time period (T) divided by the cube of the major axis from the sun is the same for all planets.

From Kepler's laws, one can easily deduce Newton's law of gravitation, as shown below.

Assuming that the orbit of a planet around the sun is a circle of radius R, the centripetal force acting on the planet is given by

$$F = mR\omega^2$$

where $\omega = \frac{2\pi}{T}$, and m is the mass of the planet.

Therefore $$F = mR\left(\frac{2\pi}{T}\right)^2 = \frac{4\pi^2 mR}{T^2} \quad \text{(i)}$$

According to Kepler's third law

$$T^2 \propto R^3$$

or $$T^2 = KR^3$$

where K is a constant. Substituting the value of T in the above Eq. (i), we get

$$F = \frac{4\pi^2 mR}{KR^3} = \frac{4\pi^2 m}{KR^2} = K_1 \frac{m}{R^2}$$

where $K_1 = \frac{4\pi^2}{K}$ = constant. Therefore,

$$F \propto \frac{m}{R^2}$$

The force of attraction acting on a planet is, thus

(i) directly proportional to its mass, and

(ii) inversely proportional to the square of its distance from the sun.

However, since the force of attraction is mutual and directed along the line joining the two bodies, the force must be proportional to the mass of the other body, that is, the sun. Thus

$$F \propto \frac{Mm}{R^2} = \frac{GMm}{R^2}$$

where G is the universal gravitational constants and M is the mass of sun. Newton formulated the law of gravitation for any two bodies of masses M and m.

EXAMPLE 6.15

Show that the velocity of a planet or comet, moving in an elliptic orbit around the sun, at its turning points can be written as

$$v = \frac{Gm_s m_c}{L} (\epsilon \pm 1)$$

Here + and – signs correspond to $r_{\min}$ and $r_{\max}$ respectively and m_s and m_c are the masses of the sun and the planet or comet.

From Eq. (6.97), the total energy, E of any particle moving under a central force is given by

$$\epsilon^2 = 1 + (2L^2/mk^2) E$$

where m is effective mass of the system,

$$m = \mu = \frac{m_s m_c}{m_s + m_c} \simeq m_c$$

because the mass of the planet or comet (m_c) is much smaller than the mass of the sun (m_s). Therefore

$$E = \frac{mk^2}{2L^2}(\epsilon^2 - 1)$$

$$= \frac{m_c m_s^2 m_c^2 G^2}{2L^2}(\epsilon^2 - 1)$$

Now,

$$= \frac{G^2}{2L^2} m_s^2 m_c^3 (\epsilon^2 - 1)$$

Total energy = kinetic energy of moving body + potential energy of the system + centripetal energy

or $$E = \left(\frac{1}{2}\right) m_c v^2 - \frac{Gm_s m_c}{r} + \frac{L^2}{2m_c r^2}$$

$\therefore$ $$\frac{1}{2} m_c v^2 = \frac{G^2 m_s^2 m_c^3}{2L^2}(\epsilon^2 - 1) + \frac{Gm_s m_c}{r} - \frac{L^3}{2m_c r^2}$$

For a planet or a comet L is small and r is large so that the centripetal energy is negligible. Therefore

$$v^2 = Gm_s\left[\frac{Gm_s m_c^2}{L^2}(\epsilon^2 - 1) + \frac{2}{r}\right]$$

or $$v = \frac{Gm_s m_c}{L}\left[(\epsilon^2 - 1) + \left(\frac{L^2}{Gm_s m_c^2}\right)\frac{2}{r}\right]^{1/2}$$

Now at turning points, r becomes $r_{\min}$ and $r_{\max}$, which are given by Eq. (6.103) as

$$\frac{1}{r_{\max}} = \frac{Gm_c^2 m_s}{L^2}(1 - \epsilon)$$

$$\frac{1}{r_{\min}} = \frac{Gm_c^2 m_s}{L^2}(1 + \epsilon)$$

The velocities corresponding to $r_{\max}$ and $r_{\min}$ will be $v_{\min}$ and $v_{\max}$, and therefore,

$$v_{\min} = \frac{Gm_s m_c}{L}\left[(\epsilon^2 - 1) + \frac{L^2}{Gm_s m_c^2}\frac{2}{r_{\max}}\right]^{1/2}$$

$$= \frac{Gm_s m_c}{L}\left[(\epsilon^2 - 1) + \frac{L^2}{Gm_s m_c^2}\frac{2Gm_s m_c^2}{L^2}(1 - \epsilon)\right]^{1/2}$$

$$= \frac{Gm_s m_c}{L}(\epsilon - 1)$$

and $$v_{max} = \frac{Gm_s m_c}{L}\left[(\epsilon^2 - 1) + \frac{L^2}{Gm_s m_c^2}\frac{2}{r_{min}}\right]^{1/2}$$

$$= \frac{Gm_s m_c}{L}\left[(\epsilon^2 - 1) + \frac{L^2}{Gm_s m_c^2}\frac{2Gm_s m_c^2}{L^2}(1+\epsilon)\right]^{1/2}$$

$$= \frac{Gm_s m_c}{L}(\epsilon + 1)$$

EXAMPLE 6.16

The first American satellite Explorer I, which was launched on Jan. 31, 1958, had mass 14 kg, an apogee (maximum distance from the earth's surface) of 2552 km and a perigee (minimum distance from the surface of the earth) of 352 km. Determine the angular momentum, energy and time period of the satellite and also its velocity at turning points. The radius of the earth is 6378 km.

Solution

The satellites move around the earth in elliptical orbits as shown in Fig. 6.16 (which is, however, not drawn to scale).

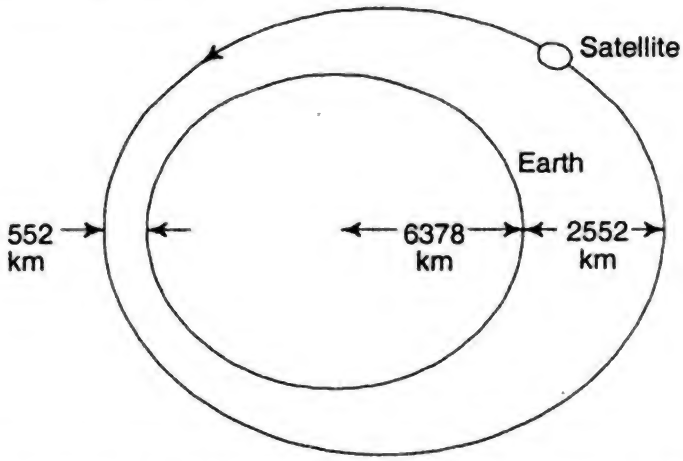


Fig. 6.16 Elliptic orbit of explorer *I* around the earth. Distances shown are in km

Major axis of the orbit

$$= \text{diameter of earth + perigee + apogee}$$

or $$2a = 2 \times 6378 + 352 + 2552 \text{ km}$$
$$= 15660 \text{ km}$$

Therefore,

Semimajor axis, $a = 7830$ km $= 7.83 \times 10^6$m

and $r_{max} = 6378 + 2552 = 8930$ km $= 8.93 \times 10^6$ m

Minor axis, $r_{min} = 6378 + 352 = 6730$ km $= 6.73 \times 10^6$ m

Taking masses of the earth and satellite as m_c and m_s, we have effective mass,

$$\mu \equiv m = \frac{m_c m_s}{m_c + m_s} \simeq m_s$$

because the mass of the satellite $m_s = 14$ kg is much smaller than the mass of the earth, $m_e = 6 \times 10^{24}$ kg. Further,

$$k = Gm_e m_s$$
$$= 6.67 \times 10^{-11} \times 6 \times 10^{24} \times 14 \text{ N m}^2$$
$$= 5.6 \times 10^{15} \text{ N m}^2$$

Now from Eq. (6.104),

$$a = \frac{L^2}{mk}\frac{1}{1-\epsilon^2}$$

which together with Eq. (6.97) yields

$$a = \frac{L^2}{mk}\frac{1}{-2EL^2/mk^2} = -\frac{k}{2E}$$

Therefore, energy of the satellite,

$$E = -\frac{k}{2a} = -\frac{5.6 \times 10^{15}}{15.66 \times 10^6} \text{ J}$$
$$= -3.6 \times 10^8 \text{ J}$$

Next, solving Eq. (6.103) simultaneously for ϵ, we have

$$\epsilon = \frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \frac{(8.93 - 6.73) \times 10^6}{(8.93 + 6.73) \times 10^6}$$
$$= 0.14$$

Again from Eq. (6.104), we have for angular momentum

$$L = [mka(1 - \epsilon^2)]^{1/2}$$
$$= [14 \times 5.6 \times 10^{15} \times 7.83 \times 10^6 (1 - 0.14^2)]^{1/2} \text{ kg m}^2\text{/s}$$
$$= 7.8 \times 10^{11} \text{ kg m}^2\text{/s}$$

Since angular momentum is conserved for motion under the inverse square force, angular momenta at the perigee and apogee will be the same as L. Further at turning points, the velocity is perpendicular to the radius vector. Accordingly, the velocity will be maximum for the perigee and minimum for the apogeee. Thus

$$L = mr_{min} v_{max} = mr_{max} v_{min}$$

Therefore

$$v_{max} = \frac{L}{mr_{min}}$$
$$= \frac{7.8 \times 10^{11}}{14 \times 6.73 \times 10^6} \text{ m/s}$$
$$= 8.28 \times 10^3 \text{ m/s}$$

$$v_{min} = \frac{r_{min}}{r_{max}} v_{max}$$
$$= \frac{6.73 \times 10^6}{8.93 \times 10^6} \times 1.818 \times 10^4 \text{ m/s}$$
$$= 6.24 \times 10^3 \text{ m/s}$$

From Eq. (6.105), the time period of a satellite is given by

$$T = 2\pi \left(\frac{ma^2}{k}\right)^{1/2}$$

$$= 2 \times 3.142 \left[\frac{14 \times (7.83 \times 10^6)^3}{5.6 \times 10^{15}}\right]^{1/2} \text{ s}$$

$$= 6.284\ [1.2 \times 10^6]^{1/2} \text{ s}$$

$$= 114.7 \text{ minutes}$$

6.16 SATELLITE MOTION

We are familiar with natural satellites in our daily life. The earth is the satellite of the sun and the moon that of the earth. It was in 1957 that the first man made or artificial satellite was hurled into space with the help of powerful rockets. The principle of motion is that the gravitational attraction on the satellite due to the larger body provides the centre-attracting centripetal force that is balanced by the fictitious centrifugal force on it, pulling it away from the centre.

If m is the mass of the satellite, M the mass of the earth and r the distance between their centres, the gravitational pull of the earth on the satellite or the centripetal force = $G\, mM/r^2$, where G is the gravitational constant.

The centrifugal force on the satellite outwards = mv^2/r, where v is the velocity of the satellite.

If ω is the angular velocity of the satellite, then $v = r\omega$. Equating the gravitational pull of the earth on the satellite and the centrifigual force on the satellite, we get

$$G\frac{mM}{r^2} = \frac{mv^2}{r} = mr\omega^2$$

$$\omega = \sqrt{\frac{MG}{r^3}}$$

Now $r = R + h$, where R is the radius of the earth and h, the height of the satellite above the surface of the earth. Therefore,

$$\omega = \sqrt{\frac{MG}{(R+h)^3}}$$

If g is the acceleration due to gravity on the earth's surface, then

$$g = \frac{MG}{R^2}$$

or $$MG = gR^2$$

Hence $$\omega = \sqrt{\frac{gR^2}{(R+h)^3}}$$

and the time period

$$T = 2\pi\sqrt{\frac{(R+h)^3}{gR^2}}$$

This result, for the case of the orbit lying close to the earth, i.e. $h < R$ will approximate to

$$T = 2\pi\sqrt{\frac{R}{g}}$$

The orbital speed v is given by

$$\begin{aligned} v &= \omega r \\ &\approx \omega R \\ &= R\sqrt{\frac{g}{R}} \\ &= \sqrt{gR} \end{aligned}$$

Let us estimate the speed by putting the values: $R = 6.4 \times 10^8$ cm and $g = 980$ cm/s^2. Therefore,

$$\begin{aligned} v &= \sqrt{980 \times 6.4 \times 10^8} \\ &= 7.92 \times 10^5 \text{ cm/s} \end{aligned}$$

EXAMPLE 6.17

Satellites always seem to stay over the same point of the earth's surface if their angular velocity is exactly the same as that of the earth. Calculate the height at which a satellite must revolve in its orbit around the earth, concentric and coplanar with the equator.

Solution

Let r be the radius of orbit of such a satellite. Now in order to be at the same point over the earth's surface, its angular velocity is the same as that of the earth.

Now for a satellite,

$$\omega = \sqrt{\frac{GM}{r^3}}$$

Thus

$$r = \left(\frac{GM}{\omega^2}\right)^{1/3}$$

Substituting

$$\begin{aligned} G &= 6.67 \times 10^{-s} \text{ (in cgs units)} \\ M &= 5.98 \times 10^{27} \text{ g} \\ \omega &= 7.28 \times 10^{-5} \text{ rad/s} \end{aligned}$$

We have

$$\begin{aligned} r &= \left[\frac{6.67 \times 10^{-8} \times 5.98 \times 10^{27}}{(7.28 \times 10^{-5})^2}\right]^{1/3} \\ &= 4.23 \times 10^9 \text{ cm} \end{aligned}$$

Now $r = (R + h)$, where R is the radius of the earth and h the height of the satellite.

Thus

$$\begin{aligned} h &= r - R \\ &= 4.23 \times 10^9 - 6.38 \times 10^8 \\ &= 3.59 \times 10^9 \text{ cm} \end{aligned}$$

The satellite will revolve around the earth at a height of 3.59×10^9 cm. Such satellites are called stationary and are used for communication purposes.

QUESTIONS

6.1 Why are gravitational and coulombic forces called inverse square forces? Show that these forces are central and long-range forces.
6.2 What are weak forces? Comment on the fact that these are short-range forces.
6.3 Justify the term 'contact potential' for the weak interaction.
6.4 Why are nuclear interactions called strong interactions? Discuss their spatial dependence to bring out the fact that these are short-range forces.
6.5 What makes it necessary to introduce dimensionless coupling constants for the comparison of various forces of nature?
6.6 Compare the space dependence of the four forces of nature.
6.7 Bring out the significance of studying inverse square law forces.
6.8 Compare the intrinsic strengths of the four forces in nature.
6.9 Prove that the centre of mass of a two-body system interacting through central forces always has constant velocity.
6.10 Define the reduced mass of a two-body system. Does it depend on the nature of the forces acting between two bodies?
6.11 Why do we reduce a two-body problem to a one-body problem by introducing the concept of effective mass?
6.12 Discuss the motion of reduced mass under the influence of the inverse square law force.
6.13 Show that energy is conserved in equivalent one-body motion under the influence of the inverse square force.
6.14 Starting from the expression for radial acceleration in planar motion, obtain the relationship between r and t for a two-body system.
6.15 Show graphically the variation of the inverse square law potential energy, centripetal energy and the sum of the two, with distance between two bodies. Use these curves to discuss the nature of motion under inverse square forces.
6.16 What is effective potential energy V'? Under what conditions is it positive and when can it be negative?
6.17 'The gravitational force is attractive in nature, but still, the motion of a particle under this force can be unbounded.' Discuss.
6.18 Starting from the expressions for r and θ [Eqs (6.78) and (6.79)], obtain the equation for the trajectory of the particle moving under inverse square law force.
6.19 What are the turning points? Find their positions in terms of the total energy, inverse square law force constant and angular momentum of the two body system having effective mass m.
6.20 Show that the shape of the trajectory of a particle moving under the inverse square law force, depends on the relationship between total energy and angular momentum.
6.21 Show that the energy of a particle moving in an orbit of eccentricity ϵ, is given by

$$E = \frac{G^2 m_1^3 m_2^3}{2(m_1 + m_2) L^2} (\epsilon^2 - 1)$$

[*Hint:* Simplify Eq. (6.97).]
6.22 Discuss the motion of a particle having total energy greater than zero and moving under the influence of an inverse square law force.

6.23 A particle is in the bound state with respect to another particle exerting inverse square force on it. Discuss the nature of its motion. Under what conditions will the trajectory be circular?

6.24 Starting from Eq. (6.83), show that radius of a circular orbit under inverse square force is given by L^2/km.

6.25 The earth's gravitational force acting on an artificial satellite of mass m is $-(GmM/r^2)$. Find the necessary relation between its energy and angular momentum so that its orbit is circular.

6.26 Can a particle having total energy E equal to zero move under the influence of an inverse square law force? If yes, discuss the nature of its motion. If not, suggest some means which can help in imparting motion to it.

6.27 State Kepler's laws of planetary motion and prove these by treating the motion of planets as one body equivalent problem.

6.28 Show that Kepler's second law of motion is a direct consequence of conservation of angular momentum under central forces.

6.29 How does Kepler's third law of planetary motion provide evidence that the force between a planet and sun obeys inverse square law?

6.30 Show that for an elliptical orbit,

$$\varepsilon = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}$$

6.31 In the discussion of motion of a planet around the sun, one comes across the terms 'apihelion' and 'perihelion', and in the motion of satellites of the earth these terms give place to 'epogee' and 'perigee'. Correlate these terms with the contents of this chapter and show that their magnitudes depend on the total energy and angular momentum of the system under discussion.

6.32 A light particle of mass m is moving in an elliptical orbit under the influence of force $= -(A/r^2)$ such that the centre of attraction is a focus of the ellipse. Show that the period of this motion will be $T = (4\pi^2\ ma^3/A)^{1/2}$, where a is the semi-major axis of the orbit.

6.33 According to Kepler's third law, the period of revolution of a planet around the sun depends on semi-major axis of its orbit as $a^{3/2}$ and is independent of its mass. Comment on this in the light of Eq. (6.105).
[*Hint:* Mass of the sun = 2.0×10^{30} kg.]
Mass of the heaviest planet, Jupiter = 1.90×10^{27} kg.

6.34 Show that the semi-major axis a of an elliptical orbit is related to the energy of a planet through

$$a = k/(-2E)$$

where $\quad k = Gm_1m_2$.

[*Hint:* Use Eqs (6.97) and (6.104) of the text.]

6.35 Enunciate Kepler's laws and show how they may be deduced from Newton's law of gravitation.

6.36 Employing the first two Kepler's laws of planetary motion and Newton's laws of motion, show that the force acting on a planet is directed toward the sun, is directly proportional to the product of the masses of the sun and the planet, and inversely proportional to the square of its distance from the sun.

6.37 Explain the terms: gravitational potential and gravitational field. Obtain expressions for the gravitational potential and gravitational field at a point (i) inside and (ii) outside a hollow spherical shell.

6.38 Calculate the gravitational potential and gravitational field due to a sphere at a point (i) outside, (ii) on the surface, and (iii) inside the sphere. Show that the potential inside the hollow sphere is zero.

6.39 (i) Show that the potential at the centre of the sphere is one and a half times that on its surface and (ii) inside the sphere, it is proportional to the distance from the centre of the sphere.

6.40 Define an equipotential surface and show that the field has no component along the surface and is perpendicular to it at all the points.

6.41 Show that the escape velocity from the surface of earth is $\sqrt{2}$ times the velocity of projection of an artificial satellite orbiting close to the earth.

6.42 Explain the term 'gravitational self-energy' of a body or a system of particles. Show that the gravitational self-energy of a system of n particles, each of mass m, at an average distance r from each other is given by $U_s = \dfrac{1}{2}\, Gn(n-1)\,\dfrac{m^2}{r}$.

6.43 Calculate the electrostatic self-energy of a charge q spread uniformly over the surface of a sphere of radius r.

6.44 Define the classical radius of an electron and show that it is equal to 2.81×10^{-13} cm.

6.45 Show that gravitational field due to earth is equal in magnitude and direction due to gravity.

6.46 What is the gravitational constant? What are its dimensions? Describe in detail Boys method for its determination.

PROBLEMS

6.1 A particle of mass m_1 is approaching another particle of mass m_2 located at the origin of the coordinate system. Initially when m_1 is at infinity, it has velocity v_1 along a line separated by distance d from m_2. The particle is deflected towards m_2 due to gravitational attraction and passes it at minimum distance b. Determine the value of b in terms of other parameters, by treating the problem as a reduced one-body system.
[*Hint:* Find angular momentum and energy for $r = \infty$ as well as $r = b$ and apply the laws of their conservation.]

$$Ans.\ b = \frac{1}{v_1^2}\left\{\left[(m_1 + m_2)^2\, G^2 + d^2 v_1^2\right]^{1/2} - (m_1 + m_2)G\right\}$$

6.2 The paths of two particles moving under the action of central forces are given by
1. $r(1 + 0.1 \cos \theta) = \text{const}\ (A)$,
2. $r\dot{\theta} = \text{const}$.

Find the corresponding force laws. *Ans.* 1. $F(r) = -L^2/mAr^2$
2. $F(r) = -L^2/mr^3$

6.3 Find the total energy of the earth in its orbit around the sun assuming that mass of the sun is 2×10^{30} kg and that of earth is 6×10^{24} kg. The average radius of the earth's orbit is 1.5×10^8 km. *Ans.* 2.67×10^{33} J

6.4 The central force part of the nuclear interaction can be written as

$$U(r) = -K\frac{e^{-ar}}{r}$$

where K and a are positive constants, and $U(r)$ is called the Yukawa potential. Derive an expression for the force corresponding to this potential and compare it with the

inverse square law. Discuss the nature of motion of a particle of mass m moving under the influence of such a force. Under what conditions are circular orbits possible?

Ans. $\mathbf{F} = -Ke^{-ar}\left[\dfrac{a}{r}+\dfrac{1}{r^2}\right]\hat{\mathbf{r}}$ For circular orbit

$$L^2 = Kmr_c^2 \exp(-ar_c)\left(a+\frac{1}{r_c}\right) \text{ and}$$

$$E = \frac{-K\exp(-ax_c)(a+1/x_c)}{2}$$

6.5 A particle moves in a circular orbit under the influence of attractive inverse square force, $F(r) = -K/r^2$. Suddenly, K is reduced to one-fourth its original value. Show that the trajectory of the particle will become hyperbolic.

6.6 Depending on its total energy, a particle can move in a parabolic ($E = 0$) or circular ($E = -mk^2/2L^2$) orbit under an attractive inverse square force. Show that for the same value of angular momentum, the perihelion distance (r_{min}) of the parabolic path is half the radius of the circular path.

6.7 In the text it has been shown that the angular momentum is conserved in motion under a central force. The earth is moving around the sun under a gravitational force and its orbit has semi-major axis of 1.496×10^8 km. When the earth passes closest to the sun (i.e. it is at its perihelion), its distance is 1.47×10^3 km and its orbital velocity is 0.303 km s^{-1}. Find the eccentricity of the earth's trajectory, its velocity at the aphelion and also the angular velocities at the two positions.

Ans. 0.017, 0.293 km s^{-1}, 2.06×10^{-9} rad s^{-1}, 1.93×10^{-9} rad s^{-1}

6.8 The planet Mars has an aphelion (maximum) distance of 2.485×10^5 km and perihelion (minimum) distance of 2.06×10^8 km with respect to the sun, whose own radius is nearly 7×10^6 km. Determine the eccentricity of the orbit and also the values of its angular momentum and energy taking the mass of sun = 2×10^{30} kg, mass of Mars = 6.5×10^{23} kg and G = 6.67×10^{-11} Nm^2/kg^2.

Ans. 0.093, 3.63×10^{39} kg m^2/s, -1.85×10^{32} J

6.9 An artificial satellite is revolving around the earth in an orbit with eccentricity 0.90 and period 48.6 h. Determine the apogee and perigee of the satellite from the surface of the earth. Given: mass of the earth 6×10^{24} kg, radius of the earth 6378 km, and $G = 6.67 \times 10^{-11}$ Nm^2/kg^2 *Ans.* 122243 km, 391 km

6.10 The earth revolves around the sun in an elliptic orbit with eccentricity 0.017 and semi-major axis of 1.496×10^8 km in 365.26 days. On the other hand, the moon revolves around the earth in an orbit with eccentricity 0.0549 and semi-major axis as 3.844×10^5 km in 27.32 days. Find the mass of the sun, if the mass of the earth is given to be 6×10^{24} kg. Also, determine r_{max} and r_{min} for the two orbits.

Ans. $m_s = 2.08 \times 10^{30}$ kg,

$(r_{max})_E = 1.522 \times 10^8$ km, $(r_{min})_E = 1.471 \times 10^8$ km,

$(r_{max})_M = 4.055 \times 10^5$ km, $(r_{min})_M = 3.633 \times 10^5$ km

6.11 The periodic times for Mercury, Mars and Earth are 87.97, 687.05 and 365.26 sidereal days. Find the major axes of the orbits of Mercury and Mars in terms of that of the earth. *Ans.* 0.387, 1.523

6.12 Obtain the expression for force on a particle of mass m, for which the equation for trajectory is given by $r = a \sin \theta$.

Ans. $F(r) = \dfrac{-2a^2 l^2}{mr^5}$

6.13 Show that N charged particles each carrying charge q esu and separated from each other of an average distance of r cm have an electrostatic potential energy given by $-\frac{1}{2}N(N-1)\frac{q^2}{r}$ ergs.

6.14 If the electric field everywhere within a uniformly charged spherical shell is zero, show without using the differential form of Gauss's law that the electrostatic law of force is the inverse square of the distance.

6.15 Calculate the rate of energy radiated by the sun in contracting at the rate of 25 km per year in radius without reducing effectively the mass. Given that mass of sun and the radius of sun $M_s = 2 \times 10^{33}$ gm and the radius of sun $R_s = 6.96 \times 10^{10}$ cm.
Ans. 5.95×10^{30} cal/min.

6.16 The energy received at the earth's surface is 1.7 cal per square cm per min. Calculate the rate of reduction of the sun's radius, assuming that the whole of self-energy lost due to contraction is radiated by the sun. Given the sun-earth distance = 1.5×10^{13} cm.
Ans. 25m/year

6.17 Calculate the electrostatic self-energy of a (i) conducting and (ii) non-conducting sphere of radius 20 cm carrying a charge of 200 esu.
Hint: In the case of a conducting sphere, the charge resides only on its surface, whereas in the case of a non-conducting sphere, the charge is distributed uniformly over its volume.
Ans. (i) 1,000 erg; (ii) 1,200 erg

6.18 The orbital velocity of sun about the centre of our galaxy is 3×10^7 cm/s and its distance from the axis of the galaxy is approximately 3×10^{22} cm. Estimate the mass of galaxy. Given that $G = 6.67 \times 10^{-8}$ cgs units.
Ans. 4.05×10^{44} gm

6.19 Calculate the mass of the earth from the following data:
Radius of earth = 6×10^8 cm; $G = 6.6 \times 10^{-8}$ cgs units; $g = 980$ cm/s^2.
Ans. 6×10^{27} g

6.20 Estimate the mass of sun assuming the orbit of the earth around the sun to be a circle. The distance between the sun and earth is 1.49×10^{13} cm and $G = 6.66 \times 10^{-8}$ cgs units. Take the year to consist of 365 days.
Ans. 2.0×10^{30} kg

6.21 The maximum and minimum distances of a comet from the sun are 2×10^{12} m and 8×10^{10} m, respectively. If the speed of the comet at the nearest point is 60 km/s, calculate the speed at the farthest point.
Ans. 2.4 km/s

7

Elastic and Inelastic Collisions

7.1 INTRODUCTION

One of the main themes of research activities in physics has been to understand the characteristics of various interactions and forces operating in nature at the macroscopic as well as the microscopic level. One method of investigating the interactions is to observe the motion of a particle or object in the neighbourhood of another particle, where interaction involves a particular type of force. The experiments show that such interactions affect the trajectory of the particle. It has already been seen in Chapter 6 that a particle or a body moving under the influence of potential V due to another body describes an elliptical orbit if the total energy E is negative and a hyperbolic orbit if the total energy is positive (Fig. 6.14). The former situation corresponds to the bound state of the system, whereas the latter corresponds to the unbound state, which essentially describes the scattering or collision of the particle in the field of the target body. In physics, collision in general does not necessarily imply physical contact between two particles or systems (the way it happens in collisions of two marbles). A force can come into play between the two particles or systems for a finite time, with or without any direct contact, which results in a measurable change in their relative motions. Some of the examples of such interaction are: the deflection in the path of a comet on passing near the solar system; elliptical path of planets, deflection of a charged particle on passing through an electric or a magnetic field, redistribution of the intensity of neutron beam on passing through a magnetised material, scattering of protons from protons, etc.

Historically, the interest in the systematic study of collisions dates back to 1668 when the Royal Society of London circulated a request for work on clarification of the collision phenomena. The comments submitted by Huygens, Wallis and Wren constituted the basis of what was ultimately developed as classical collision theory. Interestingly, the concepts introduced as the basic laws by the earlier workers have stood the test of time and found place even in the modern quantum theory of collision of microscopic particles. The solution of the collision problem essentially involves the conservation of energy and momentum, and both these laws are equally valid in classical and quantum physics. In this way the details of the mechanism of

scattering are not invoked and hence the theory developed in the framework of classical physics can also be used for understanding collision phenomena in the quantum world, such as scattering of α-particles, protons, neutrons, etc. from the nucleus or other particles.

The collision between two particles or systems can give rise to two alternative situations. In some cases the nature of the particles or systems taking part in the collision process is not changed. Such collisions are referred to as scattering of particles. On the other hand, many times it may happen that the final particles or systems are different from the initial particles or systems. These collisions are termed as reactions. The general pictorial representation of collision is given in Fig. 7.1, where m_1 and m_2 are the masses of two particles before collision and m_3 and m_4 are the corresponding quantities after collision.

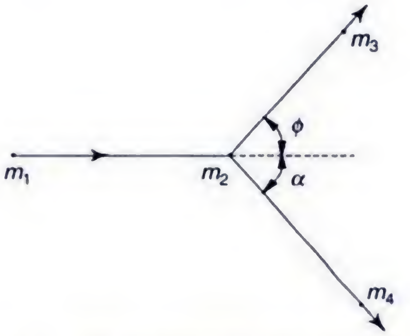


Fig. 7.1 Diagram depicting the process of collision

When an experiment on collision is carried out, the natural curiosity of the experimenter is to get information about the following two aspects:

1. The velocities, linear momenta and energies of the particles or systems before and after the collision, and hence the changes brought about in these.
2. The probability of collision with particular specifications, e.g. the probability of scattering in a given direction.

Before discussing these problems, we digress to understand the meaning of a few terms.

First, we should be familiar with what are called (i) laboratory systems, and (ii) centre-of-mass systems. A laboratory system (or lab system) means that the coordinate system is fixed in the laboratory. Because the observer is stationary with respect to the laboratory, a lab system implies that the coordinate system is stationary with respect to the observer in the laboratory.

On the other hand, a centre-of-mass system (CM system) means that the coordinate system is fixed with respect to the centre of mass. In general, the centre of mass [see Eq. (6.67)] will be moving with a certain constant velocity with respect to the laboratory or observer. Because of the constant velocity of the centre-of-mass system with respect to the lab system, the various quantities in the two systems are connected with each other through Galilean transformation* when the relative ve-

* The Galilean transformations are discussed in detail in Chapter 10.

locity is small as compared to the speed of light. The frame of reference in the lab system is stationary with respect to the observer and no kinetic energy is associated with it. In the centre-of-mass system, the frame of reference itself is moving with respect to the observer and kinetic energy is associated with it. Consequently, the value of the kinetic energy and hence the total energy will be different in the lab system and centre of mass system.

The classification of the processes of collision can be made as follows:

(a) Elastic Scattering

If the collisions in which final particles or systems are the same as the initial particles or systems, and the sum of the kinetic energies is the same after the collision as before it, the collisions are called elastic and are referred to as elastic scattering as illustrated in Fig. 7.2.

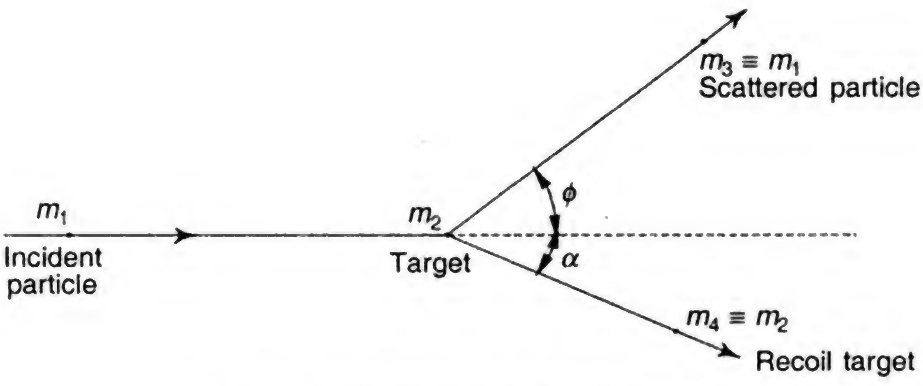


Fig. 7.2 Illustration of elastic scattering

In this case m_1 and m_2 are the masses of the incident and target particles before scattering and m_3 and m_4 the masses after scattering, which are identically the same as m_1 and m_2 respectively, i.e. $m_3 = m_1$ and $m_4 = m_2$. The incident particle of mass m_1 is scattered through an angle φ and is called the scattered particle. The target or the second particle of mass m_2 goes at an angle α with the direction of the incident particle and is called the recoil particle.

(b) Inelastic Scattering

This term is used for those collisions in which the initial and final particles or systems are identical but the total kinetic energy is either decreased or increased as a consequence of the collision. This can result in a change of the potential energy of the particles and also in the production of some other form of energy, e.g. heat, sound, etc. in the macroscopic cases and excitation, light or gamma rays, etc. in microscopic systems, such as atoms and nuclei. The case of inelastic scattering is illustrated in Fig. 7.3. In this case, scattered and recoil particles are indicated by asterisks.

(c) Reactions

Those collisions in which the outgoing particles are entirely different from the initial particles are referred to as reactions. As shown in Fig. 7.4, m_3 and m_4 are different from m_1 and m_2. In such a case m_1 and m_2 are called reactants and m_3 and

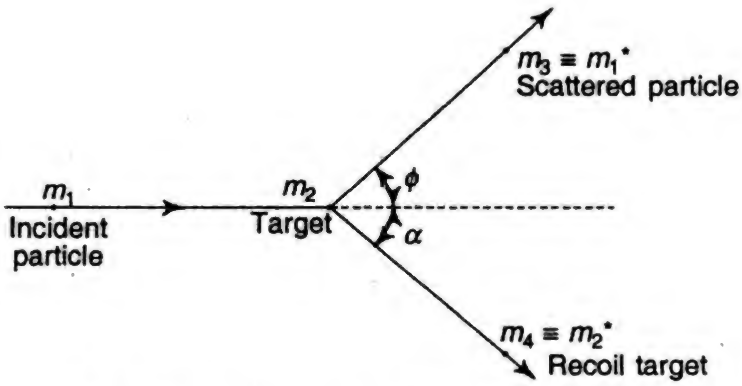


Fig. 7.3 Representation of inelastic scattering

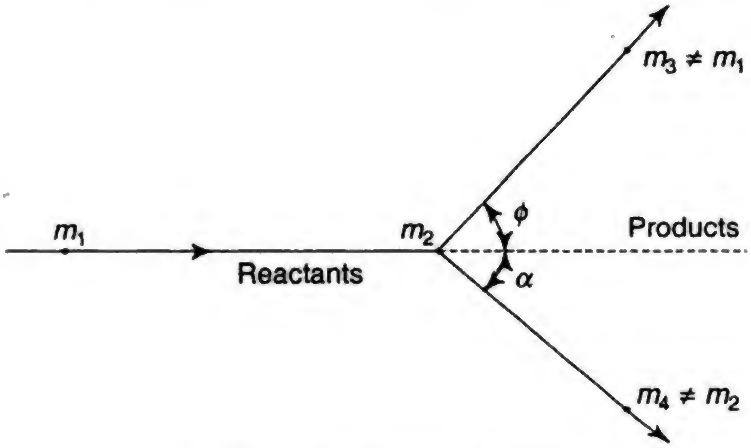


Fig. 7.4 Illustration of a reaction

m_4 are called the products. The particle represented by m_3 is specifically called the emitted particle and m_4 is called the residual particle.

7.2 CONSERVATION LAWS

There are some conservation laws which hold good in all the three cases of collisions mentioned above. In each case the interacting system is isolated so that the forces or torques operating at the time of interaction are only internal to the system and are not applied from outside. Hence the laws of conservation of linear momentum (corresponding to no external force) and conservation of total angular momentum (implying no external torque) hold good in all the three cases.

In the case of elastic scattering there is no change in both kinetic as well as potential energies, hence the sum of the potential and kinetic energies is conserved in elastic scattering. However, in inelastic scattering the sum of kinetic energies is changed along with a change in total potential energy resulting in the production of some other form of energy. In the case of reactions even internal structures are also altered. Consequently, the sum of kinetic and potential energies is not conserved in both inelastic scattering and reactions. Nevertheless, it may be mentioned that in both these cases, the total energy including the kinetic, potential and other energies is conserved.

The implications of the law of the conservation of angular momentum in collision deserve more attention as discussed below.

A body can have angular momentum due to two types of motion: (i) rotation around an axis of the body; and (ii) motion along a curvilinear path around an axis outside the body. Here we recall that the angular momentum **L** around an axis outside the body is given by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{7.1}$$

Figure 7.5 shows **r** and **p** in the lab system where **r** is the radius vector from the point of reference and **p** is the linear momentum of the moving body at a given time. The angular momentum **L** due to the curvilinear motion of a particle around a point outside the moving body is called the *orbital angular momentum.*

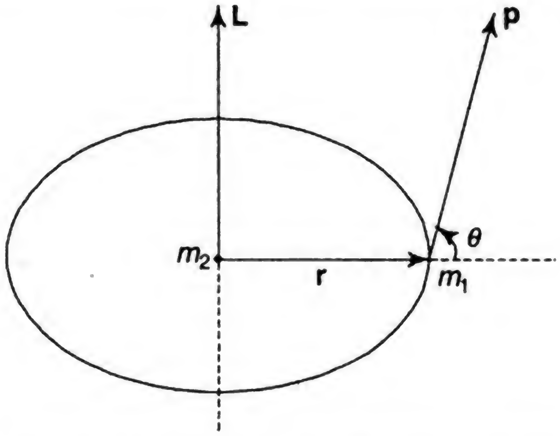


Fig. 7.5 The relationship between **L**, **r** and **p**

Sometimes, the body under consideration may posses an intrinsic angular momentum **S** (called spin) due to its rotation around an axis passing through itself. A good example of this is the rotation of the earth around its axis passing through itself through north and south poles or that of a symmetric top rotating about the axis of symmetry. Similarly, particles such as electrons, protons and neutrons possess characteristic spin value which, of course, has no classical analogue. This type of angular momentum for a macroscopic rotating body is given by

$$\mathbf{S} = I\boldsymbol{\omega} \tag{7.2}$$

where I is the moment of inertia of the body around the axis of rotation and $\boldsymbol{\omega}$ is its angular velocity.

If a body has both intrinsic and orbital angular momenta, then the total angular momentum **J** of the body will be given by

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \tag{7.3}$$

The earth revolving around the sun possesses both orbital as well as intrinsic angular momenta. Similarly, an electron, a proton, or a neutron, on being scattered from a nucleus, also possesses both types of angular momenta. However, an alpha particle does not have any intrinsic angular momentum and hence will possess only orbital angular momentum.

The total angular momentum **J** is always conserved for the whole system, whatever may be the forces operating between the interacting particles in the system. This is so because there is no external torque on the system as a whole, and hence there is no change in the total angular momentum. The space is isotropic and

homogeneous; hence unless there is an external torque, the space does not contribute any change in the angular momentum.

When two bodies interact, they exert equal and opposite torques on each other so that the total torque is zero, i.e.

$$\mathbf{\Gamma}_{12} + \mathbf{\Gamma}_{21} = \mathbf{\Gamma}'_{12} + \mathbf{\Gamma}'_{21} = 0 \tag{7.4}$$

where $\mathbf{\Gamma}_{12}$ is the torque due to particle 1 on 2 and $\mathbf{\Gamma}_{21}$ is the torque due to particle 2 on 1, before the interaction. Similarly, $\mathbf{\Gamma}'_{12}$ and $\mathbf{\Gamma}'_{21}$ are the torques after scattering. Equation (7.4) should be compared with Eq. (4.50).

If the forces of the interaction are central, then as shown in Chapter 4, the orbital angular momentum remains constant. In that case each of the four torques in the above equation are zero and hence the orbital angular momentum remains same before and after scattering. And what is the total angular momentum $\mathbf{J}$ of the system?

For particle 1,

$$\mathbf{J}_1 = \mathbf{L}_1 + \mathbf{S}_1 \tag{7.5}$$

and for particle 2,

$$\mathbf{J}_2 = \mathbf{L}_2 + \mathbf{S}_2 \tag{7.6}$$

The total angular momentum $\mathbf{J}$ of both the particles will, therefore, be

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 = (\mathbf{L}_1 + \mathbf{L}_2) + (\mathbf{S}_1 + \mathbf{S}_2) \tag{7.7}$$

When the forces of interaction are central, $\mathbf{L}_1 + \mathbf{L}_2$ is constant. $\mathbf{J}$ is also constant, because there are no external torques. Hence $\mathbf{S}_1 + \mathbf{S}_2$ remains constant, i.e. the total intrinsic angular momentum of the particles does not change.

On the other hand if the forces of interaction are non-central, then orbital angular momenta are not conserved and hence $\mathbf{L}_1 + \mathbf{L}_2$ is not constant. However, $\mathbf{J}$ is still constant because the space is isotropic. This means that $\mathbf{S}_1$ and $\mathbf{S}_2$ will change in such manner that $\mathbf{J}$ remains constant.

It may be emphasised that these comments hold good for all three categories of collisions.

EXAMPLE 7.1

A particle of mass m_1 and moving with velocity $\mathbf{u}_1$ is elastically scattered from another particle of mass m_2 at rest. After the collision, the two particles move in opposite directions with the same speed. Find the mass of the target in terms of the mass of the incident particle.

Solution

Momentum of the incident particle = $m_1\mathbf{u}_1$.

Kinetic energy of the incident particle = $(1/2)\, m_1 u_1^2$

Before collision, the target is at rest so that its momentum and kinetic energy are both zero. After the collision the particle is scattered with velocity $\mathbf{v}_1$ and the target recoils with velocity $\mathbf{v}_2$ such that these are equal and opposite, i.e.

$$\mathbf{v}_2 = -\,\mathbf{v}_1$$

Since the collision is elastic, the linear momentum and kinetic energy will be conserved. First of these gives

$$\begin{aligned} m_1\mathbf{u}_1 &= m_1\mathbf{v}_1 + m_2\mathbf{v}_2 \\ &= (m_1 - m_2)\,\mathbf{v}_1 \end{aligned}$$

For a heavy target ($m_2 > m_1$), the right-hand side is negative so that $\mathbf{v}_1$ is opposite to $\mathbf{u}_1$, i.e. the particle is scattered backward while the target moves in the forward direction with velocity $\mathbf{v}_2$. Now taking only magnitudes, we have

$$|\mathbf{v}_1| = \frac{m_1}{m_1 - m_2} |\mathbf{u}_1|$$

The conservation of kinetic energy yields

$$(1/2)\, m_1 u_1^2 = (1/2) m_1 v_1^2 + (1/2)\, m_2 v_1^2$$

or
$$m_1 u_1^2 = (m_1 + m_2)\, v_1^2$$

or
$$m_1 u_1^2 = (m_1 + m_2) \frac{m_1^2}{(m_1 - m_2)^2} u_1^2$$

or
$$(m_1 - m_2)^2 = m_1(m_1 + m_2)$$

or
$$m_1^2 + m_2^2 - 2m_1 m_2 = m_1^2 + m_1 m_2$$

or
$$m_2^2 = 3m_1 m_2$$

$$m_2 = 3m_1$$

i.e. the target is three times heavier than the incident particle.

EXAMPLE 7.2

In his work on the discovery of neutron, James Chadwick made use of the following information about scattering of these unknown neutral particles. When the particles were bombarded on the hydrogenous material (paraffin), the recoil protons had maximum velocity 3.3×10^7 ms^{-1} and when these were bombarded on the nitrogenous material (para-cyanogen), the maximum velocity of the recoil nitrogen nuclei was 4.7×10^6 ms^{-1}. Determine the mass of neutron assuming the collisions to be elastic, hydrogen and nitrogen nuclei to be at rest before collision and taking mass of proton as 1.66×10^{-27} kg. Also, find the initial velocity of the neutrons using the value of the mass of the neutron as determined in this problem.

Solution

From the simultaneous application of laws of conservation of momentum and energy in elastic collisions, it can be shown that when a particle collides with a target at rest, the velocity of the target-particle after collision is greatest if it moves in the same direction as that of the incident particle. Such a collision is called head-on collision. For convenience, we will not use vector notation.

Suppose the neutrons have mass m_n and velocity v_n. When these collide elastically with a particle of mass m_2 at rest, the velocity imparted to m_2 will be maximum for head-on collision. Let it be v_2. Then conservation of momentum gives

$$m_n v_n = m_n v_1 + m_2 v_2$$

where v_1 is velocity of the neutron after head-on collision. From conservation of energy, we have,

$$\frac{1}{2} m_n v_n^2 = \frac{1}{2} m_n {v_1}^2 + \frac{1}{2} m_2 v_2^2$$

In order to eliminate v_1 from these equations, we find v_1 from the first equation and substitute in the second; and get

$$v_2 = \frac{2m_n v_n}{m_2 + m_n}$$

When neutrons of the same velocity v_n are scattered from the hydrogenous and nitrogenous materials, the maximum velocities v_H and v_N imparted to hydrogen and nitrogen nuclei respectively will be

$$v_H = \frac{2m_n v_n}{m_H + m_n}$$

and
$$v_N = \frac{2m_n v_n}{m_N + m_n}$$

Dividing these, we have

$$\frac{v_H}{v_N} = \frac{m_N + m_n}{m_H + m_n}$$

On simplification, it gives

$$m_n = \frac{m_N v_N + m_H v_H}{v_H + v_n}$$

Now, the mass of the nitrogen nucleus is approximately 14 times that of proton, so that we take $m_N = 14m_H$. Using this fact and the given values of m_H, v_H and v_N, we get

$$m_n = \frac{14 \times 4.7 \times 10^6 - 3.3 \times 10^7}{3.3 \times 10^7 - 4.7 \times 10^6} \times 1.66 \times 10^{-27}$$

$$= 1.16 \times 1.66 \times 10^{-27}\ \text{kg} = 1.92 \times 10^{-27}\ \text{kg}$$

From the expression for v_H above, we also see that

$$v_n = \frac{v_H (m_H + m_n)}{2m_n}$$

Putting the values of various quantities, we have

$$v_n = \frac{3.3 \times 10^7\ (1.66 + 1.92) \times 10^{-27}}{2 \times 1.92 \times 10^{-27}}$$

$$= 3.1 \times 10^7\ \text{m/s}$$

7.3 LABORATORY AND CENTRE-OF-MASS SYSTEMS

(a) Laboratory System

When scattering experiments are performed in the laboratory, one of the particles (target) of mass m_2 may be taken at rest ($\mathbf{u}_2 = 0$) and the other particle of mass m_1 approaches it with velocity $\mathbf{u}_1$. The frame of reference with origin at m_2 defines the laboratory frame (Fig. 7.6). After collision, the two particles have velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ and φ and α are the angles through which m_1 and m_2 are deflected respectively with respect to the initial direction of motion of m_1. The term T_0 denotes the total kinetic energy of the two particles (i.e. the whole system) before scattering. We

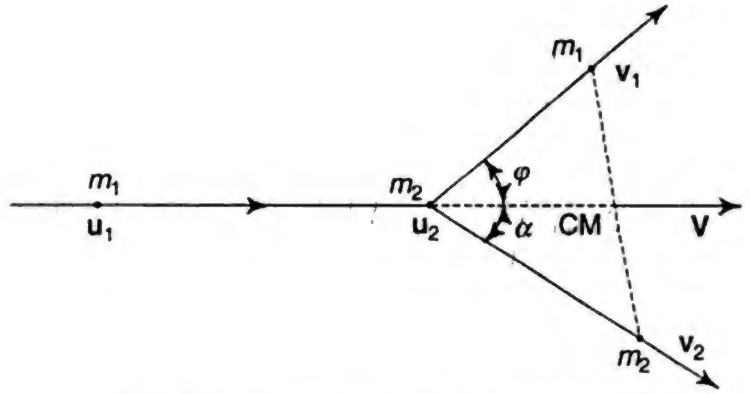


Fig. 7.6 Scattering in the laboratory system

know that E, the total energy of the system is conserved during elastic scattering. Therefore $E = T_0 + U$ is constant, where U is the potential energy. As $U \to 0$, at large distance, $E = T_0$ = constant, in the case of elastic scattering.

Let T_1, T_2 be the kinetic energies of m_1 and m_2 respectively, before scattering, and t_1 and t_2 kinetic energies of m_1 and m_2, respectively after scattering. Then

1. For elastic scattering,
$$T_0 = T_1 + T_2 = t_1 + t_2$$
2. For inelastic scattering,
$$(T_1 + T_2) > (t_1 + t_2)$$
3. For a reaction, $(T_1 + T_2)$ can be less or greater than $(t_1 + t_2)$.

(b) Centre-of-Mass System

The discussion of collisions becomes much simpler if it is referred to the centre-of-mass system in which the centre of mass of the colliding particles is at rest. In the centre-of-mass system, we generally denote,

$\mathbf{u}_1'$ and $\mathbf{u}_2'$ as the initial velocities of m_1 and m_2 respectively

$\mathbf{v}_1'$ and $\mathbf{v}_2'$ as the final velocities. Since the origin of the coordinate system is now at the centre of mass, which corresponds to $m_1\mathbf{r}_1 = -m_2\mathbf{r}_2$ as shown in Eq. (6.8), the directions of motion of the two particles will be opposite to each other before and after the collision. Therefore, the angles through which the two masses get scattered are the same (Fig. 7.7).

θ as the angle of scattering in the centre-of-mass system

T_1' and T_2' as the kinetic energies of the two particles before scattering

$T_0' = T_1' + T_2'$ as the total kinetic energy of the two particles before scattering

t_1' and t_2' as the kinetic energies of the particles after scattering.

We will use these terms in subsequent discussions.

7.3.1 Relationship between Displacements and Velocities

We can now draw the following conclusions about the relationships between various quantities in the two frames of reference.

1. The mass m_2 is stationary in the lab system, i.e. $\mathbf{u}_2 = 0$. Therefore, velocity with respect to lab system means relative velocity with respect to m_2.

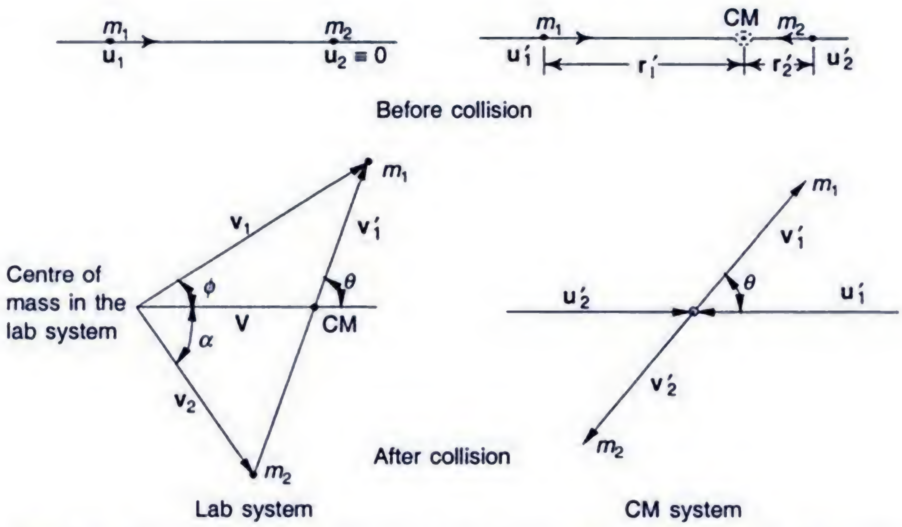


Fig. 7.7 The process of collision as observed in the lab as well as in the CM system

2. If $\mathbf{v}$ is the velocity of the centre of mass with respect to m_2 which is stationary in the lab system, then the relative velocity $\mathbf{u}_2'$ of m_2 with respect to the centre-of-mass system will be $-\mathbf{v}$.

3. As explained in the previous chapter, the velocity of the centre of mass with respect to the lab system is constant. This means that both the systems—centre-of-mass and lab systems—can be taken as inertial frames of reference and the transformations of various quantities from one system to the other system are Galilean as long as the velocities involved are small.

4. The Galilean transformation between the two systems will be given as

Lab System		*Centre-of-mass system*	
$\mathbf{r}_1$	=	$\mathbf{r}_1' + \mathbf{v}t$	
$\mathbf{r}_2$	=	$\mathbf{r}_2' + \mathbf{v}t$	(7.8)

where t is the time at which the measurements are made, assuming that at $t = 0$, the origins of the two systems are at the same point. Obviously,

$$\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_1' - \mathbf{r}_2' \tag{7.9}$$

i.e. the separation of the two particles is the same in the two frames. Again, we should realise that

$$\mathbf{u}_1 = \mathbf{u}_1' + \mathbf{v} \tag{7.10}$$

$$\mathbf{u}_2 = \mathbf{u}_2' + \mathbf{v} \tag{7.11}$$

so that $$\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{u}_1' - \mathbf{u}_2'$$

Further $\mathbf{u}_2 = 0$ because the particle with mass m_2 is stationary in the lab system. Therefore, from Eq. (7.11)

$$\mathbf{u}_2' + \mathbf{v} = 0$$

or $$\mathbf{u}_2' = -\mathbf{v} \tag{7.12}$$

which is the same thing as mentioned earlier. Hence

$$\mathbf{u}_1 = \mathbf{u}_1' - \mathbf{u}_2' \tag{7.13}$$

Similarly, $\quad \mathbf{v}_1 = \mathbf{v}_1' + \mathbf{v}$

$$\mathbf{v}_2 = \mathbf{v}_2' + \mathbf{v} \tag{7.14}$$

Therefore $\quad \mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_1' - \mathbf{v}_2' \quad (7.15)$

5. The relationship between $\mathbf{v}_1$, $\mathbf{v}_1'$ and $\mathbf{v}$ requires further discussion, as it depends on whether

$$|\mathbf{v}_1'| > |\mathbf{v}|$$

or $\quad |\mathbf{v}_1'| < |\mathbf{v}|$

(i) Consider the case in which the velocity of the scattered particle in the CM system is more than the velocity of the CM system with respect to the lab system, i.e. $|\mathbf{v}_1'| > |\mathbf{v}|$. If we represent the centre of mass by point 0, then the velocity vector $\mathbf{v}_1'$ will be represented by $\overrightarrow{OP}$, such that the magnitude is $|\mathbf{v}_1'|$ = constant and the orientation depends on the angle through which m_1 is scattered. Thus the locus of P will be a circle of radius $|\mathbf{v}_1'|$ (Fig. 7.8). Since $|\mathbf{v}| < |\mathbf{v}_1'|$, the vector $\mathbf{v}$, will be within the circle. Obviously $\mathbf{v}_1$ is the vector representing velocity in the lab system. It is also clear from the figure that for one value of φ, there is a single value of θ.

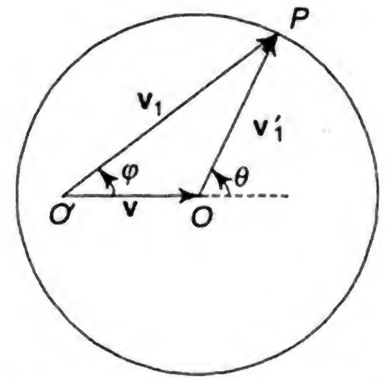


Fig. 7.8 Relations between $\mathbf{v}_1'$, $\mathbf{v}$ and $\mathbf{v}_1$ when $|\mathbf{v}_1'| > |\mathbf{v}|$

(ii) If $|\mathbf{v}_1'| < |\mathbf{v}|$, then one can draw similar diagram (Fig. 7.9) in which point O', will be outside the circle and is again the vector representing velocity in the lab system. It is evident from the figure that for a given φ, there are two values of θ represented in the diagram as θ_f (i.e. the angle in the forward direction) and θ_b (i.e. angle in the backward direction). The values of $|\mathbf{v}_1'|$ of course are the same in the two cases. But the velocity in the lab system has now two values which satisfy the relation (7.14) between $\mathbf{v}_1$, $\mathbf{v}_1'$ and $\mathbf{v}$. The velocities in the lab system are represented in the above diagram as $\mathbf{v}_{1b}$ — the velocity corresponding to the backward scattering angle θ_b, and $\mathbf{v}_{1f}$ —the velocity corresponding to the forward direction. The magnitude of $\mathbf{v}_{1f}$ is always greater than that of $\mathbf{v}_{1b}$. This means that for a given value of $\mathbf{v}_1'$ there will be two velocities $\mathbf{v}_{1b}$ and

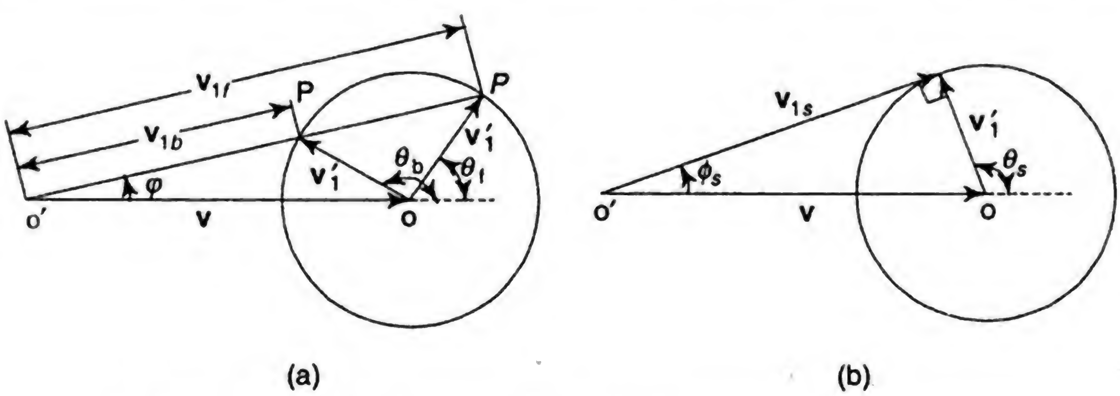


Fig. 7.9 Relations between various velocities for the case $|\mathbf{v}_1'| < |\mathbf{v}|$

$\mathbf{v}_{1l}$ both along the direction making angle φ in the lab system with the initial direction. The difference between these two values depends on the value of φ. However, there exists an angle φ_s, for which $\mathbf{v}_1'$ has one single value as shows in Fig. 7.9 (b) for $|\mathbf{v}| > |\mathbf{v}_1'|$. For this value of velocity, $\theta_b = \theta_f = \theta_s$ and the scattered particles have only one value of energy.

6. The values of various velocities for elastic scattering can now be derived.

(i) By definition, the position vector **R** of the centre of mass in the lab system is given by

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = (m_1 + m_2)\,\mathbf{R} = M\mathbf{R}$$

Differentiating with respect to time, we have

$$m_1\dot{\mathbf{r}}_1 \; m_2\dot{\mathbf{r}}_2 = M\dot{\mathbf{R}} \tag{7.16}$$

or
$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = M\mathbf{v}$$

But in the lab system $\mathbf{u}_2 = 0$, so that the above relation reduces to

$$m_1\mathbf{u}_1 = M\mathbf{v}$$

or
$$\mathbf{v} = [m_1/(m_1 + m_2)]\mathbf{u}_1 \tag{7.17}$$

This means that **v** and $\mathbf{u}_1$ have the same direction.

(ii) In the CM system, the linear momentum is not only conserved but is also zero, before and after scattering. Therefore

$$m_1\mathbf{u}_1' + m_2\mathbf{u}_2' = 0 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2' \tag{7.18}$$

Hence
$$-\mathbf{u}_1'/\mathbf{u}_2' = m_2/m_1 = -(\mathbf{v}_1'/\mathbf{v}_2') \tag{7.19}$$

The negative sign in Eq. (7.19) means that $\mathbf{u}_1'$ and $\mathbf{u}_2'$ are in opposite directions. Similarly, $\mathbf{v}_1'$ and $\mathbf{v}_2'$ are in opposite directions. We have already used this result in drawing Fig. 7.7 for the CM system.

(iii) From conservation of kinetic energy, we have

$$(1/2)\, m_1u_1'^2 + (1/2)\, m_2u_2'^2 = (1/2)\, m_1v_1'^2 + (1/2)\, m_1v_2'^2 \tag{7:20}$$

Substituting for u_1' and v_1' in terms of u_2' and v_2' from Eq. (7.19), we get

$$\frac{1}{2}m_1\left(\frac{m_2^2}{m_1^2}u_2'^2\right) + \frac{1}{2}m_2u_2'^2 = \frac{1}{2}m_1\left(\frac{m_2^2}{m_1^2}\right)v_2'^2 + \frac{1}{2}m_2v_2'^2$$

or
$$u_2'^2\left(\frac{m_2^2}{2m_1} + \frac{m_2}{2}\right) = v_2'^2\left(\frac{m_2^2}{2m_1} + \frac{m_2}{2}\right)$$

Hence $\quad u_2'^2 = v_2'^2 \qquad$ (7.21a)

Also, combining Eqs (7.19) and (7.21a) we obtain

$$|\mathbf{u}_1'| = |\mathbf{v}_1'| \tag{7.21b}$$

(iv) We have already seen that

$$\mathbf{u}_2' = -\mathbf{v} \qquad (7.12)$$

and
$$\mathbf{v} = [m_1/(m_1 + m_2)]\mathbf{u}_1 \qquad (7.17)$$

So that
$$|\mathbf{v}_1'| = |\mathbf{u}_1'| = |\mathbf{u}_1 - \mathbf{v}| \qquad (7.10)$$
$$= |\mathbf{u}_1 - [m_1/(m_1 + m_2)]\mathbf{u}_1|$$
$$= [m_2/(m_1 + m_2)]\,|\mathbf{u}_1| \tag{7.22}$$

Also,
$$|\mathbf{v}_2'| = |\mathbf{u}_2'| = |-\mathbf{v}| = [m_1/(m_1 + m_2)]|\mathbf{u}_1| \tag{7.23}$$

7.3.2 Relationship Between Angles

In order to obtain the relations between angles φ and θ for elastic scattering, we have, from Fig. 7.8,

$$|\mathbf{v}_1| \sin \varphi = |\mathbf{v}'_1| \sin \theta$$

$$|\mathbf{v}_1| \cos \phi = |\mathbf{v}'_1| \cos \theta + |\mathbf{v}| \tag{7.24a}$$

$$\tan \varphi = \frac{|\mathbf{v}'_1| \sin\theta}{|\mathbf{v}'_1| \cos\theta + |\bar{\mathbf{v}}|} = \frac{\sin\theta}{\cos\theta + \frac{|\mathbf{v}|}{|\mathbf{v}'_1|}} \tag{7.24b}$$

But from Eqs (7.17) and (7.22)

$$\frac{|\mathbf{v}|}{|\mathbf{v}'_1|} = \frac{m_1 |\mathbf{u}_1|}{m_1 + m_2} \div \frac{m_2 |\mathbf{u}_1|}{m_1 + m_2} = \frac{m_1}{m_2} \tag{7.25a}$$

Hence
$$\tan \varphi = \frac{\sin\theta}{\cos\theta + m_1/m_2} \tag{7.25b}$$

Now we can discuss the following cases:

(a) If $m_1 \ll m_2$, i.e. the target is very heavy as compared to the incident particle, then

$$\tan \varphi \simeq \tan \theta \tag{7.26}$$

or
$$\varphi \simeq \theta$$

i.e. the angles in the lab and CM systems are the same.

(b) If $m_1 = m_2$, i.e. the incident particle is as heavy as the target, then

$$\tan \varphi = \frac{\sin\theta}{\cos\theta + 1} = \tan \theta/2 \tag{7.27}$$

or
$$\varphi = \theta/2$$

(c) It may also be noted that if $m_1 < m_2$, then $|\mathbf{v}| < |\mathbf{v}'_1|$ form Eq. (7.25a), so that

$$\tan \varphi = \frac{\sin\theta}{\cos\theta + (\text{quantity less than one})} \tag{7.28}$$

It can be shown that there is one value of θ for one value of φ for this case.

(d) If $m_1 > m_2$ or $|\mathbf{v}| > |\mathbf{v}'_1|$, we have

$$\tan\varphi = \frac{\sin\theta}{\cos\theta + (\text{quantity greater than one})} \tag{7.29}$$

then there will be two values of θ (in general) for one value of φ.

(e) It can be seen from Fig. 7.8 for the case of $|\mathbf{v}'_1| > |\mathbf{v}|$ that

$$\sin \varphi_{\max} = \frac{|\mathbf{v}'_1|}{|\mathbf{v}|} = \frac{m_2}{m_1} \tag{7.30}$$

One can similarly find the relation between the scattering angle θ in the CM system (this is also the recoil angle) and the recoil angle α in the lab system for elastic scattering. For this purpose, we consider Fig. 7.10 from which it is evident that

$$|\mathbf{v}_2| \sin \alpha = |\mathbf{v}'_2| \sin \theta$$

$$|\mathbf{v}_2| \cos \alpha = |\mathbf{v}| - |\mathbf{v}'_2| \cos \theta$$

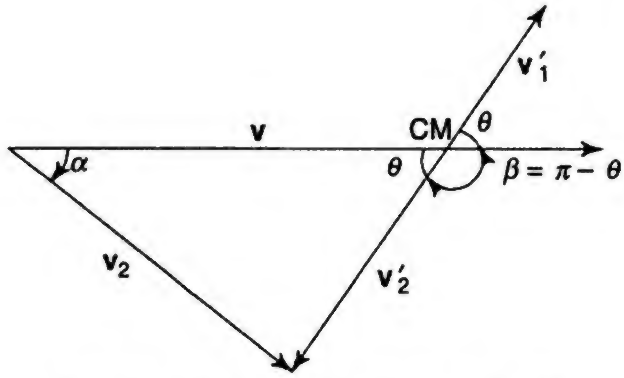


Fig. 7.10 Relation between $\mathbf{v}_2'$ $\mathbf{v}$ and $\mathbf{v}_2$

Hence
$$\tan\alpha = \frac{|\mathbf{v}_2'|\sin\theta}{|\mathbf{v}| - |\mathbf{v}_2'|\cos\theta}$$
$$= \frac{\sin\theta}{\dfrac{|\mathbf{v}|}{|\mathbf{v}_2'|} - \cos\theta}$$

But from Eq. (7.23),
$$|\mathbf{v}| = |\mathbf{v}_2'|$$
$$\tan\alpha = \frac{\sin\theta}{1-\cos\theta} \tag{7.31}$$
$$= \tan(\pi/2 - \theta/2)$$

Therefore,
$$\alpha = \pi/2 - \theta/2 \tag{7.32}$$
or
$$2\alpha = \pi - \theta = \beta \text{ (say)}$$

Equation (7.32) is independent of m_1 and m_2 and hence independent of velocities and energies. Further from Eq. (7.27), for the case $m_1 = m_2$, we can write $2\alpha = \pi - 2\varphi$,

or
$$\alpha + \varphi = \pi/2 \tag{7.33}$$
i.e. if a particle is scattered from a stationary target of the same mass, then the directions of the scattered and the recoil particles are at right angles to each other.

EXAMPLE 7.3

Th scattering angle for a heavy particle of mass m_1 colliding elastically with a light target of mass m_2 is found to be φ in the lab system and θ in the CM system. Show that φ will be maximum when $\cos\theta = -m_2/m_1$ and that

$$\tan\varphi_{\max} = \left[\frac{m_2^2}{m_1^2 - m_2^2}\right]^{1/2}$$

Solution

When a particle of mass m_1 is scattered elastically from a target of mass m_2, the angles of scattering φ and θ in the lab and CM systems are related through Eq. (7.25b), which gives

$$\tan\varphi = \frac{\sin\theta}{\cos\theta + m_1/m_2} = \sin\theta\,(\cos\theta + m_1/m_2)^{-1}$$

In order to find the condition for maximum value of φ, we use the fact that for the extremum, $d\varphi/d\theta$ will be zero. Now differentiating both sides of the above equation with respect to θ, we have

$$\sec^2 \varphi \frac{d\varphi}{d\theta} = \cos \theta (\cos \theta + m_1/m_2)^{-1} + \sin \theta \times (\cos \theta + m_1/m_2)^{-2} \sin \theta$$

$$= \frac{\cos\theta (\cos\theta + m_1/m_2) + \sin^2 \theta}{(\cos\theta + m_1/m_2)^2}$$

$$= \frac{1 + m_1/m_2 \cos\theta}{(\cos\theta + m_1/m_2)^2}$$

$d\varphi/d\theta$ will be zero, when the right hand side in the above relationship is zero, i.e.

$$1 + \left(\frac{m_1}{m_2}\right) \cos \theta = 0$$

or $$\cos \theta = -\frac{m_2}{m_1}$$

Accordingly, $$\sin \theta = (1 - \cos^2 \theta)^{1/2} = \left(1 - \frac{m_2^2}{m_1^2}\right)^{1/2}$$

This corresponds to φ_{max}. For this value of θ,

$$\tan \varphi_{max} = \frac{\sin\theta}{- m_2/m_1 + m_1/m_2}$$

$$= \frac{(1 - m_2^2/m_1^2)^{1/2}}{-m_2/m_1 + m_1/m_2}$$

$$= \frac{(m_1^2 - m_2^2)^{1/2} m_2}{(m_1^2 - m_2^2)}$$

$$= \left(\frac{m_2^2}{m_1^2 - m_2^2}\right)^{1/2}$$

It is clear from the formulae derived here that the angles θ and φ_{max} are defined only for $m_1 > m_2$, which implies that such a situation can be had only when the projectile is heavier than the target.

EXAMPLE 7.4

A particle of mass m_1 and initial velocity $\mathbf{u}_1$ collides elastically with a particle of mass $2m_1$ initially at rest. After the collision, the particle with mass m_1 is found to move at 45° and the recoil particle is moving at angle α. Determine the value of α and velocities of the two particles. Also, find these parameters in the centre-of-mass system.

Solution

A particle with mass m_1 and initial velocity $\mathbf{u}_1$ collides elastically with a stationary particle of mass $2m_1$. After the collision, the particle m_1 moves at $\varphi = 45°$ and m_2 is

moving at angle α with respect to incident direction. Let their velocities be $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Then by conservation of linear momentum,

$$m_1 |\mathbf{u}_1| = m_1 |\mathbf{v}_1| \cos 45° + 2m_1 |\mathbf{v}_2| \cos \alpha \quad \text{(i)}$$

and $$m_1 |\mathbf{v}_1| \sin 45° = 2m_1 |\mathbf{v}_2| \sin \alpha \quad \text{(ii)}$$

From conservation of kinetic energy,

$$\frac{1}{2} m_1 |\mathbf{u}_1|^2 = \frac{1}{2}(m_1) |\mathbf{v}_1|^2 + \frac{1}{2} (2m_1) |\mathbf{v}_2|^2 \quad \text{(iii)}$$

From Eq. (ii), $$|\mathbf{v}_1| = 2\sqrt{2} \, |\mathbf{v}_2| \sin \alpha$$

Substituting in Eq. (i), we have

$$|\mathbf{u}_1| = 2 |\mathbf{v}_2| \sin \alpha + 2 |\mathbf{v}_2| \cos \alpha$$
$$= 2 |\mathbf{v}_2| (\cos \alpha + \sin \alpha)$$
$$|\mathbf{u}_1|^2 = 4 |\mathbf{v}_2|^2 (1 + 2 \sin \alpha \cos \alpha)$$

and from Eq. (iii), $$|\mathbf{u}_1|^2 = 8 |\mathbf{v}_2|^2 \sin^2 \alpha + 2 |\mathbf{v}_2|^2$$

These relations give

$$2 |\mathbf{v}_2|^2 (1 + 4 \sin^2 \alpha) = 4 |\mathbf{v}_2|^2 (1 + 2 \sin \alpha \cos \alpha)$$

This equation is satisfied for $\alpha = 57° \; 10'$. Accordingly,

$$|\mathbf{v}_2| = \frac{|\mathbf{u}_1|}{2(\cos\alpha + \sin\alpha)}$$
$$= \frac{|\mathbf{u}_1|}{2(0.54 + 0.84)} = 0.36 \, |\mathbf{u}_1|$$

and $$|\mathbf{v}_1| = 2\sqrt{2} \times 0.357 \times |\mathbf{u}_1| \times \sin 57° \; 10'$$
$$= 2 \times 1.414 \times 0.357 \times 0.84 \, |\mathbf{u}_1|$$
$$= 0.86 \, |\mathbf{u}_1|$$

The velocity of the centre of mass is given by Eq. (7.17), i.e.

$$\mathbf{v} = \frac{m_1}{m_1 + m_2} \mathbf{u}_1$$
$$= \frac{\mathbf{u}_1}{3}$$

In the centre-of-mass system, velocity of the incident particle is given by

$$\mathbf{u}'_1 = \mathbf{u}_1 - \mathbf{v} \quad (7.10)$$

and the velocity of the target is given by

$$\mathbf{u}'_2 = -\mathbf{v} = -\frac{1}{3} \mathbf{u}_1 \quad (7.12)$$

The magnitude of the velocity of the scattered particle is given by

$$|\mathbf{v}'_1| = |\mathbf{u}'_1| = \frac{2}{3} |\mathbf{u}_1|$$

The magnitude of the velocity of the recoil particle is given by

$$|\mathbf{v}'_2| = |\mathbf{u}'_2| = \frac{1}{3} |\mathbf{u}_1|$$

The angle of scattering θ is given using Eq. (7.24a) as

$$\sin \theta = \frac{|\mathbf{v}_1|}{|\mathbf{v}'_1|} \sin \varphi = (0.86 \, |\mathbf{u}_1| / 0.67 \, |\mathbf{u}_1|) \sin 45°$$

$$= \frac{0.86 \times 0.707}{0.67} = 0.91$$

or $$\theta = 65°10'$$

7.4 KINETIC ENERGIES IN THE LAB AND CM SYSTEMS

Keeping in mind the difference in the application of the conservation laws of energy in the case of elastic and inelastic scatterings, we get the following relationships between kinetic energies.

(a) Lab System

(i) *Elastic scattering:* In this case the total kinetic energy before scattering is given by

$$T_0 = T_1 + T_2 = (1/2)m_1u_1^2 + (1/2)\ m_2u_2^2$$

But since $u_2 = 0$, the above relation reduces to

$$T_0 = (1/2)\ m_1u_1^2$$

On the other hand, after scattering, we have

$$t_0 = t_1 + t_2 = (1/2)\ m_1v_1^2 + (1/2)\ m_2v_2^2$$

As the kinetic energy is conserved for elastic scattering, we have

$$(1/2)\ m_1u_1^2 = (1/2)\ m_1v_1^2 + (1/2)\ m_2v_2^2 \tag{7.34}$$

(ii) *Inelastic scattering:* In inelastic scattering the scattered particles may be excited. Here the total energy—including the excitation energies—is conserved. But the kinetic energy alone is not conserved. We, therefore, have the following relationships.

As explained earlier, before scattering the total kinetic energy T_0 is given by

$$T_0 = T_1 = (1/2)\ m_1u_1^2 \tag{7.35a}$$

and is equal to the total energy. After scattering, the total kinetic energy is given by

$$t_0 = (1/2)\ m_1^*v_1^2 + (1/2)\ m_2^*v_2^2 \tag{7.35b}$$

If $$E_1 = (m_1^* - m_1)\ c^2$$

and $$E_2 = (m_2^* - m_2)\ c^2$$

are the excitation energies of m_1 and m_2 after scattering, where m_1^* and m_2^* are the masses of the particles whose internal structure has been changed due to the scattering. Then conserving the total energy, we can write

$$(1/2)\ m_1u_1^2 = (1/2)\ m_1^*v_1^2 + (1/2)\ m_2^*v_2^2 + E_1 + E_2 \tag{7.36}$$

(b) Centre-of-Mass System

(i) *Elastic Scattering:* Before scattering, the total kinetic energy is given by

$$T_0' = T_1' + T_2' = (1/2)\ m_1u_1'^2 + (1/2)\ m_2u_2'^2 \tag{7.37a}$$

After scattering,

$$T_0' = t_1' + t_2' = (1/2)\ m_1v_1'^2 + (1/2)\ m_2v_2'^2 \tag{7.37b}$$

Also, as $|\mathbf{u}_1'| = |\mathbf{v}_1'|$ and $|\mathbf{u}_2'| = |\mathbf{v}_2'|$, we have

$$T_0' = t_1' + t_2' = T_1' + T_2' \tag{7.38}$$

which is in accord with the conservation of kinetic energy. It may be realised that the centre-of-mass system is moving with respect to the lab system with velocity **v**. Therefore, the kinetic energy T_{CM}, of the centre-of-mass system with respect to lab system is given by

$$T_{CM} = (1/2)(m_1 + m_2)v^2 \tag{7.39}$$

Consequently,

$$T_0 = T_0' + T_{CM} \tag{7.40}$$

(ii) *Inelastic scattering:* We have here, before scattering

$$T_0' = T_1' + T_2'$$

whereas after scattering

$$T_0' = t_1' + t_2' + E_1 + E_2 \tag{7.41}$$

where again $\quad E_1 = (m_1^* - m_1)\, c^2$

and $\quad E_2 = (m_2^* - m_2)\, c^2$

are excitation energies of the two particles.

From here, we can proceed to find the relationship between different energies for elastic scattering in the two systems as follow. From Eqs (7.35a) and (7.37a), we have

$$\frac{T_0}{T_0'} = \frac{(1/2)\, m_1 u_1^2}{(1/2)\, m_1 u_1'^2 + (1/2)\, m_2 u_2'^2}$$

But

$$|\mathbf{u}_1'| = |\mathbf{v}_1'| = \frac{m_2}{m_1 + m_2}\, |\mathbf{u}_1| \tag{7.22}$$

$$|\mathbf{u}_2'| = |\mathbf{v}_2'| = \frac{m_1}{m_1 + m_2}\, |\mathbf{u}_1| \tag{7.23}$$

$$\frac{T_0}{T_0'} = \frac{m_2 + m_1}{m_2} = 1 + \frac{m_1}{m_2} \tag{7.42}$$

This means that T_0' is always less than T_0. The rest of the energy goes to T_{CM}.

Again using Eqs (7.21) and (7.22), we have

$$T_1' = \frac{1}{2}\, m_1 u_1'^2 = \frac{1}{2}\, m_1 v_1'^2$$

$$= \frac{1}{2} m_1 \left(\frac{m_2}{m_1 + m_2}\right)^2 u_1^2$$

$$= \left(\frac{m_2}{m_1 + m_2}\right)^2 T_0 \tag{7.43}$$

Similarly, from Eqs (7.21) and (7.23), we get

$$T_2' = \frac{1}{2}\, m_2 u_2'^2 = \frac{1}{2}\, m_2 v_2'^2$$

$$= \frac{1}{2}\, m_2 \left(\frac{m_1}{m_1 + m_2}\right)^2 u_1^2$$

$$= \frac{m_1 m_2}{(m_1 + m_2)^2}\, T_0 \tag{7.44}$$

Therefore, $$\frac{T_1'}{T_2'} = \frac{m_2}{m_1} \tag{7.45}$$

i.e. the kinetic energies of the two colliding particles in the CM system are inversely proportional to their respective masses.

Since $t_1 = (1/2)\, m_1 v_1^2$

$t_2 = (1/2)\, m_2 v_2^2$

and $T_0 = (1/2)\, m_1 u_1^2$

we have $$t_1/T_0 = v_1^2/u_1^2 \tag{7.46a}$$

and $$t_2/T_0 = v_2^2/u_1^2 \tag{7.46b}$$

Now from Fig. 7.7,

$$v_1'^2 = v_1^2 + v^2 - 2\,|\mathbf{v}_1|\,|\mathbf{v}| \cos\varphi \tag{7.47}$$

Therefore $$\frac{t_1}{T_0} = \frac{v_1^2}{u_1^2} = \frac{v_1'^2 - v^2 + 2|\mathbf{v}_1||\mathbf{v}|\cos\varphi}{u_1^2} \tag{7.48}$$

From Eq. (7.22)

$$\frac{|\mathbf{v}_1'|}{|\mathbf{u}_1|} = \frac{m_2}{m_1 + m_2} \tag{7.22}$$

and from Eq. (7.23)

$$\frac{|\mathbf{v}|}{|\mathbf{u}_1|} = \frac{m_1}{m_1 + m_2} \tag{7.23}$$

Further from Fig. 7.8

$$\frac{2|\mathbf{v}_1||\mathbf{v}|}{u_1^2}\cos\varphi = 2\left(\frac{|\mathbf{v}_1'|}{|\mathbf{u}_1|}\frac{\sin\theta}{\sin\varphi}\right)\frac{|\mathbf{v}|}{|\mathbf{u}_1|}\cos\varphi$$

$$= 2\,\frac{\sin\theta\cos\varphi}{\sin\varphi}\frac{|\mathbf{v}_1'|}{|\mathbf{u}_1|}\frac{|\mathbf{v}|}{|\mathbf{u}_1|}$$

$$= 2\,\frac{\sin\theta\cos\varphi}{\sin\varphi}\frac{m_2}{m_1 + m_2}\frac{m_1}{m_1 + m_2} \tag{7.49}$$

where we have used Eqs (7.24a), (7.22) and (7.23). But, from Eq. (7.25b),

$$\frac{\sin\theta\cos\varphi}{\sin\varphi} = \frac{\sin\theta}{\tan\varphi} = \cos\theta + \frac{m_1}{m_2}$$

Therefore $$\frac{2|\mathbf{v}_1||\mathbf{v}|}{u_1^2}\cos\varphi = \frac{2m_1m_2}{(m_1+m_2)^2}\left(\cos\theta + \frac{m_1}{m_2}\right) \tag{7.50}$$

Substituting the above results in Eq. (7.48), we obtain

$$\frac{t_1}{T_0} = \frac{m_2^2}{(m_1+m_2)^2} - \frac{m_1^2}{(m_1+m_2)^2} + \frac{2m_1m_2}{(m_1+m_2)^2}$$

$$\times\left(\cos\theta + \frac{m_1}{m_2}\right) = 1 - \frac{2m_1m_2}{(m_1+m_2)^2}(1 - \cos\theta) \tag{7.51}$$

Furthermore, $$\frac{t_2}{T_0} = 1 - \frac{t_1}{T_0} = \frac{2m_1 m_2}{(m_1 + m_2)^2}(1 - \cos\theta) \tag{7.52}$$

When $m_1 = m_2$, $\theta = 2\varphi$ from Eq. (7.27), we get

$$\frac{t_1}{T_0} = \cos^2\varphi \tag{7.53}$$

and $$\frac{t_2}{T_0} = \sin^2\varphi \tag{7.54}$$

EXAMPLE 7.5

A particle of mass m moving with a velocity $\mathbf{u}$ collides with a stationary target of mass $5m$. As a result of the elastic collision, the scattered particle moves backward while the recoil particle advances in the forward direction. Determine velocities of the two particles as well as that of the centre of mass after collision. Also, find the total kinetic energy of the two particles and the kinetic energy of the incident particle in the CM system.

Solution

In the lab system, the incident particle of mass m has velocity $\mathbf{u}$, whereas the target mass is $5m$. After collision, the scattered particle has velocity $-\mathbf{v}_1$ and the recoil particle velocity is $\mathbf{v}_2$; $\mathbf{v}_1$ has been taken negative because the scattered particle is moving backward. The conservations of linear momentum and energy in the lab system yield,

$$m\mathbf{u} = -m\mathbf{v}_1 + 5m\mathbf{v}_2$$

or $$\mathbf{u} = -\mathbf{v}_1 + 5\mathbf{v}_2$$

and $$1/2\, mu^2 = 1/2\, mv_1^2 + 1/2\, 5mv_2^2$$

or $$u^2 = v_1^2 + 5v_2^2$$

Squaring the relationship for u, we have

$$\begin{aligned} u^2 &= v_1^2 + 25v_2^2 - 10\, \mathbf{v}_1 \cdot \mathbf{v}_2 \\ &= v_1^2 + 25v_2^2 - 10\, |\mathbf{v}_1||\mathbf{v}_2| \end{aligned}$$

Equating the expressions for u^2, we get

$$20v_2^2 - 10\, |\mathbf{v}_1||\mathbf{v}_2| = 0$$

$$\therefore \quad |\mathbf{v}_1| = 2|\mathbf{v}_2|$$

Putting in the relation of $\mathbf{u}$ and remembering that all the vectors are colinear, we obtain

$$|\mathbf{u}| = 3\, |\mathbf{v}_2|$$

$$\therefore \quad |\mathbf{v}_2| = \frac{|\mathbf{u}|}{3} \quad \text{and} \quad |\mathbf{v}_1| = \frac{2}{3}\, |\mathbf{u}|$$

In fact, $$\mathbf{v}_1 = \frac{2}{3}\,\mathbf{u} \quad \text{and} \quad \mathbf{v}_2 = \frac{\mathbf{u}}{3}$$

From Eq. (7.17), the velocity of the centre of mass is

$$\mathbf{v} = \frac{m}{m + 5m}\mathbf{u} = \frac{\mathbf{u}}{6}$$

which is along the same direction as the direction of the incident particle.

In order to find the velocities of the particles in the CM system we note that their values after collision are given by Eq. (7.14). Accordingly, using primes for this case, we have

$$\mathbf{v}_1' = \mathbf{v}_1 - \mathbf{v} = -\frac{2\mathbf{u}}{3} - \frac{\mathbf{u}}{6}$$
$$= -\frac{5\mathbf{u}}{6}$$
$$\mathbf{v}_2' = \mathbf{v}_2 - \mathbf{v} = \frac{\mathbf{u}}{3} - \frac{\mathbf{u}}{6}$$
$$= \frac{\mathbf{u}}{6}$$

The total kinetic energy of the two particles in the CM system, after collision is given by Eq. (7.37b) which gives

$$T_0' = (1/2)mv_1'^2 + (1/2)\, 5mv_2'^2$$
$$= (1/2)m\,(25u^2/36) + (5/2)m\,(u^2/36)$$
$$= (5/12)\, mu^2$$

Note: It can be seen that the kinetic energy of the centre of mass in the lab system is given by

$$(1/2)\,(m + 5m)\,(u^2/36) = (1/12)\, mu^2$$

which when added to T_0' gives the total kinetic energy in the lab system as mentioned in Eq. (7.40).

The kinetic energy of the incident particle in the CM system is given by Eq. (7.51).

$$\therefore \qquad t_1 = \left[1 - \frac{2m \cdot 5m}{(m+5m)^2}(1 - \cos\theta)\right] T_0$$
$$= \left[1 - \frac{5}{18}(1 - \cos\theta)\right] T_0$$

But $\qquad T_0 = (1/2)\, mu^2$

and the scattering angle θ in the CM system is related to the scattering angle φ in the lab system through Eq. (7.25b)

$$\tan\varphi = \frac{\sin\theta}{\cos\theta + (m/5m)}$$

Since $\varphi = 180°$, $\tan\varphi = 0$ which implies that $\sin\theta = 0$.

and hence $\qquad \theta = 180°$

Consequently $\qquad t_1 = 4/9\ T_0 = 2/9\ mu^2$

The $\theta = 0$ case corresponds to no scattering in the CM system and therefore, it has been discarded.

7.5 CROSS-SECTION OF ELASTIC SCATTERING

In the discussion of scattering of particles from some target, one requires not only the relationships between velocities, angles and energies as we have done so far, but

also the probability of scattering. In general parlance, probability is defined as the ratio of the actual events taking place to the total possible number of events. In the case of scattering, one can define the probability of scattering as follows:

Probability of elastic scattering

$$= \frac{\text{number of elastically scattered particles}}{\text{total number of incident particles}} \tag{7.55}$$

If every particle falling on the target is elastically scattered, then the probability of elastic scattering is unity. However, if some of the particles are inelastically scattered or go unscattered, then the probability of elastic scattering will be less than unity.

Now suppose that the target has n particles in it and its area on which the particles are incident is A (Fig. 7.11). If I is the number of particles incident per unit area per unit time (flux density) and N_{sc} is the number of particles scattered per unit scatterer per unit time

$$\text{Probability of scattering} = \frac{nN_{sc}}{IA} \tag{7.56}$$

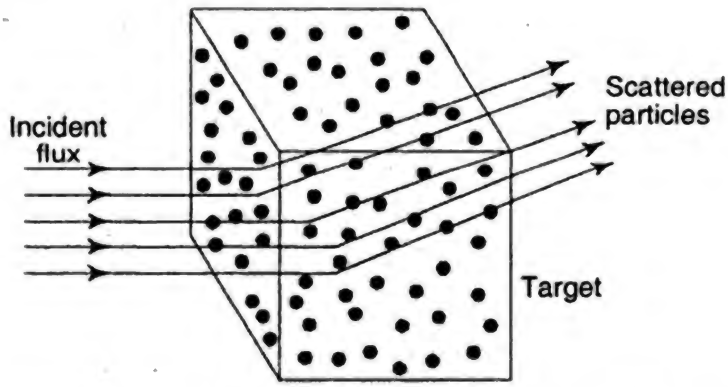


Fig. 7.11 The illustrations of a scatter in the path of a beam

In the scattering of nuclear or atomic particles, the scattering probability is generally expressed in terms of an effective cross-section or area which the target particle offers to the incoming beam. It is evident that greater the effective area which the target particle offers, greater the probability of scattering. In this way the effective cross-section of the target and the probability of scattering get connected together.

Suppose each target particle in Fig. 7.11 offers an effective cross-section σ_{sc} to the incoming beam. Then the total effective area which the particles will offer will be $n\sigma_{sc}$. If the whole area of the target is A, then we can express the probability of scattering as

$$\text{Probability of scattering} = \frac{n\sigma_{sc}}{A} \tag{7.57}$$

where σ_{sc} is the cross-section for scattering of a given kind of particles.

Equating the expressions given in Eqs (7.56) and (7.57), we have

$$\frac{n\sigma_{sc}}{A} = \frac{nN_{sc}}{IA}$$

or
$$\sigma_{sc} = \frac{N_{sc}}{I} \tag{7.58}$$

Thus the scattering cross-section is the number of particles scattered per unit scatterer per unit time per unit incident flux. It can be seen from Eq. (7.58) that σ_{sc} has the dimensions of area because N_{sc} has the dimensions of number per second, and I has the dimensions of number per second per unit area, which justifies its name as scattering cross-section.

Owing to the interaction, the target particles have a certain capacity to scatter the incident particles. If the interaction is strong, the cross-section σ_{sc} is higher and more particles are scattered. If the interaction is weak, less number of particles are scattered and the cross-section is less. It should be noted that σ_{sc} will be of the order of the area of the scattering particles for strong interactions. It should also be realised that we have assumed in the above discussion that the scattering sample is such that total area presented by the n particles does not overlap.

In Eq. (7.58), N_{sc} and σ_{sc} are directly proportional to each other and represent the same nature of the measurement, if $N_{sc}(\theta)$ is the number of scattered particles per scatterer at angle θ per unit solid angle per unit time, then the corresponding cross-section is written as $\sigma(\theta)$ and is called the differential cross-section given by

$$\sigma(\theta) = \frac{N_{sc}(\theta)}{I}$$

or
$$\sigma(\theta)\, d\Omega = \frac{N_{sc}(\theta)}{I}\, d\Omega \tag{7.59}$$

where $d\Omega$ is the solid angle into which $N_{sc}(\theta)\, d\Omega$ particles are scattered. Similarly, if $N_{sc}(t)$ is the total number of scattered particles in all directions, then $\sigma_{sc}(t)$ represents the total scattering cross section. If N_{inel} represents the inelastically scattered particles, then σ_{inel} is the cross-section for inelastic scattering, and so on.

The cross-section will depend upon the energy of the incident particle as well as on the charge and the mass of the scatterer.

7.6 RUTHERFORD SCATTERING

One of the most famous experiments on scattering was performed by Rutherford who studied scattering of alpha particles from various materials and determined the differential cross-sections of elastic scattering, especially at backward angles. It was the angular dependence of the alpha scattering from various elements which laid the foundation for the nuclear model of the atom as we know it today. Let us assume that the incident and target atoms are only point masses and let the potential energy between the incident and target particles be given by

$$U(r) = +\frac{qq'}{r} \tag{7.60}$$

where q is the charge on the alpha particle which is $+2e$, e being the magnitude of the charge on electron, $+q'$ the charge of the nucleus of the target atom, and r the instantaneous distance between them. The charges of electrons in atoms have been neglected because we are assuming that scattering takes place only from the nucleus. Figure 7.12 illustrates the scattering event. The details of the trajectories

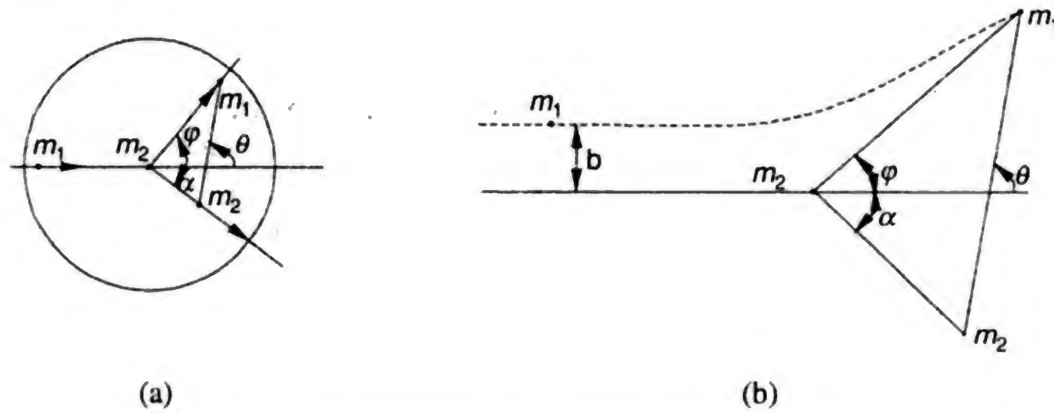


Fig. 7.12 The Rutherford scattering in the lab system

are shown in Fig. 7.12b and the details of scattering are marked in circle in Fig. 7.12a.

We are assuming that the incident particle is not necessarily travelling to make a head-on collision, but is travelling initially along a line at perpendicular distance b called the *impact parameter*.

The angles φ and α used in Fig. 7.12 are the same as defined earlier, i.e. they are the angles of scattering of the incident and target particles respectively in the lab system. Similarly, angle θ is the angle of scattering in the centre-of-mass system.

Figure 7.13 shows the details of the scattering process in the CM system where θ is the angle of scattering. As the particles are coming in a beam of uniform flux, the number of incident particles having an impact parameter between b and $(b + db)$ is given by

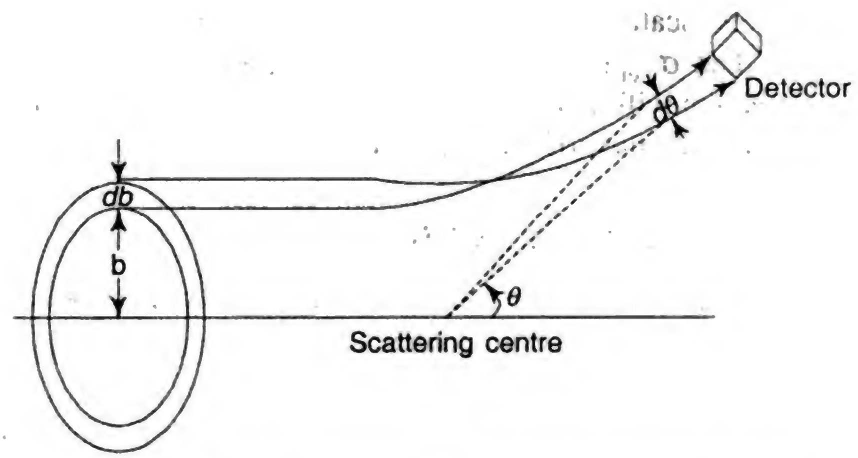


Fig. 7.13 Detection of particles scattered from a centre of force

$$IdA = I\,2\pi b\, db \tag{7.61}$$

Let $\sigma(\theta)$ be the differential cross section as explained earlier, then the number of particles $N(\theta)$ scattered between θ and $(\theta + d\theta)$ per unit solid angle per unit time per scatterer is given by:

$$\sigma(\theta) = \frac{N(\theta)}{I} \tag{7.62}$$

If our detector subtends an angle $d\theta$ in the centre of mass system, the number of particles scattered into the detector is given by

$$N(\theta)\, d\Omega = I\sigma(\theta)\, d\Omega$$

where $$d\Omega = \sin\theta\, d\theta\, d\varphi \tag{7.63}$$

If the detector has the type of dimensions shown in the Fig. 7.13; $d\varphi$ will have some finite value, say $\Delta\phi$. If the detector were annular around the direction of the beam, then $\Delta\phi$ will be 2π. For our argument, let us assume that the detector is annular so that $\Delta\phi = 2\pi$. These particles will be scattered into the angles between θ and $\theta + d\theta$. The particles with larger b will be scattered through smaller angles as shown in the diagram. This happens because larger b means less interaction. For very large b the particles will not be much deflected from their path and will go nearly straight. Now $N(\theta)\, d\Omega$ as given in Eq. (7.63) is the number of particles scattered in the solid angle $d\Omega$. As shown in Fig. 7.13 for a certain value of $d\theta$, there is a certain value of db. Further, when b increases, the scattering angle decreases or for a positive value of db, there is a negative value of $d\theta$. The total number of incident particles falling on the annular circular ring having radii between b and $b + db$ is given in Eq. (7.61). These particles will be scattered into angles between θ and $\theta + d\theta$. In this manner the two expressions given in Eqs (7.61) and (7.63) may be equated, i.e.

$$I 2\pi b db = -I\sigma(\theta)\, 2\pi \sin\theta d\theta \tag{7.64}$$

The negative sign on the right-hand side of Eq. (7.64) expresses the fact that db and $d\theta$ have opposite signs. Accordingly,

$$\sigma(\theta) = \frac{b}{\sin\theta}\frac{db}{d\theta} \tag{7.65}$$

We have not written the negative sign in Eq. (7.65) because $\sigma(\theta)$ is an area whose magnitude is taken to be positive.

Since the interaction involved here obeys the inverse square law of force, we can make use of the results of the two-body problem discussed in Sec. 6.5. The angle θ there, corresponds to the angle between the initial radial vector and the final radial vector and shall be denoted by Θ in this section. It is related to θ used in this chapter, as shown in Fig. 7.14 and explained below. From Eqs (6.25) and (6.26), we get

$$d\Theta = \frac{L dr}{mr^2\,[(2/m)(E - U(r) - L^2/2mr^2)]^{1/2}} \tag{7.66}$$

or $$\Theta = \int_{r_{min}}^{r_{max}} \frac{L/r^2\, dr}{[2m(E - U(r) - L^2/2mr^2)]^{1/2}} \tag{7.67}$$

where $r_{min} = b$ and r_{max} is infinity in the case of scattering.

Furthermore, in our problem when treated in the centre-of-mass system, the initital radial vector corresponds to the radial vector from the centre of mass to the positions of m_1 in the incident beam and final radial vector from the centre of mass to a position of m_1 after scattering, say at A in Fig. 7.14. Then limits of r will be from $r = -\infty$ to $r = r_{min}$. In this case, the angle between the two radial vectors will be taken anticlockwise and will be equal to $2\pi - \Theta$. However, if we take the limits from r_{min} to $r = \infty$ as we have done in Eq. (7.67), we take the angle Θ in the

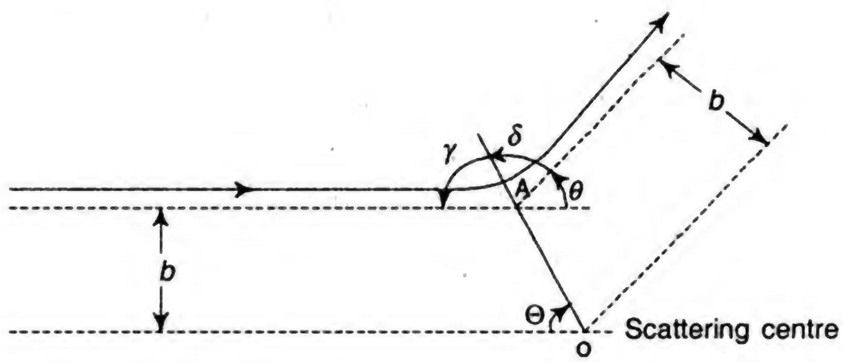


Fig. 7.14 Position of the scattered particle m_1 at closest approach wrt the scattering centre

clockwise manner as shown in Fig. 7.14. As we are only going to use cos Θ in our subsequent discussions, it does not matter which convention is used. We also see from Fig. 7.14 that $\Theta = \gamma$. Because of the symmetry of the trajectory around the line OA, we have

$$\Theta = \gamma = \delta \tag{7.68}$$

But $$2\gamma + \theta = \pi$$

Hence $$\theta = \pi - 2\gamma = \pi - 2\Theta \tag{7.69}$$

Now, from the definition of angular momentum as the moment of linear momentum, we can write

$$\begin{aligned} L &= m_1 u_1 b \\ &= [(1/2)\, m_1 u_1^2 \times 2m_1]^{1/2}\, b \\ &= b(2m_1 T_0)^{1/2} \end{aligned} \tag{7.70a}$$

where T_0 is the kinetic energy in the lab system. In the centre-of-mass system, we replace T_0 by T_0' and m by the reduced mass μ, so that in the CM system;

$$L = b(2\mu T_0')^{1/2} \tag{7.70b}$$

The integration of Eq. (7.67) can be carried out easily. Remembering that $r_{max} \to \infty$ at $E = T_0'$, we get from Eqs (7.67) and (7.70b),

$$d\theta = \frac{(b/r^2)\, dr}{[1 - b^2/r^2 - U/T_0']^{1/2}} \tag{7.71}$$

But $$U = k/r = +\, qq'/r$$

Then $$d\theta = \frac{(b/r)\, dr}{[r^2 - b^2 - (kr/T_0']^{1/2}} \tag{7.72}$$

The integration of the right hand side gives

$$\Theta = \cos^{-1}\left[\frac{x/b}{[1 + (x/b)^2]^{1/2}}\right] \tag{7.73a}$$

where $$x = \frac{k}{2T_0'} \tag{7.73b}$$

From Eq. (7.73a),

$$\cos\Theta = \frac{x/b}{[1 + (x/b)^2]^{1/2}}$$

$$\tan \Theta = \frac{1}{x/b} = b/x \tag{7.74}$$

or $b^2 = x^2 \tan^2 \Theta$. But $\Theta = \pi/2 - \theta/2$, therefore Eq. (7.74) leads to

$$b = x \cot(\theta/2) \tag{7.75}$$

From this,

$$\frac{db}{d\theta} = -\frac{x}{2}\frac{1}{\sin^2(\theta/2)}$$

From Eq. (7.65),

$$\sigma(\theta) = \frac{x^2 \cos(\theta/2)}{2\sin\theta \sin^3(\theta/2)} \tag{7.76}$$

Now $\qquad \sin\theta = 2\sin(\theta/2)\cos(\theta/2)$,

$$\sigma(\theta) = \frac{x^2}{4}\frac{1}{\sin^4(\theta/2)} = \frac{k^2}{(4T_0')^2}\frac{1}{\sin^4(\theta/2)} \tag{7.77}$$

This is the famous Rutherford scattering formula. Its experimental verification led to the conclusion that the scattering centre can be considered to be a point mass, with charge $q' = Ze$, where Z is the atomic number of the scattering atom. From this formula, conclusions were drawn about the theory of the atom, according to which the massive nucleus sits at the centre, and the electrons revolve around it.

EXAMPLE 7.6

In one of their experiments on scattering of α-particles, Geiger and Marsden bombarded 7.7 MeV alpha particles on a gold ($Z = 79$, $A = 197$) target. Determine the impact parameter of the α-particles which are scattered through an angle equal to or greater than 10°.

Solution

The impact parameter b is related to the scattering angle θ through Eq. (7.75), viz.

$$b = x\cot(\theta/2)$$

$$= \frac{k}{2T_0'}\cot(\theta/2)$$

where the value of x has been substituted from Eq. (7.73b). In the present problem,

$$k = qq' = 79 \times 2 \times (4.8 \times 10^{-10})^2 \text{ [cgs units]}$$

$$= 3.64 \times 10^{-17} \text{ [cgs units]}$$

The kinetic energy T_0 of α-particles in the laboratory system

$$T_0 = 7.7 \text{ MeV}$$

$$= 12.3 \times 10^{-6} \text{ ergs}$$

Mass of the α-particles $= m_1 = 4 \times 1.67 \times 10^{-24}$ g

$$= 6.68 \times 10^{-24} \text{ g}$$

Magnitude of the velocity of the incident α-particles

$$|\mathbf{u}_1| = \left(\frac{2T_0}{m_1}\right)^{1/2}$$

$$= \left(\frac{2 \times 12.3 \times 10^{-6}}{6.68 \times 10^{-24}}\right)^{1/2}$$

$$= 1.92 \times 10^9 \text{ cm/s}$$

Mass of the target particle (Au) $= m_2 = 197 \times 1.67 \times 10^{-24}$ g

$$= 3.29 \times 10^{-22} \text{ g}$$

Magnitude of velocity of the centre of mass,

$$|\mathbf{v}| = \frac{m_1}{m_1 + m_2} |\mathbf{u}_1|$$

$$= \frac{4}{201} \times 1.92 \times 10^9 \text{ cm/s}$$

$$= 3.82 \times 10^7 \text{ cm/s}$$

Kinetic energy of the centre of mass,

$$T_{CM} = \frac{1}{2} (m_1 + m_2) |\mathbf{v}|^2 \tag{7.39}$$

$$= \frac{1}{2} \times 201 \times 1.67 \times 10^{-24} \times (3.82 \times 10^7)^2 \text{ ergs}$$

$$= 2.45 \times 10^{-7} \text{ ergs}$$

The kinetic energy of α-particles in the centre-of-mass system is given by

$$T_0' = T_0 - T_{CM} = 12.1 \times 10^{-6} \text{ ergs}$$

For α-particles scattered through an angle equal to or greater than 10°,

$$\theta = 10°$$

Hence, we have

$$b = \frac{3.64 \times 10^{-17}}{2 \times 12.1 \times 10^{-6}} \cot 5°$$

$$= 0.15 \times 10^{-11} \times 11.4 \text{ cm}$$

$$= 1.7 \times 10^{-11} \text{ cm}$$

EXAMPLE 7.7

Find the scattering cross-section $\sigma(\theta)$ of Pb (Z = 82, A = 207) for 7 MeV α-particles, corresponding to $\theta = 30°$; given

$$1 \text{ a.m.u} = 1.67^{-27} \text{ kg}$$

Solution

From Eq. (7.77),

$$\sigma(\theta) = \frac{k^2}{(4T_0')^2} \operatorname{cosec}^4 \frac{\theta}{2}$$

In the given problem,

Kinetic energy T_0 of α-particles in the lab system

$$= 7 \text{ MeV}$$

$$= 11.2 \times 10^{-13} \text{ J}$$

Mass of α-particle,

$$m_1 = 4 \times 1.67 \times 10^{-27} \text{ kg}$$

$$= 6.68 \times 10^{-27} \text{ kg}$$

Velocity of incident α-particle,

$$|\mathbf{u}_1| = \left[\frac{2T_0}{m_1}\right]^{1/2}$$

$$= \left(\frac{2 \times 11.2 \times 10^{-13}}{6.68 \times 10^{-27}}\right)^{1/2}$$

$$= 1.8 \times 10^7 \text{ m s}^{-1}$$

Mass of target,

$$m_2 = 207 \times 1.67 \times 10^{-27}$$
$$= 3.46 \times 10^{-25} \text{ kg}$$

Velocity of the centre of mass

$$|\mathbf{v}| = \frac{m_1}{m_1 + m_2} |\mathbf{u}_1|$$

$$= \frac{4}{211} \times 1.8 \times 10^7 \text{ m s}^{-1}$$

$$= 3.4 \times 10^5 \text{ m s}^{-1}$$

Kinetic energy of CM $= T_{CM} = (1/2)\,(m_1 + m_2)\,|\mathbf{v}|^2$

$$= (1/2) \times 211 \times 1.67 \times 10^{-27} \times (3.4 \times 10^5)^2$$
$$= 2 \times 10^{-14} \text{ J}$$

Kinetic energy of α-particles in the CM system

$$T_0' = T_0 - T_{CM} = 11 \times 10^{-18} \text{ J}$$

Furthermore, $k = qq'$ as introduced in the text for cgs units needs being converted to mks units, where it becomes

$$k = qq'/4\pi\epsilon_0 = 82 \times 2e^2/4 \times 3.14 \times 8.85 \times 10^{-12}$$
$$= 164 \times (1.6 \times 10^{-19})^2/1.1 \times 10^{-10}$$
$$= 3.8 \times 10^{-26}$$

Therefore,

$$\sigma(30°) = \frac{(3.8 \times 10^{-26})^2}{(4 \times 1.1 \times 10^{-12})^2} \operatorname{cosec}^4 15° \text{ m}^2$$

$$= 0.74 \times 10^{-28} \times (3.86)^4 \text{ m}^2$$
$$= 1.8 \times 10^{-26} \text{ m}^2$$

QUESTIONS

7.1 Define the term 'collision' and bring out the usefulness of the study of collisions in understanding the forces in nature.

7.2 What is scattering? When is it elastic and inelastic?

7.3 What is a laboratory frame of reference? How will two such frames be related to each other when the observers are in two adjoining rooms?

7.4 Define 'centre-of-mass system'. How does it differ from the lab system?

7.5 Under what conditions can collision be termed as a reaction? Illustrate your answer by either considering a chemical reaction or nuclear reaction.

7.6. In what respects are inelastic collisions different from the elastic collisions?

7.7 In nuclear physics elastic scattering as well as inelastic scattering are taken as special categories of reactions. Justify this type of classification.

7.8 Justify the statement: 'Conservation of linear and angular momenta holds good in all types of collisions'.

7.9 Dicuss the law of conservation of energy as it should be applied to elastic, inelastic and reactive collisions giving their expressions in the lab system.

7.10 What is orbital angular momentum? Show that this is conserved in elastic collisions but not in inelastic collisions.

7.11 Under what conditions can the results of the lab and CM systems be taken to be same?

7.12 Define total angular momentum and argue to show that it is conserved in collisions.

7.13 Draw a labelled figure bringing out the results of the general collision process as seen by the observers in the lab and CM systems.

7.14 Show that the separation of the two colliding particles as observed in the lab and CM systems will be the same.

7.15 Prove that a target which is at rest in the lab system will have velocity $\mathbf{v}$ with respect to the CM system.

7.16 A scattered particle of mass m_1 is found to have velocities $\mathbf{v}_1$ and $\mathbf{v}_1'$ in the lab and CM systems, which themselves have relative velocity $\mathbf{v}$. Show that there will be unique value of $\mathbf{v}_1$ for particular $\mathbf{v}_1$ if $|\mathbf{v}_1'| \gg |\mathbf{v}|$.

7.17 Consider the above questions for the situation $|\mathbf{v}_1'| \ll |\mathbf{v}|$. Prove that the two different angles of scattering in the CM system can yield one angle of scattering in the lab system. Also, bring out the condition when these will be one-to-one correspondence of the angle in the two frames.

7.18 Prove the following relationships:

(i) $|\mathbf{v}_1'| = |\mathbf{u}_1'| = [m_2/(m_1 + m_2)]\ |\mathbf{u}_1|$

(ii) $|\mathbf{v}_2'| = |\mathbf{u}_2'| = |-\mathbf{v}| = [m_1/(m_1 + m_2)]\ |\mathbf{u}_1|$

(iii) $|\mathbf{v}_1'|^2 = |\mathbf{v}_1|^2 + |\mathbf{v}|^2 - 2|\mathbf{v}_1'|\ |\mathbf{v}| \cos\theta$

where various symbols are as defined in the text.

7.19 The angle of scattering in the lab and CM systems are represented as φ and θ respectively. Prove that

$$\varphi = \tan^{-1} \tan\theta / \left(1 + \frac{m_1}{m_2}\sec\theta\right)$$

What will be the form of this relationship for $m_1 \ll m_2$, $m_1 < m_2$, $m_1 = m_2$ and $m_1 > m_2$?

7.20 Derive the relationship between the recoil angles θ and α in the CM and lab systems.

7.21 If the angle of scattering in the CM system is θ, then show that it is related to the recoil angle α in the lab system through $\alpha = (\pi - \theta)/2$.

7.22 Show that the total kinetic energy in the lab system is always greater than the total kinetic energy in the CM system.

7.23 Prove that the kinetic energies of two colliding particles in the CM system are inversely proportional to their masses.

7.24 Kinetic energies of two particles after collision as seen in the lab system are t_1 and t_2. Show that these are related to the total kinetic energy T_0 through

$$t_1 = T_0 - \frac{2m_1 m_2}{(m_1 + m_2)^2}(1 - \cos\theta)\, T_0$$

$$t_2 = \frac{2m_1 m_2}{(m_1 + m_2)^2}(1 - \cos\theta)\, T_0$$

where θ is the angle of scattering in the CM system.

7.25 Define the scattering cross-section and express it in terms of the number of particles scattered per unit time per scatterer and the incident flux density. Hence show that it has dimensions of area.

7.26 'The number of target particles as well as that of the projectiles should be quite large for meaningful determination of scattering cross-section'. Discuss.

7.27 How can scattering cross-section be taken as a measure of the strength of interaction?

7.28 Define an impact parameter. Argue to show that a larger impact parameter leads to a smaller scattering angle.

7.29 What is Rutherford scattering? Show that the scattering cross-section for Rutherford scattering is given by

$$\sigma(\theta) = (k^2/4T_0'^2)\ \text{cosec}^4\,(\theta/2)$$

7.30 The Rutherford experiment was performed with an α-particle so that the force between the incident particles and the target nucleus was repulsive [see Eq. 7.69]. Discuss the possible effect of attractive force on the expression for the scattering cross-section.

PROBLEMS

7.1 Suppose that atomic beams of helium and carbon moving with average speed $|\mathbf{u}|$ in opposite direction are passed through an evacuated tube and studied after collision. What will be the average velocity of the two sets of atoms. Take the collisions to be elastic and masses of helium and carbon atoms as 4 amu and 12 amu respectively.

[*Ans.* $v_{He} = -2\,|\mathbf{u}|$, $v_c = 0$]

7.2 The speed of bullets can be determined with the help of a ballistic pendulum, which consists of a block of mass M suspended with a strong string of length L. When the bullet hits the block, it is stopped within the block and the latter swings through an angle θ to a height H above its equilibrium position (just like a simple pendulum). Find an expression for the velocity of the bullet in terms of its mass m, mass of the ballistic pendulum block M and height H. Use the result so obtained to find the value of v from the following data:

$$m = 25\text{ g},\ M = 2.5\text{ kg, and } H = 5\text{ cm}$$

Hint: The kinetic energy of the block-and-bullet combination is converted into poential energy of the swing of the block]

$$\left[\textit{Ans. } v = \left(1 + \frac{M}{m}\right)\sqrt{2gH}\ ;\ 100\ m/s\right]$$

7.3 In a road accident on a crossing, a car of mass 1000 kg moving with velocity 100 km/h towards east collided with a truck of mass 6000 kg going towards north. The car which struck almost at the centre of the truck got locked with it and the two moved at an angle of 60° with the east. Find the initial velocity of the truck and the fraction of kinetic energy which is carried by the locked system. [*Ans.* 28.6 km/hr; 0.38]

7.4 A particle of mass m_1 moving with velocity u_1 collides with another particle of mass m_2, which is at rest. As a result of the collision, the two particles stick together to form a particle of mass $(m_1 + m_2)$. Find an expression for the kinetic energy of the combined particle and hence show that this is an inelastic collision.

$$\left[\textit{Ans. } T_0' = \frac{m_1}{m_1 + m_2} T_1\right]$$

7.5 A neutron moving with velocity 10^9 cm/s collides elastically with a carbon nucleus at rest. Evaluate their initial velocities in the centre of mass frame. After the collision, the

recoil nucleus is found to move at an angle of 30°. Determine the final velocity and angle of scattering for the neutron in the lab system.

[*Ans.* $|\mathbf{u}'_n| = 9.23 \times 10^8$ cm/s, $|\mathbf{u}'| = -7.7 \times 10^7$ cm/s
$|\mathbf{v}_n| = 8.87 \times 10^8$ cm/s, $\phi = 64° \ 18'$]

7.6 In an experiment, 4MeV protons from the cyclotron are scattered from stationary protons in a target. In the laboratory frame, the two outgoing protons are observed to be moving at right angles to each other. Justify this observation and determine the velocity of the protons observed at 30° to the direction of the centre-of-mass system. [*Hint:* Here $m_1 = m_2$.] [*Ans.* $|\mathbf{v}_1| = 2.4 \times 10^9$ cm/s, $|\mathbf{v}'_1| = 1.02 \times 10^9$ cm/s]

7.7 A particle of mass m_1 and initial velocity $\mathbf{v}_1$ collides elastically with a particle of mass m_2 coming from the opposite direction. As a result of the collision, m_1 moves at right angles to the incident direction with half its initial speed and m_2 moves off at 45° to the incident direction. Find m_2 in terms of m_1. Also, determine the final velocity of m_1 in the centre of mass system
[*Hint:* The velocity of the centre of mass is given by

$$\mathbf{v} = \frac{m_1\mathbf{u}_1 + m_2\mathbf{u}_2}{m_1 + m_2}$$

and not by Eq. (7.17) which assumes m_2 to be at rest].

$$\left[Ans.\ m_2 = \frac{1}{3}\ m_1,\ |\mathbf{v}'_1| = \frac{5}{8}\ |\mathbf{u}_1| \text{ inclined at } 126°\ 52' \text{ with } \mathbf{u}_1\right]$$

7.8 In an experiment on elastic scattering of particles of mass m and velocity $\mathbf{u}$ from target particles of mass M, it is found that the scattered particles bounce back with a speed (9/11) $|\mathbf{u}|$ while the target moved forward with speed (2/11) $|\mathbf{u}|$. Find:
(i) the mass of the target in terms of the projectile mass,
(ii) the total kinetic energy of the two particles, after collision in the CM system, and
(iii) the kinetic energy of the target in the lab system.

[*Ans.* (i) $M = 10\ m$
(ii) (55/121) mu^2
(iii) (20/121) mu^2]

7.9 In an experiment on the scattering of α-particles from paraffin, the scattering angle and initial kinetic energy of α-particles in the centre-of-mass system are 90° and 1.5 MeV respectively. Find the corresponding quantities in the laboratory system. Take mass of α-particles to be four times that of protons. [*Ans.* $\varphi = 14°$ $T_0 = 7.5$ MeV]

7.10 A beam of neutrons is passed through paraffin and the scattered neutrons are studied at an angle θ in the centre-of-mass system. Show that the velocity $|\mathbf{v}_1|$ of the scattered neutrons in the laboratory system is given by

$$|\mathbf{v}_1| = \cos\frac{\theta}{2}\ |\mathbf{u}_1|$$

Also, determine the velocity of the recoil protons in the lab system.
Hint: The mass of the neutrons is nearly the same as that of protons. Use Eqs (7.30), (7.62) and (7.63)].

$$\left[Ans.\ |\mathbf{v}_2| = \sin\frac{\theta}{2}\ |\mathbf{u}_1|\right]$$

7.11 In one of the experiments on the scattering of α-particles, the 5.5 MeV α-particles from ${}^{214}_{83}$Bi were scattered from a foil of silver for which atomic number is 47 and the

atomic weight is 108. Find the impact parameter for the particles scattered at angles equal to or greater than 8°. [*Ans*. 1.82×10^{-13} m]

7.12 Suppose we have an experimental arrangement that can detect α-particles with a minimum angular separation of 2°. It means that the term 'scattered α-particles' will be used for those that are scattered by an angle equal to or more than 2°. Determine the impact parameter corresponding to this angle for 7.0 MeV α-particles being scattered from lead [Z = 82, A = 207]. [*Ans*. 9.9×10^{-13} m]

7.13 $^{210}_{84}$Po emits α-particles of energy 5.3 MeV, which are scattered from iron (Z = 26, A = 56). Find the scattering cross-section corresponding to θ = 20°.
[*Ans*. 1.5×10^{-26} m^2]

7.14 Alpha-particles from a $^{226}_{88}$Ra source with energy 4.8 MeV are scattered from gold (Z = 79, A = 197) film. Determine the scattering cross-section corresponding to the scattering angles of 60° and 120°. [*Ans*. 2.3×10^{-27} m^2, 2.6×10^{-28} m^2]

8

Dynamics of Rigid Bodies

8.1 INTRODUCTION

We have discussed the dynamics of a many-particle system in Chapter 4. In practice, one comes across three types of many-particle systems:

1. *Gases contained in vessels:* Here the positions and momenta of the particles change randomly and therefore only their averages are meaningful. We have already discussed the virial theorem which is applicable to such situations.

2. *Fluids which can be looked upon as continuous media:* Though a fluid consists of many particles, having a combination of random and regular motions, it is easier to deal with it classically as consisting of a continuous medium. A completely new subject called fluid mechanics has been developed for this purpose. We shall, however, deal with it in Chapter 15.

3. *Rigid bodies in which the constituent particles have nearly a fixed distance from each other and execute only small oscillatory motion about their mean positions*: We can, of course, define an idealised rigid body for which the distances between different particles are constant. As we shall see subsequently, such a situation can be handled somewhat easily.

The problems of a rigid body can be divided into two parts:

1. The problem of static equilibrium, in which the external forces operate so as not to change the coordinates of different points in the rigid body. In practice, one comes across such situations in fixing structures involving beams, pillars, walls, etc., so that the whole structure is stable. It is a very interesting problem and is of great concern to a mechanical or civil engineer. We will, however, deal with this problem in Chapter 15.

2. The problem of the dynamics of rigid bodies, i.e. their motion under the influence of external forces. This is the basic concern of this chapter. A rigid body may undergo translation or rotation around an axis (or many axes) passing through the rigid body or revolution around one or many axes outside the body. The purpose of the topic of dynamics of rigid bodies is to obtain the relationships between coordinates, momenta and time under appropriate external forces.

Before considering the effect of an external force or torque on the rigid body, we should understand the role of internal forces acting between the constituent particles of the rigid bodies. We have already seen in Chapter 4 that for a stable many-body

system, which will be the case for a rigid body, the vector sum of the internal forces is zero, i.e.

$$\sum_{ij}{}' \mathbf{F}_{ij}^{\text{int}} = 0 \tag{8.1}$$

This is, of course, expected on general physical grounds because otherwise a stable rigid body will start undergoing motion without an external force. This is against the first law of motion.

Similarly, as shown for many-body systems in Eq. (4.49), the vector sum of torques due to internal forces in a rigid body is zero, i.e.

$$\sum_{ij}{}' \mathbf{r}_{ij} \times \mathbf{F}_{ij}^{\text{int}} = 0 \tag{8.2}$$

This is also expected on physical grounds, from the conservation of the angular momentum for a rigid body in the absence of external torques.

Further, the particles in a rigid body oscillate like a simple harmonic oscillator so that the sum of the kinetic and potential energies is constant, as will be proved in Chapter 9.

We, therefore, conclude that only external forces (for translation) and external torques (for rotation) are effective for the motion of a rigid body.

8.2 ELEMENTARY TREATMENT OF RIGID BODIES

Before we undertake in the next few sections a general discussion on the various dynamical problems of the rigid bodies, we recapatulate in this section, the concepts of dynamics discussed in Chapter 4 and their simple application to rigid bodies.

(a) Torque and Moment of Inertia

We have already discussed the concepts of angular momenta, **L** and torque, $\boldsymbol{\Gamma}$, as applied to a particle or system of particles in Sec. 4.3. All the equations derived for a system of particles are applicable to rigid bodies if one assumes that the angular velocity ω_i for all particles is the same. If there is only one axis of rotation, then the angular velocity $\boldsymbol{\omega}_i$ of all the particles is the same, say $\boldsymbol{\omega}$ and one can write, as in Eq. (4.42)

$$\mathbf{L} = I\boldsymbol{\omega} \tag{8.3}$$

where $I \equiv \sum_i m_i r_i^2$ is called the moment of inertia.

On the other hand, if one has more than one axis of rotation, e.g. in the case of a freely rotating ball, then one has to modify Eq. (4.42). This case is dealt below.

(b) Couple

A quantity called couple which is analogous to the torque is often used in mechanics. Many times two equal and opposite forces, say **F** act on a body separated by a distance $|\mathbf{r}|$ as shown in Fig. 8.1. It is evident that under the influence of the forces, the body will rotate. Because the two forces are equal, there will be no translatory motion. The torque of the two forces, around a point, say P_1 is given

$$\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F} \tag{8.4}$$

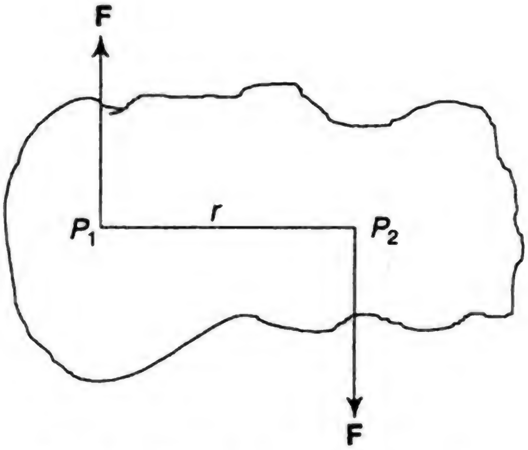


Fig. 8.1 Illustration of a couple

It can be seen that for such a couple, the value of the torque is the same around any point of rotation between P_1 and P_2. In this manner, a couple may be defined as a pair of equal forces which are parallel but opposite to each other, applied to a body with a certain distance between them. The moment of such a couple is given by the product of one of the forces and thc perpendicular distance between them.

EXAMPLE 8.1

The moment of inertia of the earth is 9.8×10^{44} g-cm^2 and its angular velocity is 7.3×10^{-5} rad/s. If one wants to stop it from rotating in one year, how much tangential force should be applied?

Solution

Time needed for the purpose is Δt = 1 year = 3.2×10^7 s. If the earth is stopped from rotation, the change in angular velocity $\Delta\omega = -7.3 \times 10^{-5}$ rad/s. The negative sign means that we are working against the earth's rotation. The angular acceleration of the earth would be

$$|\alpha| = \frac{\Delta\omega}{\Delta t}$$

$$= \frac{-7.3 \times 10^{-5}}{3.2 \times 10^{7}} = -2.3 \times 10^{-12} \text{ rad/s}$$

The torque needed for stopping the earth from rotation is given by

$$|\Gamma| = I\alpha|$$

$$= -9.8 \times 10^{44} \times 2.3 \times 10^{-12}$$

$$= -2.3 \times 10^{33} \text{ dynes-cm}$$

The radius of the earth, R *is* 6.4×10^8 cm. The tangential force required is

$$|\mathbf{F}| = \frac{|\Gamma|}{R}$$

$$= \frac{-2.3 \times 10^{33}}{6.4 \times 10^{8}} = -3.6 \times 10^{24} \text{ daynes}$$

Thus if one can apply 3.6×10^{24} dynes of force against the rotation of the earth, one can stop it from rotating in one year.

(c) Work Done by Torque

It can be easily shown that in the case of a rotating rigid body around one axis, the work done by force F in moving a mass-point on the rigid body from S_1 to S_2 (Fig. 8.2) is given by

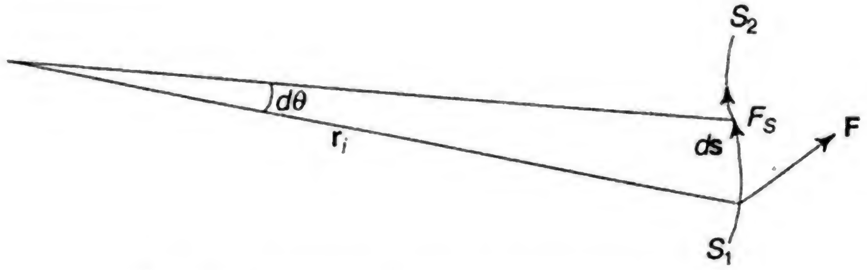


Fig. 8.2 Representation of work done by torque

$$\int_{s_1}^{s_2} \mathbf{F} \cdot \mathbf{ds} = \int_{s_1}^{s_2} F_s \cdot ds$$

$$= \int_{\theta_1}^{\theta_2} F_s r_i d\theta \tag{8.5}$$

where F_s denotes the component of force along distance **ds** and r_i is the distance of the point mass from the axis of rotation.

Now $$\Gamma_i = F_s r_i \tag{8.6}$$

is the torque of the force about the axis and perpendicular to both F_s and r_i. Therefore, the work done on ith point mass is given by

$$W_i = \int_{\theta_1}^{\theta_2} F_s r_i d\theta = \int_{\theta_1}^{\theta_2} \Gamma_i d\theta \tag{8.7}$$

Though the torque Γ_i is a vector quantity, it has the same direction if F_s is the same for all points, as will be the case in a rigid body. Hence the torques for all the mass points can be added arithmetically to give the total torque. In other words,

$$\Gamma = \sum_i \Gamma_i \tag{8.8}$$

Hence work done on the whole body is given by

$$W = \sum_i W_i = \int_{\theta_1}^{\theta_2} \left(\sum_i \Gamma_i \right) d\theta = \int_{\theta_1}^{\theta_2} \Gamma \, d\theta \tag{8.9}$$

It may be mentioned that while the value of Γ_i may vary from point to point, $d\theta$ is the same for all the mass points.

Now $$\Gamma = I\alpha = I\omega \frac{d\omega}{d\theta} \tag{8.10}$$

$\therefore$ $$W = \int_{\theta_1}^{\theta_2} I\omega \frac{d\omega}{d\theta} \, \mathrm{d}\theta = \int_{\omega_1}^{\omega_2} I\omega \, d\omega \tag{8.11}$$

where ω_1 and ω_2 are angular velocities at angles θ_1 and θ_2.

Hence
$$W = \left|\frac{1}{2} I\omega^2\right|_{\omega_1}^{\omega_2}$$

$$= \frac{1}{2} I\left(\omega_2^2 - \omega_1^2\right) \tag{8.12}$$

This is the work done when the angular velocity is changed from ω_1 to ω_2. If a rotating body is stopped when it was rotating with an angular velocity ω, then it can be easily seen that work done is given by

$$W = -\frac{1}{2} I\omega^2 \tag{8.13}$$

The minus sign shows that the work is done by the body. Conversely, if starting from rest, a rotating body acquires an angular velocity ω, then work done on the body is $\frac{1}{2} I\omega^2$. It shows that in analogy to the linear case, the work in this case also equals the increase in kinetic energy.

(d) Moments of Inertia of Different Bodies

How do we calculate the moment of inertia of different symmetrical bodies such as a cylinder, sphere, rectangular bar, disc, hollow cylinder, etc? We will see below that it is possible to calculate their moment of inertia around any axis, in terms of the parameters, such as radii, length, breadth, etc. of these bodies. All these bodies have a symmetry of shape around a point which is generally the centre of mass or an axis and that helps in calculation. If we have an irregular body so that there is no point or axis in the body around which the body has any symmetry, even then the moment of inertia can be obtained by taking a large number of mass elements and finding for each element, the value of $m_i r_i^2$ and then numerically adding them up. Each mass element should be as small in size as possible.

The basis of calculating the moments of inertia of regular bodies is, of course, the formula

$$I = \sum_i m_i r_i^2 \tag{8.14}$$

If the body has uniform density and symmetrical shape, we can replace the summation in the above equation by an integral in the following manner: As stated above, one should divide the body into a larger number of mass elements, the size of each element being as small as possible. One should then be able to write

$$I = \sum_{\substack{i \\ m_i \to 0}} m_i r_i^2 \tag{8.15}$$

As $m_i \to 0$, we express this a little differently to make it more instructive. We express each mass element written till now as m_i by Δm_i. In this way we bring out the smallness of the mass element which can tend towards zero. Then for one element

$$\Delta I = \Delta m_i r_i^2 \tag{8.16}$$

and for the whole body,

$$I = \sum_i \Delta m_i r_i^2 \tag{8.17}$$

Sometime, it is convenient to write this equation in such a manner that the whole mass of the body appears separately, e.g.

$$I = \sum_i m_i r_i^2 = \left(\sum_i \Delta m_i\right) K^2 = MK^2 \tag{8.18}$$

Then K *is* called the radius of gyration. One can, therefore, express K^2 as

$$K^2 = \frac{\sum_i \Delta m_i r_i^2}{M} \tag{8.19}$$

Physically, it means that if we assume the whole mass of the body concentrated at distance K from the axis, then the moment of inertia of this imaginary case will be the same as that of the real body. One should remember that when $\Delta m_i \to 0$, the total number of points is very large, which may be expressed as $i \to \infty$. In this limit, we may express the moment of inertia of an element as

$$dI = r^2 dm$$

or

$$I = \int r^2 dm \tag{8.20}$$

where dI is the moment of inertia of the mass element dm. For a uniform density ρ, $dm = \rho dV$, where dV is the volume element of mass dm.

Hence

$$dI = r^2 \rho dV$$

$$I = \int r^2 \rho dV = \rho \int r^2 dV \tag{8.21}$$

We have brought the density ρ outside the integral because it is independent of the position of the mass element. For obtaining the expression for the moment of inertia for different cases, one obtains the expression for $\int r^2 dV$ in a suitable manner.

These steps can be summarised as follows:

1. Select a mass element dm in such a manner that the addition of such mass elements (by varying one of the parameters) makes the whole mass.

2. This mass element should have a certain symmetry with respect to the axis of rotation, so that its moment of inertia dI around that axis can be represented by an expression containing one variable. Generally, it should be possible to write

$$dI = dm r^2$$

and

$$dm = \rho dV \tag{8.22a}$$

where ρ is the density and dV is the volume element. Alternatively, one may write

$$dm = \sigma dl \tag{8.22b}$$

where σ is the mass per unit length and dl the length of the element.

3. Then $I = \sigma \int r^2 dl$ gives the total moment of inertia. The limits of integration should vary between the physical limits of the body under consideration.

(e) Moments of Inertia of Different Symmetrical Bodies

Before discussing some actual cases, we will state and explain two theorems for the relationship of moments of inertia around two parallel or perpendicular axes. These theorems are applicable basically for lamina or plane surfaces, and are useful for deriving the moment of inertia of light bars, or discand so forth.

(i) Theorem of Perpendicular Axes

This is a theorem applicable to the case where one wants to find the moment of inertia of a plane surface like the disc, lamina and rectangular bars and so forth. It states that the moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of moments of inertia of the lamina about any two perpendicular axes in its plane, intersecting each other at the point through which the perpendicular axis passes.

It may be emphasized that this theorem does not apply to spherical or cylindrical bodies like a sphere or a cylinder.

Let us give its proof. It is intended to find the moment of inertia of the whole lamina around the perpendicular axis Oz. Take any point P at a distance r_i from O, on the lamina and also draw the x- and y-axes in the plane of the lamina, as shown in Fig. 8.3.

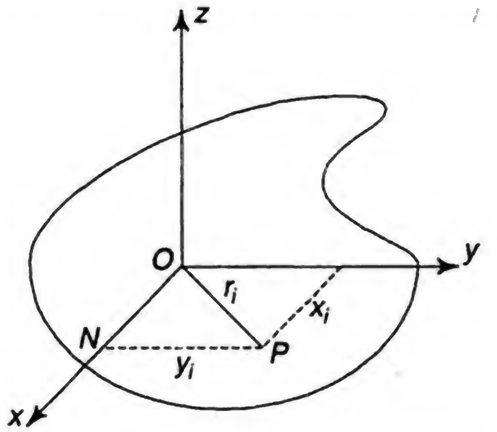


Fig. 8.3 The theorem of perpendicular axes

The moment of inertia of mass element Δm_i at P around Oz is given by

$$dI_z = \Delta m_i \,(OP)^2 = \Delta m_i r_i^2 \qquad (8.23)$$

where x_i is the perpendicular distance of P from axis OY and y_i is the perpendicular distance of P from axis OX. For the whole body, we get

$$I_z = \sum_i \Delta m_i r_i^2$$

$$I_z = \sum_i \Delta m_i x_i^2 + \sum_i \Delta m_i y_i^2 \qquad (8.24)$$

Now $\sum_i \Delta m_i x_i^2 = I_y$ is the moment of inertia of the whole body around y-axis and $\sum_i \Delta m_i y_i^2 = I_x$ is the moment of inertia of the whole body around x-axis. $\sum_i \Delta m_i r_i^2 =$ I_z is the moment of inertia of the whole body around z-axis. Thus

$$I_z = I_x + I_y \qquad (8.25)$$

Eq. (8.25) is the statement of the theorem of perpendicular axes.

(ii) Theorem of Parallel Axes

According to this theorem, the moment of inertia of a body about an axis is equal to its moment of inertia about a parallel axis passing through its center of mass, plus the product of mass of the body and the square of distance between the two axes. The axis of rotation may be in the plane of lamina or perpendicular to it. This theorem is applied not only to the case of a lamina but also to a cylinder. It is generally very useful for the cases when the axis of rotation is perpendicular to the length of lamina or cylinder.

We prove the theorem by taking the axis perpendicular to lamina. Now draw an axis OZ, passing through the center of mass of the lamina and $O'Z'$ at a distance d from O, the center of mass but parallel to the axis OZ. We take an arbitrary point P

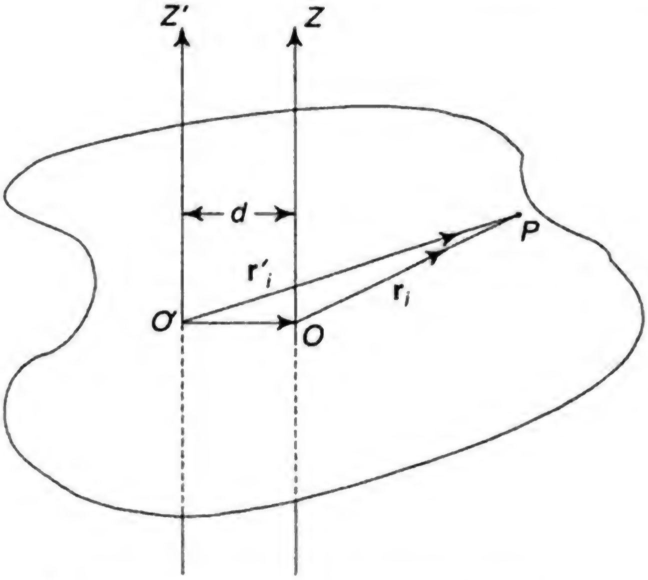


Fig. 8.4 The theorem of parallel axes

at a distance r_i from OZ and r'_i from $O'Z'$ (Fig. 8.4); then the moment of inertia of a mass element at point P, about OZ is given by $dI = \Delta m_i r_i'^2$. Therefore, the moment of inertia for the whole body around OZ is given by $I = \sum_i \Delta m_i r_i'^2$. Similarly, the moment of inertia of mass element at P around $O'Z'$ is given

$$\begin{aligned} I' &= \sum_i \Delta m_i r_i'^2 \\ &= \sum_i \Delta m_i (\mathbf{r}_i + \mathbf{d})^2 \\ &= \sum_i \Delta m_i (\mathbf{r}_i + \mathbf{d}) \cdot (\mathbf{r}_i + \mathbf{d}) \\ &= \sum_i \Delta m_i r_i^2 + 2 \sum_i \Delta m_i \mathbf{r}_i \cdot \mathbf{d} + \sum_i \Delta m_i d^2 \\ &= I + Md^2 + 2\sum_i \Delta m_i \mathbf{r}_i \cdot \mathbf{d} \end{aligned}$$

The value of $\sum_i \Delta m_i \mathbf{r}_i \cdot \mathbf{d} = 0$, because the product of $\mathbf{r}_i \cdot \mathbf{d} = r_i d \cos\theta$ is symmetric around OZ so that for every point with a positive value for $\cos\theta$, there will be a point with a negative value, and thus the sum will be zero.

Therefore,
$$I' = I + Md^2 \tag{8.26}$$

Eq. (8.26) is a statement of the theorem of parallel axes.

Some typical examples illustrating the methods of derivation of moment of inertia for symmetric bodies are given below.

(i) *Moment of inertia of a rectangular lamina: Axis perpendicular to length* We imagine a lamina with length a, breadth b and negligible thickness as shown in Fig. 8.5. We want to find the moment of inertia of such a lamina around an axis YY' which is perpendicular to a, but parallel to b and passes through O, the centre of mass.

Let us consider the mass element shown shaded in the Fig. 8.5 with a thickness dx and length b. Its area is given by bdx. Let σ be the mass of the lamina per unit length, i.e.

$$\sigma = \frac{M}{a}$$

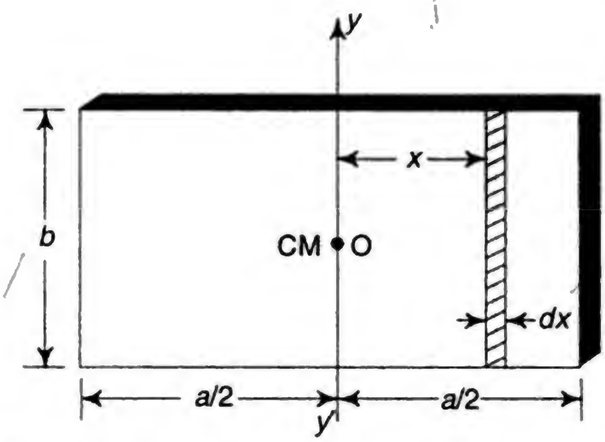


Fig. 8.5 Moment of inertia of a rectangular lamina around an axis perpendicular to length

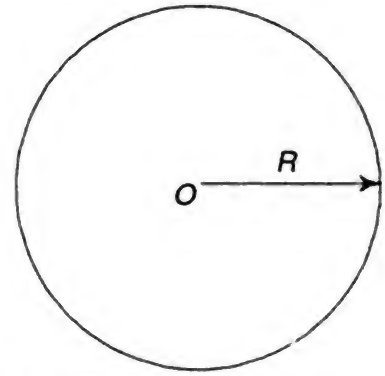


Fig. 8.6 Moment of inertia of a ring around an axis passing through the centre, perpendicular to the plane of the ring

Then moment of inertia of the mass element dI_y around YY' is given by

$$dI_y = \left(\frac{M}{a}dx\right)x^2$$

Therefore, for the whole lamina, it is given by

$$I_y = \frac{M}{a}\int_{-a/2}^{a/2} x^2 dx = \frac{M}{a}\left[\frac{x^3}{3}\right]_{-a/2}^{+a/2} = \frac{Ma^2}{12} \tag{8.27}$$

(ii) Moment of inertia of a ring: Axis passing through the centre and perpendicular to its plane (*Fig. 8.6*): Let the rim of the ring have any regular shape, i.e. its cross-section may be circular, rectangular or elliptical, etc. However, this shape must remain the same throughout. We take a small element dl of the ring. If σ is the mass per unit length of the rim of the ring, then $\sigma dl = dm$, will be the mass of the mass element. The moment of inertia of the mass element around the axis is then given by

$$dI = R^2 dm = R^3 \sigma dl$$

For the whole ring, the moment of inertia around the axis perpendicular to the plane of the ring and passing through the centre is then given by

$$\begin{aligned} I &= R^2\sigma \int dl \\ &= R^2\sigma(2\pi R) = MR^2 \end{aligned} \tag{8.28}$$

We have used the relation $(2\pi R)\ \sigma = M$ = the total mass of the ring. It may be noted that in this case it was not necessary to obtain the expression of dI in terms of dV. Instead, we obtained it in terms of dl. One should further realise that the mass

element has been taken in such a manner that it is symmetrical with respect to the axis of rotation and the value of R^2 measured from any mass element to the axis of rotation is the same for all mass elements.

(iii) Moment of inertia of a solid circular disc: Axis perpendicular to its plane and passing through centre: Let the disc have density ρ and the radius R. As will be seen below, it is not important to know its thickness as long as it is uniform. However let us still assume that it has a thickness t (Fig. 8.7). Let us draw a ring of radius x and width dx, as in Fig. 8.7. The area of the whole ring will be then given by $2\pi x dx$ and volume will be $2\pi t x dx$.

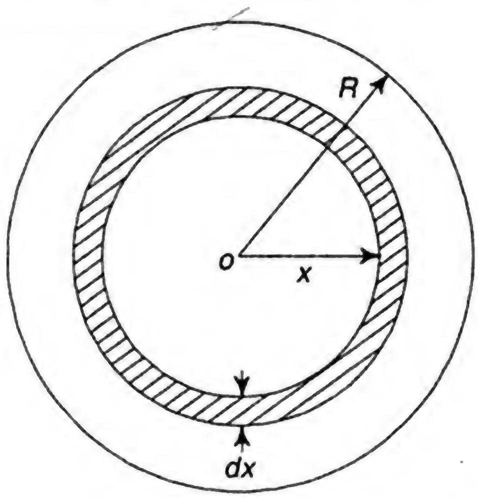


Fig. 8.7 Moment of inertia of a solid circular disc, around an axis perpendicular to its plane passing through the centre

The mass of this ring will therefore be given by

$$dm = \rho(2\pi t x dx) = \rho dV$$

This ring has been selected in such a manner that each small element in the ring is symmetrical about the axis. The moment of inertia of the ring around the axis may, therefore, be written as

$$dI = x^2 \rho dV = x^2 \rho (2\pi x dx)\, t$$
$$= \rho(2\pi t)\, x^3 dx$$

Therefore, the total moment of inertia of the disc can be taken as the sum of the moments of inertia of such rings whose radii may lie anywhere from $x = 0$ to $x = R$. If dx is taken very small approaching zero, we can use integral instead of summation.

Therefore,

$$I = \int dI = 2\pi t \rho \int_{x=0}^{x=R} x^3 dx$$
$$= \pi t \rho R^4/2$$

Now the mass of the whole disc is given by

$$M = \pi R^2 t \rho$$

Therefore $$I = \frac{MR^2}{2} \tag{8.29}$$

It may be mentioned that in the final expression for I, the value of t does not enter. As a matter of fact, one could have used the concept of mass per unit area and solved the above problem without using t at all. That will give the same result.

EXAMPLE 8.2

A thin uniform disc of radius 25 cm and mass 1 kg has a hole of radius 5 cm at a distance of 10 cm from the center of the disc. Calculate the moment of inertia of the disc about an axis perpendicular to the plane and passing through the center of the hole.

Solution

Let M_1 and M_2 stand for the mass of the disc and of circular hole respectively. Then

$$M_1 = \pi(25)^2\,\sigma$$

and

$$M_2 = \pi(5)^2\,\sigma$$

where σ is the mass per unit area. Let G be center of gravity of the disc, then

$$GO = x$$
$$GO' = 10 - x$$

Taking moments of M_1 and M_2 placed at O and O', about G; we get

$$M_1 x = -(10 - x)\,M_2$$

or

$$x = -\frac{10}{24}\ \text{cm}$$

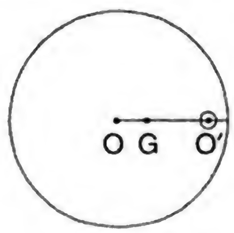


Fig. E8.2

so the center of gravity of the disc is to the left of O.

Now

$$I = \frac{1}{2}MR^2 = \frac{1}{2}\left(1000 \times \frac{600}{625}\right) \times \left(25 - \frac{10}{24}\right)^2$$
$$= 2.9 \times 10^5\ \text{gm cm}^2.$$

(iv) *Uniform slender rod: Axis perpendicular to length:* We refer to Fig. 8.8 for various quantities. Here the axis of rotation RR' is not passing through the centre of mass but at point A at an orbitrary distance h from one end. Select an element of length dx at a distance x from the axis.

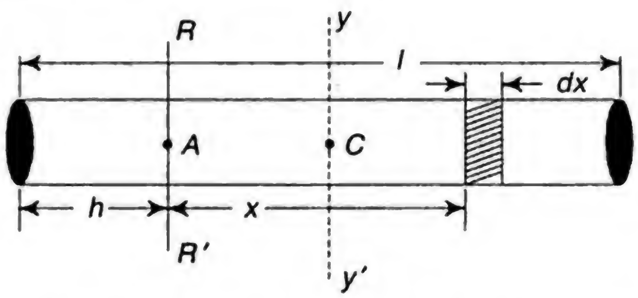


Fig. 8.8 Moment of inertia of a uniform slender rod, with axis perpendicular to length

Its mass dm is given by

$$dm = \rho dV = \rho s dx = \frac{\rho s l}{l}\,dx = \rho\,\frac{V}{l}\,dx = \frac{M dx}{l}$$

where s = cross-sectional area of the rod, l is the length of the rod and M is the mass of the whole rod.

The moment of inertia is, therefore, obviously given by

$$I = \int_{-h}^{l-h} x^2 dm$$

$$= \frac{M}{l} \int_{-h}^{l-h} x^2 dx$$

$$I = \frac{M}{3} (l^2 - 3lh + 3h^2) \tag{8.30}$$

If the axis of rotation is passing through the centre, $h = (1/2)\, l$, then

$$I = \frac{1}{12} Ml^2 \tag{8.31}$$

If the axis is passing through the left end, $h = 0$. Hence from Eq. (8.30), we get

$$I = \frac{1}{3} Ml^2 \tag{8.32}$$

If the axis is passing through the right end, $h = l$, and again from Eq. (8.30), we get

$$I = \frac{1}{3} Ml^2$$

We want to emphasise two points in these derivations:

1. The derivation of the moment of inertia as given in Eq. (8.30) does not assume any special shape for the cross-section of the rod. We have only assumed that the area of this cross-section is constant throughout, i.e. the rod is uniform. Therefore, Eqs (8.31) and (8.32) hold good for any type of a long rod as long as it is uniform, e.g. for a cylindrical rod, or with rectangular, oblong or even an irregular cross-section. The rod, however, should be long and slender because we have assumed that dx of the mass element is much smaller than the length of the rod. Further, we have assumed that the moment of inertia of the mass element is dependent on x^2, which means that the width of the cross-section of each mass element are much smaller than x.

2. Equations (8.31) and (8.32) illustrate the theorem of moment of inertia for parallel axes. The moment of inertia I around an axis at any end may also be written as:

$$I = I_c + Md^2 \tag{8.33}$$

where I_c is the moment of inertia around the centre of mass.

Here $$I_c = \frac{1}{12} Ml^2 \quad \text{and} \quad d = (1/2)\, l$$

$$I = \frac{1}{12} Ml^2 + \frac{1}{4} Ml^2 = \frac{1}{3} Ml^2$$

This is the same result as given in Eq. (8.32).

(v) *Uniform solid sphere: Axis passing through centre:* We refer to Fig. 8.9. Let x-axis be the axis of rotation of the sphere. The mass element in this case is a disc of thickness dx at a distance x from the centre. Let r be the radius of the disc. Then it is easy to see from the diagram that

$$r = (R^2 - x^2)^{1/2}$$

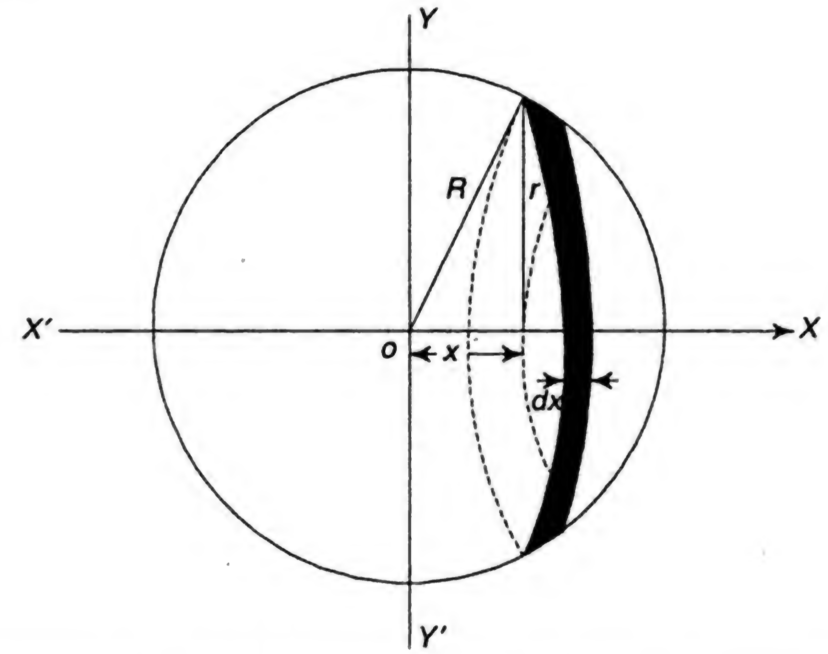


Fig. 8.9 Moment of inertia of a uniform solid sphere, with axis passing through centre

$$dV = \pi r^2 dx$$

Therefore, $$dm = \rho dV = \pi\rho(R^2 - x^2)\, dx$$

The moment of inertia for this disc representing the mass element dm around an axis passing through the centre and perpendicular to the plane of the disc is, therefore, given by [see Eq. (8.29)]

$$dI = \frac{r^2}{2}\, dm = \frac{\pi}{2}\, \rho(R^2 - x^2)\, dx$$

It may be seen that adding such thin discs with thickness dx will make a sphere. We can, therefore, write the total moment of inertia I of the whole sphere as

$$I = \frac{\pi\rho}{2} \int_{-R}^{R} (R^2 - x^2)^2\, dx$$

$$= \frac{2\pi\rho}{2} \int_{0}^{R} (R^2 - x^2)^2\, dx$$

$$= \pi\rho \left| R^4 x - \frac{2R^2 x^3}{3} + \frac{x^5}{5} \right|_0^R$$

$$= \pi\rho \left[R^5 - \frac{2R^5}{3} + \frac{R^5}{5} \right] = \frac{8}{15}\, \pi\rho R^5$$

Now remembering that the mass of the whole sphere is given by

$$M = \rho V = \frac{4\pi R^3 \rho}{3}$$

we can write for a uniform solid sphere

$$I = \frac{2}{5} MR^2 \tag{8.34}$$

8.3 ANGULAR MOMENTUM OF A RIGID BODY AND INERTIA TENSOR

We are now ready to discuss the general case. In the previous section, we have discussed a special and simple case of rotation around one axis. In that case, the expressions for angular momentum; and the moment of inertia were quite simplified. In general, however, a rigid body may rotate in a complicated way, in which case; one may analyse the motion in terms of three axes of rotation. The expressions for angular momentum; the moment of inertia and their relationships become quite complex; and require newer mathematical concepts like matrices. We will discuss their general case in this and the subsequent section.

We have already seen in Eq. (4.41) that in a many-body system, the angular momentum of the system, in general, is given by

$$\mathbf{L} = \sum_i m_i r_i^2 \boldsymbol{\omega}_i - \sum_i m_i \mathbf{r}_i (\mathbf{r}_i \cdot \boldsymbol{\omega}_i) \tag{8.35}$$

where summation is carried over all the particles. The special case of rotation around a fixed axis, corresponding to $\mathbf{r}_i \cdot \boldsymbol{\omega}_i = 0$, has also been discussed [Eq. (4.42)]. Here we want to discuss the general case for which $\mathbf{r}_i \cdot \boldsymbol{\omega}_i \neq 0$. For this purpose, we consider the xyz coordinate system fixed in the body and write Eq. (8.35) in the component form. Since $\boldsymbol{\omega}$ is the same for all the particles, the subscript i is redundant for it. We can then write the three components of $\mathbf{L}$ as follows:

$$\begin{aligned} L_x &= \sum_i m_i r_i^2 \omega_x - \sum_i m_i x_i (x_i \omega_x + y_i \omega_y + z_i \omega_z) \\ &= \sum_i (m_i r_i^2 \omega_x - m_i x_i^2 \omega_x) - \sum_i m_i x_i y_i \omega_y - \sum_i m_i x_i z_i \omega_z \\ &= \sum_i [m_i (r_i^2 - x_i^2)\, \omega_x - m_i x_i y_i \omega_y - m_i x_i z_i \omega_z] \end{aligned} \tag{8.36a}$$

Similarly
$$L_y = \sum_i [-m_i y_i x_i \omega_x + m_i (r_i^2 - y_i^2)\, \omega_y - m_i y_i z_i \omega_z] \tag{8.36b}$$

and
$$L_z = \sum_i [-m_i z_i x_i \omega_x - m_i z_i y_i \omega_y + m_i (r_i^2 - z_i^2)\, \omega_z] \tag{8.36c}$$

One can write Eq. (8.36) in a brief form as

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \tag{8.37a}$$
$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \tag{8.37b}$$
$$L_z = L_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \tag{8.37c}$$

where
$$I_{xx} = \sum_i m_i (r_i^2 - x_i^2) = \sum_i m_i (y_i^2 + z_i^2) \tag{8.38a}$$

$$I_{yy} = \sum_i m_i (r_i^2 - y_i^2) = \sum_i m_i (x_i^2 + z_i^2) \tag{8.38b}$$

$$I_{zz} = \sum_i m_i (r_i^2 - z_i^2) = \sum_i m_i (x_i^2 + y_i^2) \tag{8.38c}$$

$$I_{xy} = -\sum_i m_i x_i y_i = I_{yz} \tag{8.38d}$$

$$I_{yz} = -\sum_i m_i y_i z_i = I_{zy} \tag{8.38e}$$

$$I_{zx} = -\sum_i m_i z_i x_i = I_{xz} \tag{8.38f}$$

Equation (8.37) can also be written in the marix form as

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{8.39}$$

Equation (8.37) or (8.39) can be expressed in a more compact form by using the symbols 1, 2, 3 for x, y, z, respectively. This gives

$$L_\mu = \sum_{\nu=1}^{3} I_{\mu\nu}\omega_\nu,\ \mu = 1, 2 \text{ and } 3 \tag{8.40}$$

The above equation can be further put into elegant vector form as

$$\mathbf{L} = \mathbf{I}\ \boldsymbol{\omega} \tag{8.41}$$

Here ω is vector with three components ω_x, ω_y, ω_z and **I** is tensor of rank two with nine components:

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \tag{8.42}$$

The tensor **I** is called the moment-of-inertia tensor or simply the inertia tensor and $I_{\mu\nu}(\mu,\nu$ for 1, 2, 3 or x, y, $z)$ are its nine elements. The elements $I_{\mu\mu}$ or I_{xx}, I_{yy} and I_{zz} are called the principal moments of inertia around the x-axis, y-axis and z-axis respectively, while $I_{\mu u}$ or I_{xy}, I_{xz} and I_{yz} are called the products of inertia. Equations (8.38)–(8.40) relate, in a general manner the angular momenta L_μ's with different elements of the inertia tensor and the components of angular velocity.

Some of the properties of the moment-of-inertia tensor are listed below:

1. The moment-of-inertia tensor is symmetric, i.e. the elements of the inertia tensor for all μ and ν obey the relation

$$I_{\mu\nu} = I_{\nu\mu} \tag{8.43}$$

This is true both for regular and irregular bodies. Its validity is easily seen from the basic definitions as given in Eq. (8.38). An implication of this property is that there are only six independent components, i.e. I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{xz} and I_{yz}.

2. One can define axes xyz in the body in such a way that the products of inertia $I_{\mu\nu}$ are zero for all μ, ν $(\mu \neq \nu)$. Such axes are called principal axes of inertia and the

description of this case requires only three components I_{xx}, I_{yy}, I_{zz} which are then written as I_x, I_y, I_z.

3. For a rigid body with cylindrical symmetry, the axis of the cylinder may be taken as the principal z-axis and x- and y-axes are symmetric. Then

$$I_{xx} = I_{yy}$$

or
$$I_x = I_y \tag{8.44}$$

Any rigid body other than that having cylindrical shapes which satisfy Eq. (8.44) is called a symmetric top.

4. For a sphere, all three axes are symmetric. Therefore,

$$I_{xx} = I_{yy} = I_{zz}$$

or
$$I_x = I_y = I_z \tag{8.45}$$

A rigid body satisfying this condition is called a spherical top.

5. If $I_x \neq I_y \neq I_z$, the rigid body is labelled as an asymmetric top.

6. A body for which

$$I_x = I_y \quad \text{and} \quad I_z = 0 \tag{8.46}$$

is called a rotor and is exemplified by a diatomic molecule.

It has been mentioned above that for the principal axes, $I_{xy} = I_{yz} = I_{zx} = 0$. Now $I_{xy} = \sum_i m_i x_i y_i$ and the zero value of this sum means that the expression contains positive and negative terms which cancel each other. This implies that corresponding to a positive y_i there is negative y_i for the same x_i. Similar conclusions are drawn from $I_{yz} = I_{zx} = 0$. Such a situation can arise only if the body is symmetrical about the axes of rotation.

Also, when the symmetric body is rotating around the x-axis, y and z components, ω_y and ω_z of angular velocities are zero so that $\boldsymbol{\omega}_x = \omega_x \mathbf{i}$, and $L_x = I_{xx}\omega_x$; $L_y = L_z = 0$ so that $\mathbf{L} = I_{xx}\omega_x\mathbf{i}$. Accordingly, the angular momentum has the same direction as that of rotation. Similarly, for rotations around y- and z-axes $\boldsymbol{\omega}_y = \omega_y\mathbf{j}$ and $\boldsymbol{\omega}_z = \omega_z\mathbf{k}$ respectively. In these cases also, the angular momenta are parallel to angular velocities. For a general axis of rotation having any orientation with respect to the principal axes of the body, the angular momentum is given by Eq. (8.41).

If the body is irregular, then there are no symmetry axes and the full set of equations as given in Eq. (8.37) are used. However, if we choose three perpendicular axes of rotation along x-, y- and z-axes; then condsidering, say, x-axis for rotation $\boldsymbol{\omega} = \omega_x\mathbf{i}$ and $\omega_y = \omega_z = 0$, and from Eq. (8.37), we can write

$$L_x = I_{xx}\omega_x;\ L_y = I_{xy}\omega_x \text{ and } L_z = I_{zx}\omega_x \tag{8.47}$$

This means that in such a case, the angular momentum $L = L_x\mathbf{i} + L_y\mathbf{j} + L_z\mathbf{k}$ and angular velocity $\boldsymbol{\omega} = \omega_x\mathbf{i}$ are not in the same direction. Similar comments hold for rotation, around y- and z-axes.

EXAMPLE 8.3

Consider a cube of mass M, each side a and density ρ and define a coordinate system with the origin at one corner and three axes along the three adjacent edges of the cube. Calculate the inertia tensor for this cube with respect to this coordinate system.

Solution

For a collection of masses m_i, the components of the inertia tensor are given by Eq. (8.38). However, a rigid body contains such a large number of closely packed atoms that summation can be replaced by integration.

If the density of the material is ρ, then the mass of an element at (x, y, z) having volume $dxdydz$ will be $\rho\, dxdydz$. Therefore,

$$\Delta I_{xx} = \rho\, dxdydz\, (y^2 + z^2)$$

The component for the whole body will be

$$I_{xx} = \int \Delta I_{xx} = \iiint \rho\, dxdydz\, (y^2 + z^2)$$

Similarly, the other components of the inertia tensor become

$$I_{yy} = \iiint \rho(z^2 + x^2)\, dxdydz$$

$$= I_{zz} \iiint \rho(x^2 + y^2)\, dxdydz$$

$$I_{xy} = I_{yx} = -\iiint \rho xy\, dxdydz$$

$$I_{yz} = I_{zy} = -\iiint \rho yz\, dxdydz$$

$$I_{zx} = I_{xz} = -\iiint \rho zx\, dxdydz$$

In the given problem, the coordinate system and cube are as shown in Fig. 8.10.

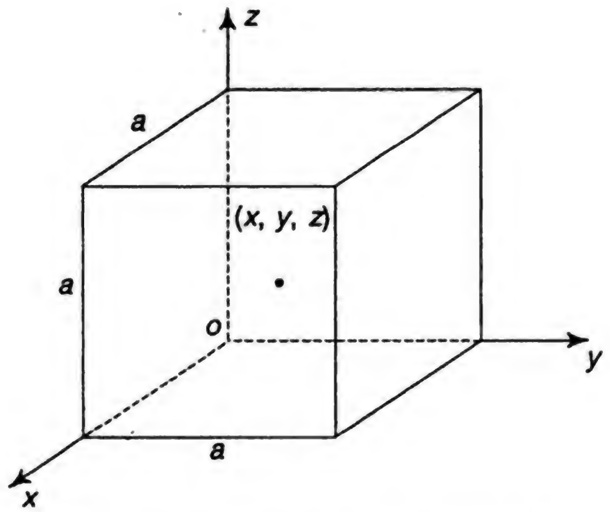


Fig. 8.10 Diagram illustrating Example 8.1

The limits of integration over x, y, z are 0 to a. Also ρ is constant so that it can be taken out of the integral sign. Thus

$$I_{xx} = \rho \int_0^a\int_0^a\int_0^a (y^2 + z^2)\, dxdydz$$

$$= \rho \int_0^a dx \left[\int_0^a y^2 dy \int_0^a dz + \int_0^a dy \int_0^a z^2 dz\right]$$

$$= \rho a\, [(a^3/3)\, a + a \cdot (a^3/3)]$$

$$= \rho a^3\, (2a^2/3)$$

$$= 2/3\, Ma^2$$

because a^3 is the volume of the cube and $\rho a^3 = M$, the mass of the cube.
Similarly

$$I_{yy} = \rho \int_0^a dy \left[\int_0^a dx \int_0^a z^2 dz + \int_0^a dz \int_0^a x^2 dx \right]$$

$$= \rho a[a \cdot (a^3/3) + a\,(a^3/3)]$$

$$= 2/3\ Ma^2$$

$$I_{zz} = \rho \int_0^a dz \left[\int_0^a x^2 dx \int_0^a dy + \int_0^a dx \int_0^a y^2 dy \right]$$

$$= \rho a[(a^3/3)\, a + a\,(a^3/3)]$$

$$= 2/3\ Ma^2$$

$$I_{xy} = I_{yx} = -\rho \int_0^a xdx \int_0^a ydy \int_0^a dz$$

$$= -\rho(a^2/2)\,(a^2/2)\, a$$

$$= -\frac{1}{4} Ma^2$$

$$I_{yz} = I_{zy} = -\rho \int_0^a dx \int_0^a ydy \int_0^a zdz$$

$$= -\rho a \cdot \frac{a^2}{2} \cdot \frac{a^2}{2}$$

$$= -\frac{1}{4} Ma^2$$

$$I_{zx} = I_{xz} = -\rho \int_0^a xdx \int_0^a dy \int_0^a zdz$$

$$= -\rho \frac{a^2}{2} \cdot a \cdot \frac{a^2}{2}$$

$$= -\frac{1}{4} Ma^2$$

Hence the inertia tensor is

$$\bar{\mathbf{I}} = Ma^2 \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix}$$

Note: Since the cube is symmetric with respect to its centre, it should be possible to define the principal axes for this. In fact, the coordinate system with centre of the cube as the origin and axes parallel to the faces constitutes the principal coordinate system (see Problems at the end of this chapter).

8.4 ANGULAR MOMENTA AND ROTATIONAL KINETIC ENERGY

If the linear velocity of the ith particle of mass m_i due to its rotational velocity $\boldsymbol{\omega}$ is

$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$, then its kinetic energy is given by $\frac{1}{2} m_i |\mathbf{v}_i|^2$ so that the rotational kinetic energy of the rigid body takes the form

$$T = \sum_i \frac{1}{2} m_i |\mathbf{v}_i|^2 \tag{8.48}$$

Now, from vector algebra, it is known that

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{A} \cdot [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] \tag{8.49}$$

so that

$$\begin{aligned} 2T &= \sum_i m_i \boldsymbol{\omega} \cdot [\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)] \\ &= \sum_i \boldsymbol{\omega} \cdot m_i [\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)] \\ &= \sum_i \boldsymbol{\omega} \cdot m_i [\mathbf{r}_i \times \mathbf{v}_i] = \boldsymbol{\omega} \cdot \sum_i m_i (\mathbf{r}_i \times \mathbf{v}_i) \\ &= \boldsymbol{\omega} \cdot \mathbf{L} \end{aligned} \tag{8.50}$$

or

$$\begin{aligned} T &= \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L} \\ &= \frac{1}{2} \sum_\mu \omega_\mu L_\mu \\ &= \frac{1}{2} \sum_{\mu\nu} \mathrm{I}_{\mu\nu} \omega_\nu \omega_\mu \end{aligned}$$

since μ, ν can have values x, y, z, the above expression can be written as

$$T = \frac{1}{2} [I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2I_{xy}\omega_x\omega_y + 2I_{yz}\omega_y\omega_z + 2I_{xz}\omega_x\omega_z] \tag{8.51}$$

If the symmetric rigid body is rotating around, say the x-axis, passing through the centre of mass, then

$$\boldsymbol{\omega} = \omega_x \mathbf{i};\ \omega_y = 0, \quad \omega_z = 0, \quad \text{and} \quad I_{xy} = I_{yz} = 0$$

For this case,

$$T = \frac{1}{2} (I_{xx}\omega_x^2) \tag{8.52}$$

Combining the relationships given in Eqs (8.50) and (8.41), we can also write:

$$T = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I}\, \boldsymbol{\omega} \tag{8.53}$$

As mentioned in Sec. 8.3, when the axes are the principal axes, I_{xx}, I_{yy}, I_{zz} can be written as I_x, I_y, I_z and $I_{xy} = I_{xz} = I_{yz} = 0$. Therefore, in this case

$$T = \frac{1}{2} (I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2) \tag{8.54}$$

EXAMPLE 8.4

A cube of mass M and each side a is rotating with angular velocity $\boldsymbol{\omega}$ around one of its edges, called the x-axis. Find expressions for its angular momentum and kinetic energy.

Solution

It is given that the cube is rotating around the x-axis which coincides with one of its edges. Therefore,

$$\boldsymbol{\omega} = \omega_x \mathbf{i}$$

or $$\omega_x = |\boldsymbol{\omega}|;\ \omega_y = \omega_z = 0$$

These, together with Eq. (8.37) for the components of angular momenta give,

$$L_x = I_{xx}\omega_x$$
$$L_y = I_{yx}\omega_x$$

and $$L_z = I_{zx}\omega_x$$

Substituting for components of the inertia tensor, from Ex. 8.3, we get

$$L_x = 2/3\ M\ |\boldsymbol{\omega}|\ a^2$$
$$L_y = -1/4\ M\ |\boldsymbol{\omega}|\ a^2$$
$$L_z = -\ 1/4\ M\ |\boldsymbol{\omega}|\ a^2$$

Therefore $$\mathbf{L} = L_x\mathbf{i} + L_y\mathbf{j} + L_z\mathbf{k}$$
$$= (2/3\ \mathbf{i} - 1/4\ \mathbf{j} - 1/4\ \mathbf{k})\ M\ |\boldsymbol{\omega}|\ a^2$$

Next, substituting the values of ω_x, ω_y and ω_z in Eq. (8.51) for kinetic energy, we get

$$T = 1/2\ I_{xx}\ |\boldsymbol{\omega}|^2$$

Since $I_{xx} = 2/3\ Ma^2$, the kinetic energy of rotation is given by

$$T = 1/3\ M\ |\boldsymbol{\omega}|^2\ a^2$$

EXAMPLE 8.5

Derive an expression for the rise of temperature of earth, if it suddenly stops rotating.

Solution

If the earth suddenly stops rotating, then its rotational kinetic energy will be completely converted into heat. Thus

$$\frac{1}{2}\ I\omega^2 = MSt$$

where I = moment of inertia of earth about its axis
ω = angular velocity of earth
S = specific heat of earth
t = rise in temperature of earth

If R is the radius of earth, then

$$I = \frac{2}{5}\ MR^2$$

thus $$\frac{1}{2} \times \frac{2}{5}\ MR^2\omega^2 = MSt$$

or $$\frac{R^2\omega^2}{5} = St$$

Therefore, $$t = \frac{R^2\omega^2}{5S}$$

8.5 INDEPENDENT COORDINATES OF A RIGID BODY AND EULER ANGLES

Till now, we have described the angular momenta and kinetic energies of the rigid body in terms of the coordinate system attached to it. In this case, in general, the coordinate axes rotate along with the body. However, generally one is interested in describing the motion of the rigid body from the point of view of the observer or lab system. This requires prescribing the coordinates of the rigid body with respect to the observer or lab system, which is stationary with respect to the observer.

For an *N-body* system, in general, there should be *3N* coordinates. In a rigid body of a macroscopic size, there are billions of particles. It is, therefore, nearly impossible to describe the system with such a large number of coordinates, or is it necessary? Though there are a large number of particles in a rigid body, they are all constrained in their positions. It is known from experience that a rigid body can execute basically only two types of motion, namely translation and rotation.

It can be shown that these two motions can be described by six independent coordinates. In general, a set of three fixed non-collinear points in the rigid body suffice to define the position and orientation of a rigid body. This is so because one point fixes the translatory motion and the body can rotate about any axis through this point. For two fixed points, the rigid body can rotate through an axis passing through these two points. If we choose the third point not in line with the other two points, the orientation and position of the rigid body are specified. Since each point, in general, can be described by three coordinates, one requires nine coordinates in all. However, the distances between three points of the rigid body are fixed, giving three equations of constraint. Therefore, six coordinates are sufficient to describe the motion of a rigid body.

It will be seen below that one can define three suitable (not necessarily normal to each other) axes fixed to the body so that the general rotation can be described. The orientation of each axis can be described by an angle of orientation. This gives three coordinates for rotation. The centre of coordinates for these axes serves as a point, which can also take care of translation. This will also have three coordinates. In this way, we have six coordinates in all to describe the translation and rotation of a rigid body.

Such a system has been developed. We introduce three axes fixed in space and call them (x, y, z) space-axes and other three axes fixed in the body and call them (1, 2, 3) body-axes. The origin of the body-axes is fixed with respect to space coordinate system (x, y, z). If we are considering only rotation and no translation at all, then the origins of the two coordinate systems may coincide.

How do we select the (1, 2, 3) system which is fixed to the body? It should be desirable to select these axes, in the same symmetric manner as (x, y, z) axes. However, no such axes perpendicular tc each other have been found to be convenient. Therefore, one selects (1, 2, 3) axes in such a manner as to be most convenient in describing the rotation of the many such schemes. One such way of defining the axes of rotation has been suggested by Euler which has been accepted and found to be very convenient. This scheme of (1, 2, 3) body-axes along with (x, y, z)-axes in space is shown in Fig. 8.11.

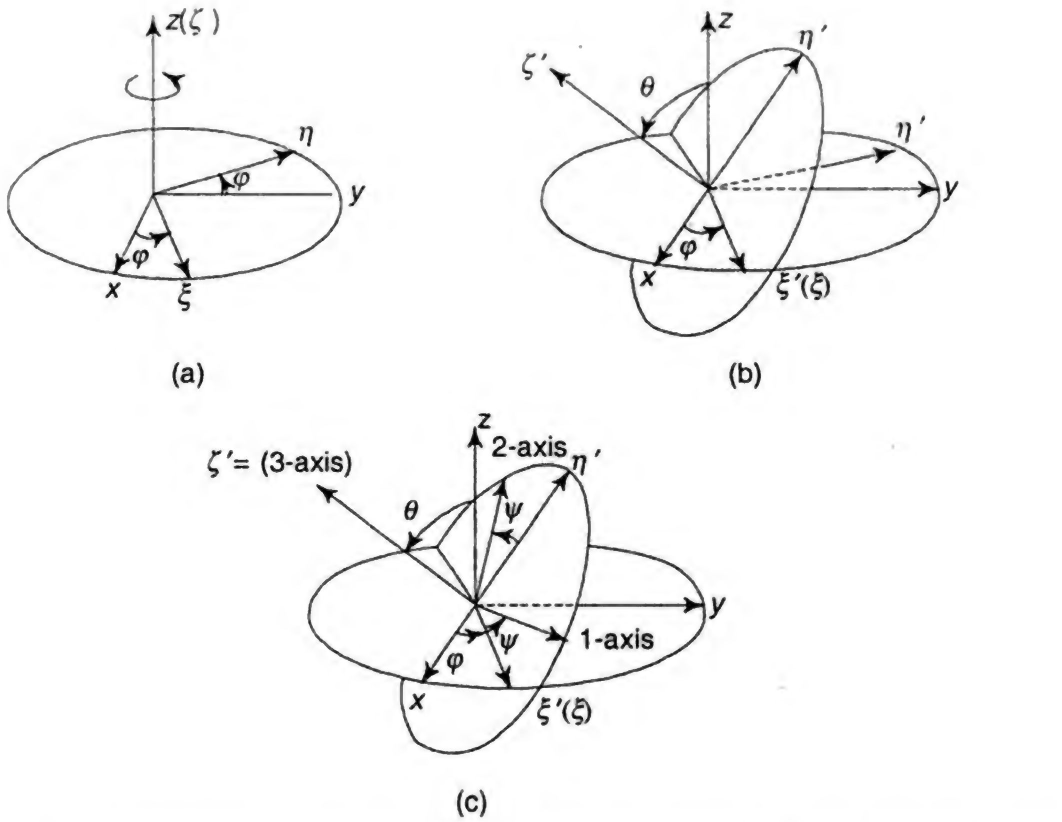


Fig. 8.11 The relationship of various Euler angles (a) shows rotation around z-axis (b) shows rotation around ξ-axis; (c) shows rotation around ζ-axis or 3-axis

In Fig. 8.11a we have shown (x, y, z)-axes, from which we obtain (ξ, η, ζ) axes by rotation (x, y)-axes around the z-axis in an anti-clockwise manner by an angle φ, so that $x \to \xi$, $y \to \eta$, $z \to z = \zeta$

In Fig. 8.11b we next go from (ξ, η, z) to (ξ', η', ζ'), by rotating η-z axes around ξ-axis by an angle θ, in an anti-clock-wise direction; so that $\eta \to \eta'$ and $z \to \zeta'$; and $\xi \to \xi' = \xi$. The ξ-axis, which is at the intersection of the xy-plane and ξ-η plane is called the line of nodes.

Next as shown in Fig. 8.11c we rotate around the ζ-axis by an angle, ψ, again in a anti-clockwise sense, so that we go from (ξ', η', ζ') to (1, 2, 3) axes in such a way that $\xi \to$ 1-axis; $\eta' \to$ 2-axis and $\zeta' \to$ 3-axis.

In this manner, we get, in three steps, from x, y, z space axes to the 1, 2, 3 body axes.

The angles φ, θ and ψ associated with these rotations are called Euler angles. It may be emphasised that:

1. φ is the rotation around the z-axis,
2. θ is the rotation around the ξ-axis, and
3. ψ is the rotation around the ζ or 3-axis.

It should be realised that the z-axis is perpendicular to the ξ-axis, and the ξ-axis is perpendicular to the 3-axis, but the z-axis and 3-axis make an angle θ with each other. Also, it should be noted that the $\zeta'(=3)$ axis is perpendicular to the $\zeta'\eta'$-plane or 1—2-plane.

We may also realise that we have obtained the final 1, 2, 3 axes by using intermediate system of ξ'–η'–ζ' axes of which the ξ'-axis is in the line of modes, ζ' coincides with the body 3-axis, and η' is in the 1—2-plane.

In general, the 3-axis is taken along the symmetry axes of the body. In view of this, the tilt of the rotating 3-axis, from the z-axis, given by θ, can be easily identified. If the external forces have symmetry (e.g. gravitational field) then the axis of symmetry of the forces which will be fixed in space is taken along the z-axis.

We will demonstrate the use of the Euler angles by calculating the angular velocity $\boldsymbol{\omega}$ of a body (with 1, 2, 3-axes) around any arbitrary axis, so that all the three Euler angles φ, θ and ψ are changing. However, before doing that, let us have a look at the individual components. If θ alone changes, while φ and ψ are constant, the body rotates around the ξ-axis (the line of nodes) with angular velocity $\dot{\theta}\,\hat{\xi}$. This motion is called *nutation*. If only φ changes, then the body rotates around the z-axis, with angular velocity $\dot{\varphi}\,\hat{\mathbf{z}}$ and the motion is said to be *precession*. However, if only ψ is changing, then the angular velocity is given by $\dot{\psi}\,\hat{\mathbf{e}}_3$ and describes the *spin motion*. These aspects are illustrated in Fig. 8.12.

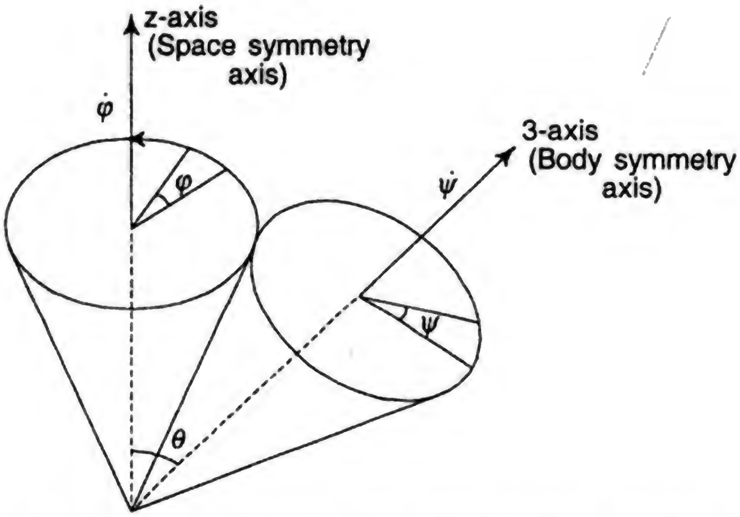


Fig. 8.12 The diagram illustrates the precession of angular velocities

Now if an arbitrary primed system is rotating around the z-axis with angular velocity $\dot{\phi}\hat{\mathbf{z}}$, and the (ξ', η', ζ') system rotates with respect to primed system with angular velocity $\dot{\theta}\hat{\xi}$ around the ξ-axis, then according to the theorem of addition of angular velocities vectorially, the angular velocity of the (ξ', η', ζ')' system is given by $\dot{\varphi}\hat{\mathbf{z}} + \dot{\theta}\hat{\xi}$. Further, if (1, .2, 3) system rotates with angular velocity $\dot{\psi}\hat{\mathbf{e}}_3$ with respect to (ξ', η', ζ') then the total angular velocity of (1, 2, 3) system (i.e. the body) is given by

$$\boldsymbol{\omega} = \dot{\varphi}\hat{\mathbf{z}} + \dot{\theta}\hat{\xi} + \dot{\psi}\hat{\mathbf{e}}_3 \tag{8.55}$$

From Fig. 8.11, it can be seen that,

$$\hat{\xi} = \hat{\mathbf{e}}_1 \cos\psi - \hat{\mathbf{e}}_2 \sin\psi \tag{8.56a}$$

$$\hat{\eta} = \hat{\mathbf{e}}_1 \sin\psi + \hat{\mathbf{e}}_2 \cos\psi \tag{8.56b}$$

$$\hat{\zeta} = \hat{\mathbf{e}}_3$$

and

$$\hat{\mathbf{z}} = \hat{\xi} \cos\theta + \hat{\eta} \sin\theta$$

$$= \hat{\mathbf{e}}_1 \sin\theta \sin\psi + \hat{\mathbf{e}}_2 \sin\theta \cos\psi + \hat{\mathbf{e}}_3 \cos\theta \tag{8.56d}$$

Substituting these values in Eq. (8.55), we get

$$\boldsymbol{\omega} = \dot{\varphi}\,(\hat{\mathbf{e}}_1 \sin\theta \sin\psi + \hat{\mathbf{e}}_2 \sin\theta \cos\psi + \hat{\mathbf{e}}_3 \cos\theta)$$

$$+ \dot{\theta}\,(\hat{\mathbf{e}}_1 \cos\psi - \hat{\mathbf{e}}_2 \sin\psi) + \dot{\psi}\,\hat{\mathbf{e}}_3$$
$$= (\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\,\hat{\mathbf{e}}_1 + (\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\hat{\mathbf{e}}_2$$
$$+ (\dot{\varphi}\cos\theta + \dot{\psi})\,\hat{\mathbf{e}}_3$$
$$= \omega_1\hat{\mathbf{e}}_1 + \omega_2\hat{\mathbf{e}}_2 + \omega_3\hat{\mathbf{e}}_3 \tag{8.57}$$

where ω_1, ω_2 and ω_3 are the components of angular velocity along the (1, 2, 3) axes respectively. Obviously,

$$\omega_1 = \dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi \tag{8.58a}$$
$$\omega_2 = -\dot{\theta}\sin\psi + \dot{\varphi}\sin\theta\cos\psi \tag{8.58b}$$
$$\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$$

Keeping in mind that (1, 2, 3) are the body axes, we can obtain the kinetic energy of the body in terms of ω_1, ω_2 and ω_3 from Eq. (8.54) as

$$T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

For the case of the symmetric top, $I_1 = I_2$ and the above expression takes the form

$$T = \frac{1}{2}I_1\left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2}I_3\omega_3^2$$
$$= \frac{1}{2}I_1(\dot{\theta}^2 + \varphi^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \varphi\cos\psi)^2 \tag{8.59}$$

EXAMPLE 8.6

Find an expression for the kinetic energy of rotation of a spherical top in terms of derivatives of Euler's angles.

Solution

For a spherical top

$$I_1 = I_2 = I_3 = I \text{ (say)}$$

Therefore, the kinetic energy of rotation of such a body will be

$$T = \frac{1}{2}I\left(\omega_1^2 + \omega_2^2 + \omega_3^2\right)$$

Now, ω_1, ω_2, and ω_3 are related to the time derivatives of Euler angles θ, φ and ψ through Eq. (8.58). Substituting for these, we get

$$T = \frac{1}{2}I\,[\,(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)^2 +$$
$$(-\dot{\theta}\sin\psi + \dot{\varphi}\sin\theta\cos\psi)^2 + (\dot{\psi} + \dot{\varphi}\cos\theta)^2]$$
$$= \frac{1}{2}I\,[\dot{\theta}^2\cos^2\psi + \dot{\varphi}^2\sin^2\theta\sin^2\psi + 2\dot{\theta}\dot{\varphi}\sin\theta\sin\psi\cos\psi$$
$$+ \dot{\theta}^2\sin^2\psi + \dot{\varphi}^2\sin^2\theta\cos^2\psi - 2\dot{\theta}\dot{\varphi}\sin\theta\sin\psi\cos\psi$$
$$+ \dot{\psi}^2 + \dot{\varphi}^2\cos^2\theta + 2\dot{\varphi}\dot{\psi}\cos\theta]$$
$$= \frac{1}{2}\,[\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta]$$

8.6 EQUATION OF MOTION OF A RIGID BODY: EULER EQUATIONS

In general, a rigid body may be under the influence of a force or a set of forces, which can lead to both translational and rotational motion. The translatory motion of the rigid body is governed by Eq. (4.36), as discussed earlier. If an external force $\mathbf{F}^{ext}$ is acting on the body, its motion is described by the equation

$$\mathbf{F}^{ext} = M \frac{d^2\mathbf{R}}{dt^2} = \frac{d\mathbf{P}}{dt} \tag{8.60}$$

where $\mathbf{P}$ is the linear momentum associated with the motion of the centre of mass (the point at which the whole mass is concentrated).

The rotation of the rigid body, on the other hand, is governed by Eq. (4.43), according to which

$$\begin{aligned}\Gamma_{space} &= \sum_i \frac{d}{dt}(\mathbf{r}_i \times \mathbf{p}_i) = \sum_i \frac{d\mathbf{L}_i}{dt} \\ &= \left(\frac{d\mathbf{L}}{dt}\right)_{space}\end{aligned} \tag{8.61}$$

where $$\mathbf{L} = \sum_i \mathbf{L}_i$$

Now in the case of a rigid body, we have seen from Eq. (8.9) that

$$\mathbf{L}_{body} = (\mathbf{I}\boldsymbol{\omega})_{body} \tag{8.62}$$

In Eq. (8.61) the quantities Γ and $\mathbf{L}$ are with reference to the space coordinates because they are measured with respect to the observer who is stationary in the lab or space. However, the tensor I refers to the body system because x_i, y_i, z_i in Eq. (8.38) are defined in the body system. We should, therefore write the equation to correlate the quantities in the free space and body system.

It will be shown in Chapter 10 [Eq. (10.28)] that operationally the derivative of any vector physical quantity of a body rotating with the angular velocity $\boldsymbol{\omega}$ in the space coordinate system is related to the derivative in the body coordinate system by the following operational equation:

$$\left(\frac{d}{dt}\right)_{space} = \left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega} \times \tag{8.63}$$

In view of this Eq. (8.57) can be written as

$$\begin{aligned}\Gamma_{space} &= \left(\frac{d\mathbf{L}}{dt}\right)_{space} \\ &= \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} \\ &= \frac{d}{dt}(\mathbf{I}\boldsymbol{\omega}) + \boldsymbol{\omega} \times \mathbf{L}\end{aligned} \tag{8.64}$$

It may be realised here, that $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ are always measured in rotating body system. Hence these suffixes have been dropped from Eq. (8.64). $\mathbf{L}$ in the equation is, of course measured in space coordinate system.

Equation (8.64) is the general equation of motion of rotation for a rigid body of any shape rotating around any axis or set of axes. It can, therefore, be used to describe the motion of bodies with regular and irregular shapes, which may or may not have symmetry associated with them. For irregular shapes, we can choose any axes of rotation, whereas for regular shapes, one normally selects principal axes of symmetry as axes of rotation.

For a symmetric body where the axes of rotation 1, 2 and 3 coincide with principal axes of symmetry, Eq. (8.64) assumes a symmetric form. For rotation around 1-axis of symmetry,

$$\Gamma_1 = | I_1 \dot{\omega}_1 + (\boldsymbol{\omega} \times \mathbf{L})_1|$$
$$= |I_1 \dot{\omega}_1 + [\boldsymbol{\omega} \times (\mathbf{I} \cdot \boldsymbol{\omega})_1|$$
$$= I_1 \dot{\omega}_1 + (\omega_2 I_3 \omega_3 - \omega_3 I_2 \omega_2)$$

or
$$\Gamma_1 = I_1 \dot{\omega}_1 + (I_3 - I_2)\, \omega_2 \omega_3 \tag{8.65a}$$
Similarly,
$$\Gamma_2 = I_2 \dot{\omega}_2 + (I_1 - I_3)\, \omega_3 \omega_1 \tag{8.65b}$$
and
$$\Gamma_3 = I_3 \dot{\omega}_3 + (I_2 - I_1)\, \omega_1 \omega_2 \tag{8.65c}$$

Equations (8.65) are called Euler equations of motion of a rigid body. Here use has been made of the fact that the components of inertia tensor are constant in the body system.

In principle, it should be possible to solve these equations for a symmetric body, to obtain ω_1, ω_2 and ω_3 if Γ_1, Γ_2 and Γ_3 are given. But these are coupled equations and can only be solved numerically. For special cases, however, they may be solved analytically.

I. For a uniform sphere,

$$I_1 = I_2 = I_3 = I \tag{8.66a}$$
Hence
$$I\dot{\omega}_1 = \Gamma_1;\ I\dot{\omega}_2 = \Gamma_2 \text{ and } I\dot{\omega}_3 = \Gamma_3 \tag{8.66b}$$

Now the three equations are uncoupled and one can solve these to find the values of ω_1, ω_2 and ω_3 if the expressions for Γ_1, Γ_2 and Γ_3 are known.

For a special case, when $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$, it is easy to see that

$$I\dot{\omega}_1 = I\dot{\omega}_2 = I\dot{\omega}_3 = 0 \tag{8.66c}$$
Hence
$$\omega_1 = \omega_2 = \omega_3 = \omega = \text{const} \tag{8.66d}$$

i.e. the angular velocity is constant for a torque-free rotation of a sphere.

2. When the external torque is zero, it is known from the conservation of angular momentum that, angular momentum **L** is constant so that $(d\mathbf{L}/dt)_{\text{body}} = 0$. This fact, together with $\Gamma = 0$, when substituted in Eq. (8.64) leads to

$$\boldsymbol{\omega} \times \mathbf{L} = \mathbf{0}$$

This is possible only if $\boldsymbol{\omega}$ and **L** are in the same direction, i.e. the angular velocity vector is along the principal axis of the body. Since $\mathbf{L} = I\boldsymbol{\omega}$; the quantities ω and **L** being in the same direction implies that I should act as a scalar. Therefore, we can write

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} \tag{8.67}$$

where **I** is the magnitude of the moment of inertia.

3. One can calculate the rate of change of kinetic energy dT/dt from the equation of motion. For this purpose, we consider the dot product of $\boldsymbol{\omega}$ with Eq. (8.64) which gives

$$\boldsymbol{\omega} \cdot \boldsymbol{\Gamma} = \boldsymbol{\omega} \cdot (I d\boldsymbol{\omega}/dt) + \boldsymbol{\omega} \cdot (\boldsymbol{\omega} \times \mathbf{L})$$

since $\boldsymbol{\omega} \cdot (\boldsymbol{\omega} \times \mathbf{L}) = (\boldsymbol{\omega} \times \boldsymbol{\omega}) \cdot \mathbf{L} = \mathbf{0}$, the above relation becomes

$$\boldsymbol{\omega} \cdot \boldsymbol{\Gamma} = \frac{d\boldsymbol{\omega}}{dt} \cdot \mathbf{I} \cdot \boldsymbol{\omega}$$

$$= \frac{1}{2}\, d/dt\, (\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega}) \tag{8.68}$$

But from Eq. (8.53)

$$T = \frac{1}{2}\, \boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2}\, \boldsymbol{\omega} \cdot \mathbf{I}\boldsymbol{\omega}$$

Therefore,

$$\boldsymbol{\omega} \cdot \boldsymbol{\Gamma} = d/dt\, (\frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}\boldsymbol{\omega}) \tag{8.69a}$$

$$= dT/dt$$

This should be compared with the expression

$$\mathbf{v} \cdot \mathbf{F} = dT/dt \tag{8.69b}$$

for the rate of change of kinetic energy in linear motion.

EXAMPLE 8.7

For a thin circular disc of uniform thickness, with mass M and radius R the inertia tensor with respect to the coordinate system having origin at its centre and the z-axis perpndicular to its plane is (see Problem 8.7)

$$\mathbf{I} = \frac{MR^2}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Find the torque acting on a disc with mass 0.1 kg, radius 0.04 m, rotating with angular velocity

$$\boldsymbol{\omega} = (3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}) \text{ rad/s}$$

where the components are expressed with respect to the principal axes Also, find the rate of change of kinetic energy.

Solution

For a disc, the principal moments of inertia are

$$I_1 = I_{xx} = \frac{MR^2}{4}, \quad I_2 = I_{yy} = \frac{MR^2}{4}, \quad I_3 = I_{zz} = \frac{MR^2}{2}$$

For the given case,

$$M = 0.1 \text{ kg}, R = 0.04 \text{ m}$$

Therefore

$$I_1 = I_2 = \frac{1}{4} \times 0.1 \times 16 \times 10^{-4} \text{ kg m}^2$$

$$= 4 \times 10^{-5} \text{ kg m}^2$$

$$I_3 = \frac{1}{2} \times 0.1 \times 16 \times 10^{-4} \text{ kg m}^2$$

$$= 8 \times 10^{-5} \text{ kg m}^2$$

Also,

$$\omega_1 = 3 \text{ rad/s}, \omega_2 = 4 \text{ rad/s}, \omega_3 = 6 \text{ rad/s}$$

From Eq. (8.65) we have, noting that ω_1, ω_2 and ω_3 are independent of time,

$$\Gamma_1 = (I_3 - I_2)\,\omega_2\omega_3 = 4 \times 10^{-5} \times 24 \text{ N m}$$
$$= 9.6 \times 10^{-4} \text{ N m}$$
$$\Gamma_2 = (I_1 - I_3)\,\omega_3\omega_1 = -4 \times 10^{-5} \times 18 \text{ N m}$$
$$= -7.2 \times 10^{-4} \text{ N m}$$
$$\Gamma_3 = (I_2 - I_1)\,\omega_1\omega_2 = 0$$

Therefore.
$$\mathbf{\Gamma} = \Gamma_1\mathbf{i} + \Gamma_2\mathbf{j} + \Gamma_3\mathbf{k}$$
$$= (9.6\mathbf{i} - 7.2\mathbf{j}) \times 10^{-4} \text{ N m}$$

Next, from Eq. (8.69a), the rate of change of kinetic energy is given by

$$dT/dt = \boldsymbol{\omega} \cdot \mathbf{\Gamma} = (3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}) \cdot (9.6\mathbf{i} - 7.2\mathbf{j}) \times 10^{-4} \text{ J/s}$$
$$= 0$$

i.e. the kinetic energy is unchanged.

8.7 FREELY ROTATING SYMMETRIC TOP

A body is said to be freely rotating if no torque is acting on it. In practice a torque might have been applied at some time to set the body rotating. Then the torque is removed and the body continues rotating freely. Examples of such motion are the rotating earth, an orbiting electron or a rotating top. In the case of a free rotating body, $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$ and Euler's equations are modified accordingly. Further, such situations often involve symmetrical bodies, such as a symmetrical top, the earth, etc. as indicated above.

We assume that the symmetry of the rotating body is such that $I_1 = I_2 \neq I_3$. A cylindrical body satisfies this condition and so do the earth and symmetrical top. Euler's Eq. (8.65) then become

$$I_1\dot{\omega}_1 = (I_2 - I_3)\,\omega_2\omega_3 \tag{8.70a}$$
$$I_2\dot{\omega}_2 = I_1\dot{\omega}_2 = (I_3 - I_1)\,\omega_3\omega_1 \tag{8.70b}$$
$$I_3\dot{\omega}_3 = 0 \tag{8.70c}$$

Integration of Eq. (8.70c) shows that ω_3 is constant, i.e. the angular velocity around the symmetry axis is constant. Further, from Eqs (8.70a) and (8.70b), we can write

$$\dot{\omega}_1 = \left[\frac{I_1 - I_3}{I_1}\,\omega_3\right]\omega_2$$
$$= \Omega\omega_2 \tag{8.71a}$$

Similarly
$$\dot{\omega}_2 = -\Omega\omega_1 \tag{8.71b}$$

where
$$\Omega = \frac{I_1 - I_3}{I_1}\,\omega_3 = \text{const} \tag{8.71c}$$

Differntiating Eq. (8.71a) and subtituting the value of $\dot{\omega}_2$ from Eq. (8.71b), we get

$$\ddot{\omega}_1 = -\Omega^2\omega_1 \tag{8.72a}$$

This equation is similar to the one obtained for the description of simple harmonic motion. Therefore, its solution may be written as

$$\omega_1 = A \sin(\Omega t + \theta_0) \tag{8.72b}$$

This together with Eq. (8.71b) gives

$$\dot{\omega}_2 = -\Omega A \sin(\Omega t + \theta_0),$$

so that

$$\omega_2 = A \cos(\Omega t + \theta_0)$$

The angular velocities ω_1 and ω_2 are along x and y directions, hence the resultant of $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ will always lie in the xy plane. One may write the resultant $\boldsymbol{\omega}_p$ as

$$\boldsymbol{\omega}_p = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} \tag{8.73a}$$

$$= A \sin(\Omega t + \theta_0)\mathbf{i}$$

$$+ A \cos(\Omega t + \theta_0)\mathbf{j} \tag{8.73b}$$

and

$$\omega_p^2 = \omega_1^2 + \omega_2^2 = A^2 \tag{8.73c}$$

The total angular velocity $\boldsymbol{\omega}$ is then given by

$$\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k} \tag{8.74a}$$

so that

$$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 \tag{8.74b}$$

$$= \omega_p^2 + \omega_3^2 = \text{const}$$

Here, in such a case, the magnitude of the total angular momentum is constant. The direction of the rotation of $\boldsymbol{\omega}$ depends on the relative values of ω_3 and $\boldsymbol{\omega}_p$. Now since ω_1 and ω_2 are changing with time and ω_3 and $\boldsymbol{\omega}$ have constant magnitudes, the direction of $\boldsymbol{\omega}$ should be continuously changing in such a way that its projection on the z-axis is constant. This is possible if $\boldsymbol{\omega}$ makes constant angle, say α with the 3-axis and processes around the 3-axis (Fig. 8.13). The cone so described is called the body cone.

The angle of preces· α is given by

$$\tan \alpha = \frac{|\boldsymbol{\omega}_p|}{\omega_3} = \frac{A}{\omega_3} \tag{8.75}$$

Further, since precession is defined by ω_1 and ω_2, which depends on $\boldsymbol{\Omega}$, the quantity Ω is called precessional angular velocity.

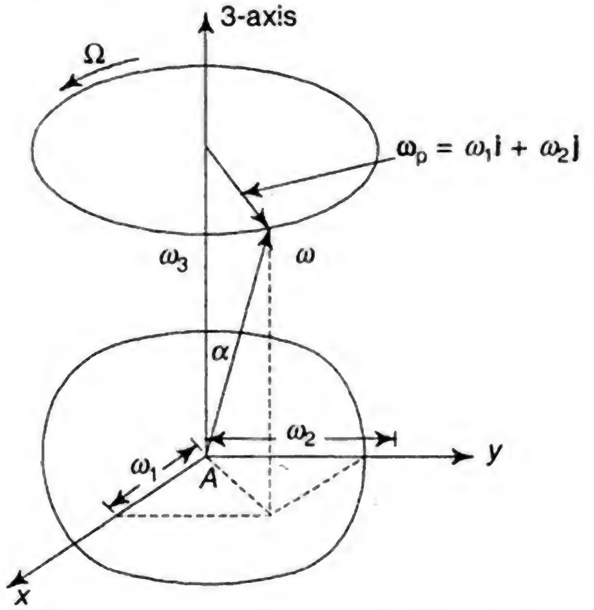


Fig. 8.13 The precession of the angular velocity about 3-axis in torque-free motion of symmetric rigid body

It is clear from Eq. (8.73a) that $\boldsymbol{\omega}_p$ is changing in direction with time with constant angular velocity Ω, but keeping its magnitude constant, equal to A. This means that $\boldsymbol{\omega}_p$ is rotating with angular velocity Ω. This precession of $\boldsymbol{\omega}$ is sometimes called *wobble.*

The values of A and ω_3 can be expressed in terms of the kinetic energy T and angular momentum L as detailed below. From Eqs (8.51) and (8.41) we have for $I_1 = I_2 \neq I_3$.

$$T = \frac{1}{2} I_1 A^2 + \frac{1}{2} I_3 \omega_3^2 \tag{8.76}$$

and

$$L^2 = I_1^2 A^2 + I_3^2 \omega_3^2 \tag{8.77}$$

These equations give

$$\omega_3^2 = \frac{L^2 - 2I_1 T}{I_3 (I_3 - I_1)} \tag{8.78}$$

and

$$A^2 = \frac{(L^2 - 2I_3 T)}{I_1 (I_1 - I_3)}$$

the angle θ between the vectors $\boldsymbol{\omega}$ and $\mathbf{L}$ is given by

$$\cos\theta = \frac{\boldsymbol{\omega}\cdot\mathbf{L}}{|\boldsymbol{\omega}||\mathbf{L}|} = \frac{\boldsymbol{\omega}\cdot\mathbf{I}\,\boldsymbol{\omega}}{|\boldsymbol{\omega}||\mathbf{L}|}$$

$$= 2T\,|\boldsymbol{\omega}|\,|\mathbf{L}| \tag{8.79}$$

This means that, in general, $\boldsymbol{\omega}$ and $\mathbf{L}$ are not in the same direction. Further, the angle between $\boldsymbol{\omega}$ and $\mathbf{L}$ is constant because $\mathbf{L}$ is constant for no external torques. This is possible if $\boldsymbol{\omega}$ precesses around $\mathbf{L}$ with θ as angle between $\boldsymbol{\omega}$ and $\mathbf{L}$. Since the vector $\mathbf{L}$ is fixed in space or laboratory, the cone obtained by the precession of $\boldsymbol{\omega}$ around $\mathbf{L}$ is called space or laboratory cone. It has also been seen earlier that $\boldsymbol{\omega}$ precesses around the 3-axis describing the body cone. These two aspects can be combined together by assuming that the body cone is rolling, without slipping around the space cone as shown in Fig. 8.14.

In physical terms, if we apply the above arguments to a freely rotating symmetric top as the earth, this means that the direction of $\boldsymbol{\omega}$ will be that of earth's axis of rotation and that direction of z will be some fixed direction in space say polar star. Then Ω will give the wobble of the earth's rotation around polar star. It may be mentioned that the earth offers the best example of a freely moving symmetric top, as torques on it due to other planets, if any, are negligible.

The tilted heavy symmetric top rotating on the earth cannot be treated in this manner because it has torque due to the earth's gravity.

EXAMPLE 8.8

Show that the precession velocity Ω can also be written.

$$\Omega = \left[\frac{(L^2 - 2I_1 T)(I_3 - I_1)}{I_1^2 I_3}\right]^{1/2}$$

where various symbols have their usual meaning.

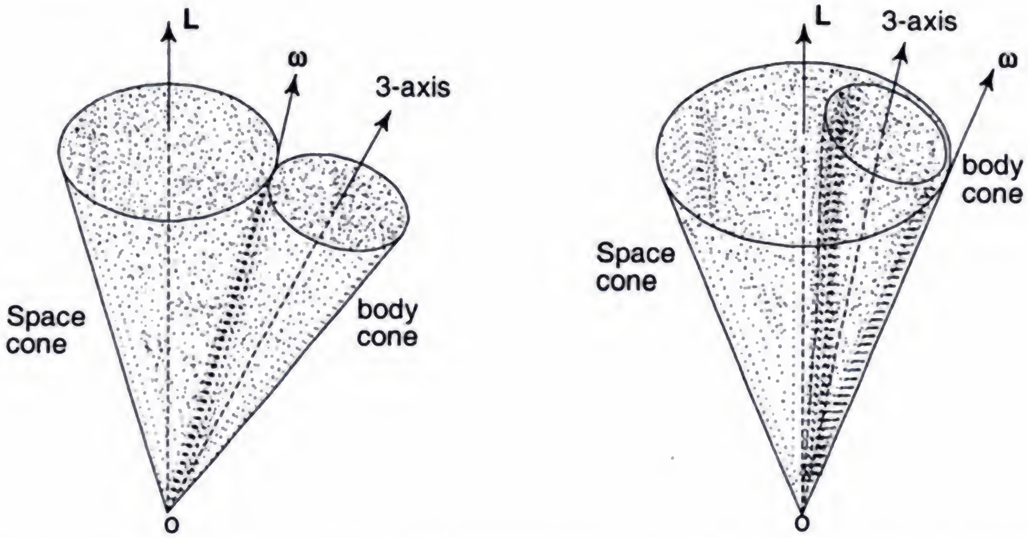


Fig. 8.14 The diagram illustrates the relationship of **L**, **ω** and 3-axis, explaining the concepts of body cone and space cone: (a) when body cone is outside space cone; (b) when body cone is inside space cone

Solution

From the expressions for angular momentum and kinetic energy as given in Eqs (8.76) and (8.77), it is clear that

$$\omega_3 = \left[\frac{(L^2 - 2I_1T)}{I_3(I_3 - I_1)}\right]^{1/2}$$

Therefore,
$$\Omega = \frac{I_1 - I_3}{I_1}\,\omega_3$$

$$= \left[\frac{(I_1 - I_3)^2\,(L^2 - 2I_1T)}{I_1^2\,I_3(I_3 - I_1)}\right]^{1/2}$$

$$= \left[\frac{(L^2 - 2I_1T)(I_3 - I_1)}{I_1^2\,I_3}\right]^{1/2}$$

EXAMPLE 8.9

A top is made by passing a light and small pin through the centre of a uniform thin disc of mass 10 g and radius 2 cm. Find the precession velocity for such a top rotating at 20 rad/s. Given, moments of inertia of a disc about its diameter and perpendicular to the plane are $\frac{1}{4}\,MR^2$ and $\frac{1}{2}\,MR^2$ respectively.

Solution

Since the disc top is rotating about the axis perpendicular to its plane,

$$I_3 = \frac{1}{2}\,MR^2$$

and
$$I_1 = I_2 = \frac{1}{4}\,mR^2$$

When the angular velocity of rotation of a free top is ω, the magnitude of velocity of precession is given by

$$\Omega = \frac{I_3 - I_1}{I_1}\,\omega$$

Substituting various values, we get

$$\Omega = \frac{\frac{1}{2}MR^2 - \frac{1}{4}MR^2}{\frac{1}{4}MR^2}\,\omega$$

$$= \omega$$

$$= 20 \text{ rad/s}$$

QUESTIONS

8.1 Define a rigid body and justify the fact that total internal forces and torques tor these are zero.

8.2 Starting from the general expression for angular momentum of a system, obtain an expression for the case of a rigid body. Hence bring out the concept of inertia tensor.

8.3 'The inertia tensor is symmetric'. Comment.

8.4 Bring out the meaning of 'principal axes of inertia'.

8.5 What is a symmetric top? Is it always cylindrical?

8.6 What is a rotor? How does it differ from a symmetric top?

8.7 A rigid body has $I_x = I_y = I_z$. What is the name given to such a body?

8.8 'Principal axes can be defined only for symmetric rigid bodies'. Discuss.

8.9 A symmetric body is rotating around the z-axis. What will be the direction of angular momentum vector?

8.10 Obtain a general expression for the kinetic energy of a rigid body.

8.11 Show that for a symmetric rigid body

$$T = \frac{1}{2}\sum_{j=1}^{3} I_j \omega_j^2$$

8.12 The motion of a rigid body can be described in terms of six coordinates'. Comment.

8.13 What are Euler angles? Bring out their meaning.

8.14 Discuss the physical meaning of time derivatives of the three Euler angles, when one of these is changing and the other two are fixed.

8.15 Obtain an expression for angular velocity $\boldsymbol{\omega}$ of a body with (1, 2, 3) axes revolving around an arbitrary axis in space.

8.16 Find an expression for kinetic energy of rotation of a rigid body with respect to the principal axes' in terms of Euler angles. Discuss the cases (i) $I_1 = I_2 \neq I_3$ and (ii) $I_1 = I_2 = I_3$.

8.17 Assuming that the components of the inertia tensor are constant in the body system, obtain the Euler equations of motion of a rigid body.

8.18 Show that angular velocity is constant for the torque-free rotation of a sphere.

8.19 'For the torque-free rotation of a rigid body, the inertia tensor can simply be taken as a scalar'. Comment.

8.20 Prove that $dT/dt = \boldsymbol{\omega} \cdot \boldsymbol{\Gamma}$, where the symbols have their usual meaning.

8.21 Discuss the motion of a freely rotating symmetric top.

8.22 Bring out the meaning of the term 'wobble'.

8.23 Explain the terms: body cone, space cone and precession.

PROBLEMS

8.1 Four point masses, each equal to m are placed at $(a, 0, 0)$, $(0, a, 0)$, $(0, 0, a)$, and (a, a, a). Evaluate the inertia tensor for this system.

$$\text{Ans. } ma^2 \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$$

8.2 Discuss the choice of the coordinate system for a homogeneous sphere of mass M and radius R such that the products of inertia are zero.

Ans. The origin of the coordinate system is at the centre of the sphere.

8.3 Calculate the inertia tensor for a cube of mass m and each side a, with respect to the coordinate system having origin at the centre of the cube and axes parallel to the faces.

Ans. $I_{xx} = I_{yy} = I_{zz} = 1/6\ Ma^2$,
$I_{xy} = I_{yx} = I_{zy} = I_{xz} = 0$

8.4 A cube of mass 0.1 kg and each edge 0.04 m is rotating at 9 rad/s around one of its principal axes, say the z-axis. Find the values of its angular momentum and kinetic energy.

Ans. $\mathbf{L} = (-3.6\mathbf{i} - 3.6\mathbf{j} + 9.6\mathbf{k}) \times 10^{-4}$ kg m^2 s^{-1}
$T = 4.32 \times 10^{-3}$ J

8.5 Consider a rectangular plate of mass M and dimensions $a \times b$. Define a coordinate system with origin at one corner; x- and y-axes along the two edges and z-axis perpendicular to the plane of the plate. Calculate the inertia tensor for this plate.
[*Hint:* For the plate, mass of an areal element is $\sigma\, dxdy$, where $\sigma = (M/ab)$ is the mass per unit area. Therefore, component I_{xx} is given by

$$I_{xx} = \int\int \sigma dxdy\,(y^2 + z^2) = \sigma \int\int y^2 dxdy \text{ because } z = 0.]$$

$$\text{Ans. } \mathbf{I} = M \begin{bmatrix} 1/3\,b^2 & -1/4\,ab & 0 \\ -1/4\,ab & 1/3\,a^2 & 0 \\ 0 & 0 & 1/3\,(a^2 + b^2) \end{bmatrix}$$

8.6 Show that the principal inertia tensor of a rectangular plate of sides l_1 and l_2 and mass M is given by

$$\mathbf{I} = \frac{M}{12} \begin{bmatrix} l_1^2 & 0 & 0 \\ 0 & l_1^2 & 0 \\ 0 & 0 & l_2^2 + l_2^2 \end{bmatrix}$$

8.7 A thin circular disc of uniform thickness has mass M and radius R. Determine inertia tensor for this disc with respect to a coordinate system having the origin at its centre and the z-axis perpendicular to its surface.
Hint: For a circular object, it is better to use circular coordinates so that $x = r\cos\theta$, $y = r\sin\theta$ and the area element $dxdy = rdrd\theta$. Accordingly, taking $z = 0$

$$I_{xx} = M/\pi r^2 \int_0^R \int_0^{2\pi} r^3\, dr \sin^2\theta\, d\theta$$

and so on.

$$\text{Ans. } \mathbf{I} = \frac{MR^2}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

8.8 Find the angular momentum and kinetic energy of a thin circular disc of mass 1 kg and radius 0.1 m, rotating with angular velocity 20π rad/s around the axis passing through its centre and perpendicular to its plane.

Ans. $\mathbf{L} = 0.314\ \mathbf{k}$ kg m^2 s^{-1}, $T = 9.88$ J

8.9 Prove that the sum of any two of the principal moments of inertia is alway greater than the third.

8.10 A rectangular plate of mass 0.24 kg and sides 0.1 m and 0.08 m is rotated in such a way, by taking the principal axes as rotation axes that the torque on it is

$$\mathbf{\Gamma} = (40.96\mathbf{i} + 32\mathbf{j} - 5.76\mathbf{k}) \times 10^{-4} \text{ N m}$$

Determine the angular velocity vector and the rate of change of kinetic energy.

Ans. $\boldsymbol{\omega} = (2\mathbf{i} - 4\mathbf{j} - 8\mathbf{k})$ rad/s, $dT/dt = 0$ J/s

8.11 The earth is essentially symmetrical about the polar axis about which it rotates at angular velocity 7.272×10^{-5} rad/s or (2π rad/day) almost freely. Further, the earth is flat at the poles and bulges at the equator, so that the moment of inertia I_3 around the polar axis is related to that (I_1) around a perpendicular axis by

$$I_3 = 1.00327\ I_1$$

Calculate the precession velocity $\mathbf{\Omega}$ of the earth. *Ans.* $\mathbf{\Omega} = 2.378 \times 10^{-7}$ rad/s

9

Oscillatory Motion

A particle is said to execute oscillatory motion when on slight displacement, it moves periodically about an equilibrium position, such as a simple pendulum or a mass attached to a spring performing small oscillations. An analogous case of electrical oscillations is provided by an electrical circuit consisting of an inductance and capacitance. Other cases include those of atoms in a solid, vibrating relative to each other and of electrons that are in rapid oscillation in a radiating (or receiving) antenna.

The most important example of oscillatory motion is provided by simple harmonic motion (SHM), since apart from the ease and simplicity with which it can be treated mathematically, it serves as an exact or approximate model for many problems in classical and quantum physics. First, we will consider the idealized case of an oscillator performing free vibrations in the absence of friction and external forces. The interplay of friction causes the motion to be damped, thus producing damped oscillations. However, when the mass is subjected to the driving force, which is a periodic function of time, the system behaves as a forced oscillator. When the frequency of the impressed force equals the natural frequency of the system, there results the phenomenon of resonance and the amplitude acquires the maximum possible value. However, this is strictly true in the case of no damping, and in actual practice the resonance frequency is slightly less than the natural frequency. The lesser the damping, the more near it is to the natural frequency.

9.1 SIMPLE HARMONIC MOTION

Let us consider a point mass m attached to a spring of negligible mass (Fig. 9.1) as an example of simple vibration of a single particle in a one-dimensional system. When the mass m is displaced slightly from its equilibrium position through a small

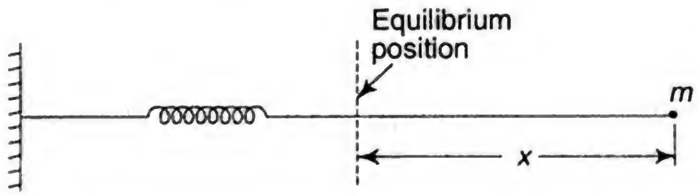


Fig. 9.1 A point mass attached to a spring of negligible mass

displacement x, the only force acting on it is an elastic restoring force proportional to x and directed towards the equilibrium position. Assuming Hook's (empirical) law of elasticity to hold and ignoring the forces of air friction and internal elastic friction, the equation of motion is

$$m \frac{d^2\mathbf{x}}{dt^2} = -S\mathbf{x} \tag{9.1}$$

where S, called the stiffness constant is the restoring force per unit displacement. The negative sign in Eq. (9.1) indicates that the restoring force is always opposite to the displacement x.
Rewriting Eq. (9.1) as

$$\begin{aligned} \frac{d^2\mathbf{x}}{dt^2} &= -\frac{S}{m}\mathbf{x} \\ &= -\omega_0^2\mathbf{x} \end{aligned} \tag{9.2}$$

where $\omega_0 = \sqrt{S/m}$ is the angular frequency of the simple harmonic motion. Let us try to solve Eq. (9.2) through the operator equivalent method which converts a differential equation into an algebraic equation. Calling the differential operator, $D \equiv d/dt$, Eq. (9.2) becomes

$$D^2\,\mathbf{x} = -\omega_0^2\mathbf{x} \tag{9.3}$$

This is an algebraic equation and its solution is

$$D = \pm\, i\omega_0 \tag{9.4}$$

The general solution of Eq. (9.1) can thus be written as

$$x = Ae^{i\omega_0 t} + Be^{-i\omega_0 t} \tag{9.5}$$

A and B are undetermined constants to be determined from the initial conditions. This is the exponential form of the solution of the harmonic force equation, Eq. (9.1). It can be expressed in the trigonometric form through the relation

$$e^{\pm i\omega_0 t} = \cos\omega_0 t \pm i\sin\omega_0 t$$

and as a result, Eq. (9.5) becomes

$$x = (A + B)\cos\omega_0 t + i\,(A - B)\sin\omega_0 t \tag{9.6}$$

Further, let us decide the nature of the constants A and B, as whether these can be real, imaginary or complex. It is evident that the displacement of a moving body must be real and this is possible provided $(A - B) = 0$ or $A = B$. Then the solution, Eq. (9.6) becomes

$$x = 2A\cos\omega_0 t \tag{9.7}$$

However, this is not a general solution of Eq. (9.1) in that it has only one arbitrary constant, whereas it should have two, being the general solution of a differential equation of the second order. The solution will again contain only one constant even if A and B are assumed to be imaginary.

Let A and B be complex numbers represented as

$$\begin{aligned} A &= C_1 + iD_1 \\ B &= C_2 + iD_2 \end{aligned} \tag{9.8}$$

The displacement x, Eq. (9.6) then becomes

$$x = (C_1 + C_2)\cos\omega_0 t - (D_1 - D_2)\sin\omega_0 t + i(D_1 + D_2)\cos\omega_0 t + i(C_1 - C_2)\sin\omega_0 t \tag{9.9}$$

Again imposing the requirement of reality on the displacement, one gets

$$D_1 + D_2 = 0 \quad \text{and} \quad C_1 - C_2 = 0$$

or

$$D_1 = -D_2 \quad \text{and} \quad C_1 = C_2$$

This leads to the following values of A and B [Eq. (9.8)]

$$A = C_1 + i D_1$$
$$B = C_1 - i D_1 \tag{9.10}$$

which are evidently the complex conjugate of each other. Thus the general solution Eq. (9.5) becomes

$$x = (C_1 + iD_1)\, e^{i\omega_0 t} + (C_1 - iD_1)\, e^{-i\omega_0 t} \tag{9.11}$$

This is called the complex exponential form of the solution. The unknown constants C_1 and D_1 are determined from the initial conditions which normally specify the values of the displacement x and the velocity dx/dt at $t = 0$. Denoting the displacement x at $t = 0$ by x_0, and the velocity at $t = 0$ by $\dot{x}_0$, one gets

$$C_1 = \frac{x_0}{2}$$

Now

$$\dot{x} = \frac{dx}{dt}$$
$$= i\omega_0 (C_1 + iD_1)\, e^{i\omega_0 t} - i\omega_0 (C_1 - iD_1)\, e^{-i\omega_0 t}$$

which at $t = 0$ becomes

$$\dot{x}_0 = i\omega_0 (C_1 + iD_1) - i\omega_0 (C_1 - iD_1)$$
$$= -2\omega_0 D_1$$

Thus

$$D_1 = -\frac{\dot{x}_0}{2\omega_0} \tag{9.12}$$

Alternatively, Eq. (9.9) can be put into the equivalent trigonometric form as follows: Dropping the imaginary part for a real displacement, Eq. (9.9) becomes

$$x = (C_1 + C_2)\cos\omega_0 t - (D_1 - D_2)\sin\omega_0 t \tag{9.13}$$

Putting $C_1 + C_2 = C_0 \sin\Phi_0$

$$-(D_1 - D_2) = C_0 \cos\Phi_0$$

Eq. (9.13) becomes

$$x = C_0 \cos\omega_0 t \,\sin\Phi_0 + C_0 \sin\omega_0 t \cos\Phi_0$$
$$= C_0 \sin(\omega_0 t + \Phi_0) \tag{9.14}$$

where

$$C_0 = [(C_1 + C_2)^2 + (D_1 - D_2)^2]^{1/2}$$

$$\Phi_0 = \tan^{-1}\frac{C_1 + C_2}{D_2 - D_1}$$

It is clear from Eq. (9.14) that C_0 is the maximum value of the displacement called the amplitude. The system oscillates between the values $\pm C_0$. The value of C_0 is determined from the total energy of the vibrating system. The angle Φ_0, called the

phase constant, defines the position in the cycle of oscillation at the time $t = 0$; when t is increased by $2\pi/\omega_0$, Eq. (9.14) for the displacement repeats itself and so the periodic time T is

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\frac{S}{m}}} = 2\pi\sqrt{\frac{m}{S}} \tag{9.15}$$

The frequency of oscillation ν is the number of complete vibrations per second, Thus

$$\nu = \frac{1}{T} = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{S}{m}} \tag{9.16}$$

As is obvious from Eq. (9.14), SHM has been expressed by a sine function and such a motion is said to be a sinusoidal function of time. Such a motion is also cosinusoidal as can be easily seen by putting $\Phi_0 = \delta + \pi/2$ Eq. (9.14). Thus

$$\begin{aligned} x &= C_0 \sin\left(\omega_0 t + \delta + \frac{\pi}{2}\right) \\ &= C_0 \cos(\omega_0 t + \delta) \end{aligned} \tag{9.17}$$

A system whose displacement can be described by either a sine or cosine function of time is said to be linear. Herein, the stiffness S is constant with displacement. However, nonlinearity is said to have set in if S does not remain constant with the displacement.

9.1.1 Examples of Simple Harmonic Oscillations

1. A Simple Pendulum

It consists of a small but heavy mass suspended by a non-extensible, light, and flexible string from a perfectly rigid support so that it can be made to oscillate about its position of equilibrium. We will show that the oscillations of this system obey the equation of SHM. Let us consider the Fig. 9.2, which shows a simple pendulum. The point 0 is the equilibrium position and the positions Q and Q' show the extreme positions of the point mass, and thus, correspond to the maximum displacements. P and P' are any arbitrary points on the two sides of the equilibrium position 0.

Let us consider the forces acting on mass m at P. Neglecting the mass of the suspension, the force acting at P due to the weight of the point mass is mg. Resolving the downward force mg into two components, we get a component $mg\cos\varphi$ along the thread and the other $mg\sin\varphi$ along the path of the mass point. The force along the thread keeps it taut during the oscillation. The other component along the path of the point mass is $mg\sin\varphi$ and its direction is towards the mean position so that when the mass is swinging away from the mean position, this force acting in opposite direction to the motion, will act as a decelerating force. The equation of motion is,

$$m\frac{d^2x}{dt^2} = -mg\sin\varphi \tag{9.18}$$

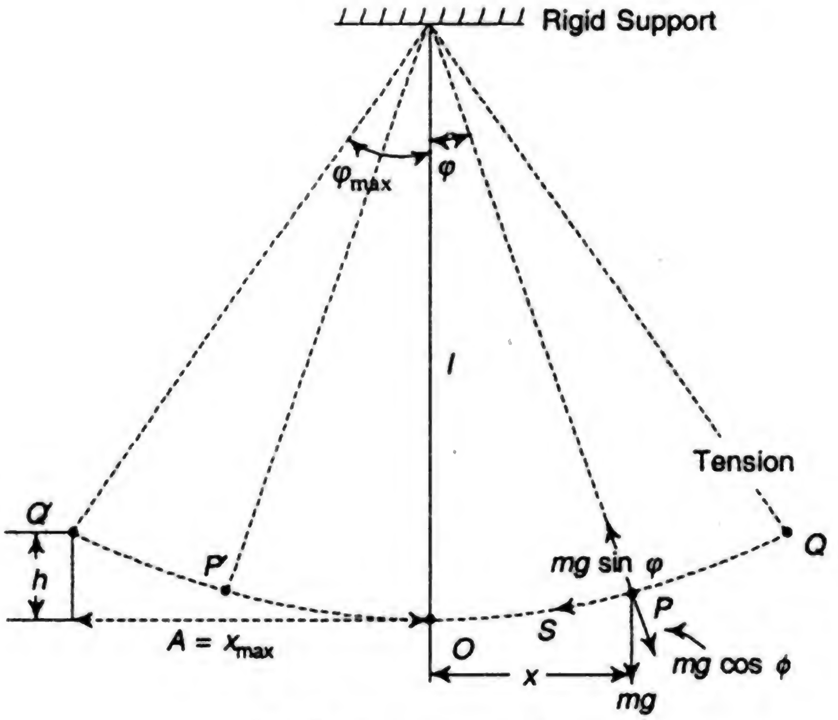


Fig. 9.2 A simple pendulum

The minus sign shows that the force is directed towards the mean position and is opposite to the direction of increase of φ

Now for small φ, say, less than 10°, we write

$$\sin\varphi \approx \varphi$$

so that Eq. (9.18) becomes

$$m\frac{d^2x}{dt^2} = -mg\,\varphi \tag{9.19}$$

also $$\varphi = \frac{S}{l}$$

where S is an arc length which the point mass makes in describing angle φ and l is the length of the suspension. For small φ, S is very nearly equal to x, the horizontal distance between O and P. Rewriting Eq. (9.19), we get,

$$m\,\frac{d^2x}{dt^2} = -\frac{mg}{l}\,x \tag{9.20}$$

This is the equation of SHM with $k = \dfrac{mg}{l}$. The time period T given by Eq. (9.15),

is $$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\dfrac{mg}{lm}}} = 2\pi\sqrt{\frac{l}{g}} \tag{9.21}$$

EXAMPLE 9.1

A pendulum is of length 50 cm. Find its period when it is susp:nded in (i) a stationary lift, (ii) a lift falling at the constant velocity of 5m/s, (iii) a lift falling at the constant, acceleration of $2\text{m}/s^2$, (iv) a lift rising at the constant velocity of 5m /s, and (v) a lift rising at the constant acceleration of 2m/s^2.

It may be remarked that F_B is the total upward force due to the spring and mg is the downward force due to load. Hence F_2 is the resultant downward force on the load.

We, therefore, conclude that the periodic motion of the vertical loaded spring is a simple harmonic motion. It should be noted that the equilibrium position itself corresponds to an extended state of the spring and the oscillations are around this state.

The value of k can be obtained from Eq. (9.22), according to which the value of k (neglecting the sign) is given by:

$$k = \frac{mg}{x_0} \tag{9.26}$$

The time of oscillation is, therefore, given by

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{x_0}{g}} \tag{9.27}$$

Example 9.2 Find the time period and the frequency of the block attached to a spring as shown in Fig. 9.3 (a), (b). The mass of the block is one kg and stretches the spring by 7.0 cm when attached to it.

Solution

The downward force on the spring is

$$mg = 1.0 \times 9.8 = 9.8 \text{ Newton}$$

also $F = -kx$ where k is the spring constant; therefore $k = \dfrac{F}{x} = \dfrac{9.8}{.07} = 140$ N/m

The time period, $\quad T = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{1.0}{140}} = 0.53\,\text{s}$

and the frequency, $\quad f = \dfrac{1}{T} = \dfrac{1}{0.53} = 1.9$ Hz

3. *Torsion Pendulum: Angular Vibrations*

An interesting application of the SHM. is the behaviour of a torsion pendulum which is shown in Fig. 9.4(a). It consists of a suspended wire to which is attached a disc. If we rotate the disc through an angle φ, this will twist the wire. The disc is capable of vibrating in the horizontal plane and the oscillations twist the wire either ways. If we consider a line AB (Fig. 9.4 (b)), on twisting the wire it takes the position AB′. Further, if we now release the disc it is found experimentally that the disc will rotate first in one direction and than in the other, through a certain angle and the time period of these back and forth oscillations is constant.

Let us see how one can explain it. If I is the moment of inertia of the disc about the wire and θ is the rotational displacement, the equation of motion is

$$I\frac{d^2\theta}{dt^2} = -\tau\theta \tag{9.28}$$

$$= 2.2 \times 10^{-2}\ \text{m} - \text{N/rad}$$

4. Compound Pendulum

A compound pendulum also called a physical pendulum, is a body of an arbitrary shape, pivoted at any point so that when the center of mass is displaced on one side, the body starts oscillating in a plane. Unlike a simple pendulum where the entire mass is considered to be situated at the centre of mass, in the case of physical pendulum, we consider the distribution of mass.

Let the distance between the pivot and the centre of gravity of the body be l. Then, if the angle of tilt of the pendulum is φ, Fig. 9.5, the torque on the body due to the weight of the pendulum acting at the centre of mass is given by

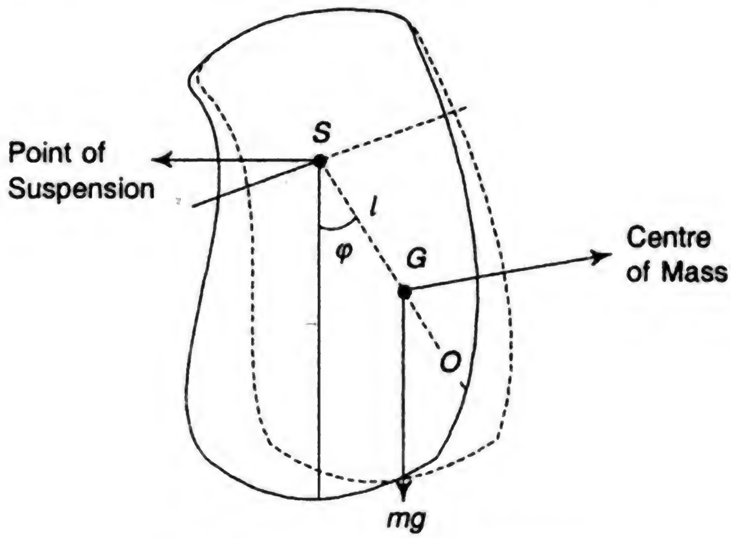


Fig. 9.5 A compound pendulum

$$\Gamma = mgl \sin \varphi$$

For small φ, $\sin \phi \approx \phi$. The torque acts opposite to the direction of increase of φ, we write the above equation as

$$\begin{aligned}\Gamma &= -mgl \sin\varphi \\ &= -k'\varphi \end{aligned} \tag{9.32}$$

where $k' = mgl$. Obviously, it is a case of SHM. Therefore, the time period of oscillation is

$$T = 2\pi\sqrt{\frac{I}{k'}} \tag{9.33}$$

where I is the moment of inertia of the compound pendulum around the axis of pivoting and Eq. (9.33) for a physical pendulum becomes,

$$T = 2\pi\sqrt{\frac{I}{mgl}} \tag{9.34}$$

Defining $$L = \frac{I}{ml}$$

we get $$T = 2\pi\sqrt{\frac{L}{g}} \tag{9.35}$$

where L is called the length of an equivalent simple pendulum. If K is the radius of gyration of the compound pendulum through the center of mass, then the moment of inertia, I, of the pendulum around an axis passing horizontally through the centre of mass is given by

$$I_c = mK^2 \tag{9.36}$$

where m is the mass of the pendulum. Then, according to the theorem of parallel axes, the moment of inertia of the pendulum, I, around the pivot is given by

$$I = I_c + ml^2 \tag{9.37}$$
$$= m\,(K^2 + l^2)$$

Hence
$$T = 2\pi\sqrt{\frac{K^2 + l^2}{gl}}$$

$$= 2\pi\sqrt{\frac{L}{g}} \tag{9.38}$$

where
$$L = \frac{I}{ml} = \frac{K^2 + l^2}{l}$$

Thus if one knows the value of radius of gyration for an irregular body around the axis through the centre of mass, the time period of the oscillation of such a body, can be calculated for different points of pivoting

EXAMPLE 9.4

A thin circular ring is suspended from a peg so that it can oscillate about it. Determine its period of oscillation if its radius is 10 cm.

Solution

Let m and R be the mass and radius of the ring respectively. The moment of inertia of the ring about the peg AB Fig. E9.4 is

$$I = I_o + I_o = mR^2 + mR^2 = 2mR^2$$

(using theorem of parallel axes)

The time period of the ring is

$$T = 2\pi\sqrt{\frac{I}{mgl}} \tag{i}$$

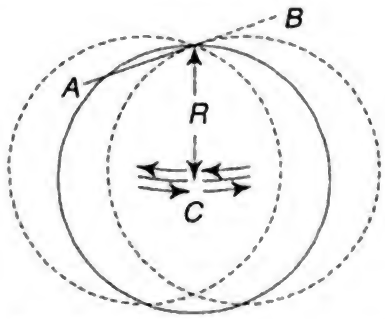


Fig. E9.4 A suspended ring

Because the distance between the point of suspension and centre of gravity is R, hence, $l = R$.

Therefore,
$$T = 2\pi\sqrt{\frac{2mR^2}{mgR}} = 2\pi\sqrt{\frac{2R}{g}} \tag{ii}$$

Here $2R$ is the length of equivalent simple pendulum. When we substitute the values of R = 10 cm and g = 980 m/s^2, we get T = 0.88 s.

Maximum and minimum time periods of a compound pendulum

Squaring Eq. (9.38), one gets

$$T^2 = \frac{4\pi^2 (K^2 + l^2)}{gl}$$

or
$$l^2 - \frac{gT^2}{4\pi^2} l^2 + K^2 = 0$$

This is a quadratic equation in l and gives two values

$$l = \frac{\frac{gT^2}{4\pi^2} \pm \sqrt{\frac{g^2T^4}{16\pi^4} - 4K^2}}{2}$$

Thus,
$$l = \frac{gT^2}{8\pi^2} + \sqrt{\frac{g^2T^4}{64\pi^4} - K^2} \quad \text{or} \quad \frac{gT^2}{8\pi^2} - \sqrt{\frac{g^2T^4}{64\pi^4} - K^2}$$

Similarly, there are two values of l on the other side of the c.m. for which T is the same as for the above two values of l on the first side.
Differentiating the expression for T^2 wrt l, one gets

$$2T\frac{dT}{dl} = \frac{4\pi^2}{g}\left(-\frac{K^2}{l^2} + 1\right)$$

Obviously, T will be a maximum or a minimum when $dT/dl = 0$, that is, when $l^2 = K^2$ or $l = \pm K$ or when $l = K$, since the negative sign has no meaning. The second derivative $\frac{d^2T}{dl^2}$ turns out to be positive when $l = K$, implying that T is a minimum at this value. The minimum time period is given by

$$T_{min} = 2\pi\sqrt{\frac{K^2 + K^2}{gK}} = 2\pi\sqrt{\frac{2K}{g}}$$

Furthermore, we see that if $l = 0$ or ∞, $T = \infty$ or a maximum. Neglecting $T = \infty$ as absurd, we see that the time period of a compound pendulum is maximum when its length is zero, that is the axis of suspension passes through its c.g. As there is no restoring torque, the pendulum will be in neutral equilibrium.

Centres of suspension and oscillation are mutually interchangeable

A point 0, on the other side of G, in line with S and G and at a distance $\frac{K^2}{l} + 1$ from S, or which is the same thing, at a distance $\frac{K^2}{l}$ from G, is called the centre of oscillation, Fig. 9.5. Axis of oscillation is the horizontal axis passing through 0 and parallel to the axis of suspension.

Calling $GS = l_1$ and $GO = l_2$ and the corresponding time periods T_1 and T_2 about the axes of suspension and oscillation respectively, we have

$$T_1 = 2\pi\sqrt{\frac{K^2/l_1 + l_1}{g}} \tag{i}$$

edges from a parallel line through the centre of gravity, G are noted. With these data and Eq. (ix) the accurate value of g can be obtained.

Advantages of a compound pendulum over a simple pendulum

(i) A compound pendulum is easily realisable in actual practice, unlike a simple pendulum.

(ii) A compound pendulum oscillates as a whole and as such there is no lag between the bob and the string as is the case in a simple pendulum.

(iii) The distance between the knife-edges, A and B, in the case of Kater's pendulum is easily measurable. The points of suspension as well as the c.g. of the bob are relatively indefinite points in the case of a simple pendulum.

(iv) A compound pendulum has large moment of inertia due to large mass, and thus, will continue to oscillate for a longer time. Thus the time period can be determined more accurately.

Fig. 9.6 Kater's pendulum

5. *Helmholtz Resonator, Longitudinal Vibrations in a Gas*

A gas column vibrating with its natural frequency is called resonator. The Helmholtz resonator consists of a spherical cavity with two necks, the wider one to receive the incoming sound and the smaller one to be inserted into the ear to hear the sound, (Fig. 9.7).

The only inertia we have to consider is that of the gas in the neck, which moves to and fro like a piston of mass ρAl where A is the cross-sectional area, l the length of the neck, and ρ the density of the gas. There is a change in pressure due to change in volume Ax caused by the movement of the air plug through a displacement x from its equilibrium position. The pressure change is calculated from the equation of state for adiabatic change, that is,

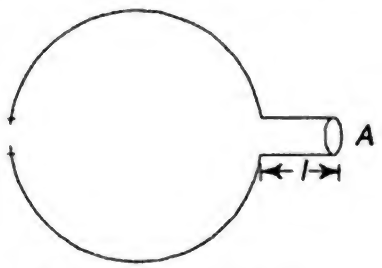


Fig. 9.7 The Helmholtz resonator

$$PV^{\gamma} = \text{constant} \tag{9.39}$$

where γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume. Taking logarithms and differentiating, we get

$$dP + \frac{\gamma}{V}\, PdV = 0$$

or
$$\frac{dP}{P} = -\frac{\gamma}{V}\, dV \tag{9.40}$$

The equation of motion for the plug of air is

$$\rho AP \frac{d^2x}{dt^2} = -\gamma P \frac{Ax}{V} A$$

where force = stress × area = $-\gamma \dfrac{PAx}{V} A$

and the voltage $$V = \frac{q_0}{C} \sin(\omega_o t + \phi_o) \tag{9.48}$$

Both the current and voltage vary harmonically with time.

The energy at a particular instant can be calculated when the condenser is charged to charge q. Thus

$$E = \frac{1}{2}CV^2$$

$$= \frac{1}{2}C\left(\frac{q}{C}\right)^2 = \frac{q^2}{2C} \tag{9.49}$$

This is electrostatic energy. The inductive energy when current I is flowing through the inductance L is

$$E = \int VIdt$$

$$= \int L\frac{dI}{dt}Idt = \int LIdI$$

$$= \frac{1}{2}LI^2 = \frac{1}{2}L\,\dot{q}^2 \tag{9.50}$$

This energy is magnetic in nature. There is an obvious similarity between the mechanical and electrical oscillators. Thus,

	Mechanical	Electrical
Equation of motion	$m\ddot{x} + Sx = 0$	$L\ddot{q} + \frac{q}{C} = 0$
Total energy	$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}Sx^2$	$\frac{1}{2}L\dot{q}^2 + \frac{1}{2}\frac{q^2}{C}$

Depending upon the values of L and C, one can generate a wide range of electrical frequencies Eq. (9.45). For the sake of illustration one can see that for L = 100mH; C = 100μF, the corresponding frequency will be 50 Hz, which lies very low in the audio range. However, for L = 1μH and C = 10pF, the frequency is approximately 50 MHz and lies in the very high frequency region.

9.2 ENERGY OF A SIMPLE HARMONIC OSCILLATOR

A simple harmonic oscillator executing free vibrations is subject to the action of a restoring force, so its energy can be both kinetic and potential. The potential energy at any instant is given by

$$\text{PE} = \int_0^x Sxdx = \frac{1}{2}Sx^2$$

$$= \frac{1}{2}m\,\omega_0^2\,C_0^2\,\sin^2(\omega_0 t + \Phi_0) \tag{9.51}$$

The kinetic energy is given by:

$$\text{KE} = \frac{1}{2}m\dot{x}^2$$

and the time period

$$T_0 = 2\pi \sqrt{\frac{m}{S}}$$

When the system is made vertical, the force of gravity, mg extends the spring in the downward direction, say through a distance x_0. The equilibrium is reached when $mg = Sx_0$.

If the spring is stretched further through a distance x, the restoring force due to the spring acting upward is given by $-S(x + x_0)$. The force due to gravity is still acting downwards and the net force acting on the spring is

$$\text{Net force} = -S(x + x_0) + mg$$
$$= -Sx \qquad \text{(ii)}$$

Equation (ii) is identical with Eq. (i) and represents the same SHM with the time period

$$T_0 = 2\pi \sqrt{\frac{S}{m}}$$

Thus the natural time period of the system is independent of the fact whether the system is horizontal or vertical.

9.3 DAMPED HARMONIC OSCILLATOR

The treatment of free oscillations in Sec. 9.1 is idealized in the sense that it was assumed that no friction was present. However, in a real situation, there is always some resistance offered to a moving body. either at the supports or by the surrounding medium like air. As a consequence, there is a gradual fall in the amplitude of the vibrating body due to some loss of energy by a resistive or viscous element. The resistance offered by the dissipative forces is called damping. When the damping force is small so as not to cause any significant modification of the undamped motion of the body, it is easily proved that the damping force is proportional to the velocity of the vibrating body *.

The equation of motion of a mass m executing simple harmonic oscillations in the presence of a damping force can be written as follows:

$$m\ddot{x} = -Sx - C\frac{dx}{dt}$$

or

$$\ddot{x} + 2r\dot{x} + \omega_0^2 x = 0 \qquad (9.58)$$

Here $\omega_o = \sqrt{S/m}$ is the angular frequency of the undamped SHM, C/m has been put equal to $2r$; r is called the damping factor and $2r$ is the damping force per unit mass at an instant when the vibrating body is moving with unit velocity.

Through the use of the differential operator $D \equiv d/dt$, Eq. (9.58) is reduced to the following algebraic equation:

$$(D^2 + 2rD + \omega_0^2)\, x = 0$$

*For a proof of this statement, refer to S. P. Puri, *Fundamentals of Vibrations and Waves*, ULP, Punjab University, Chandigarh, 1981.

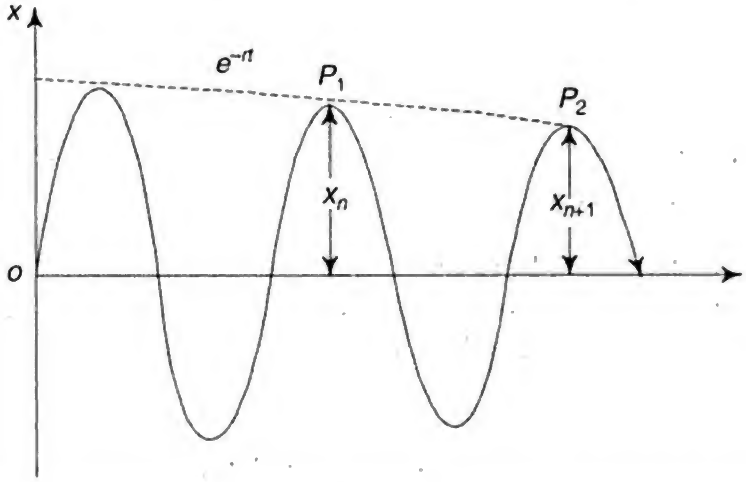


Fig. 9.11 Graphical representation of oscillatory damped motion. The amplitude decays as e^{-rt}

P_2 be the successive maximum corresponding to displacements x_n and x_{n+1} and separated by a time period = $\frac{2\pi}{q}$ (Fig. 9.11).

Thus, if the maxima at P_1 occurs at t_1, the one at P_2 will occur at $t = t_1 + \frac{2\pi}{q}$.

Hence,

$$x_n = a_o e^{-rt_1}$$

$$x_{n+1} = a_o e^{-r\left(t_1 + \frac{2\pi}{q}\right)}$$

Dividing one by the other, we get

$$\frac{x_{n+1}}{x_n} = \exp\left(-r\frac{2\pi}{q}\right) = e^{-rT} \tag{9.80}$$

Taking log on both sides, we get

$$-\log\frac{x_{n+1}}{x_n} = rT \tag{9.81}$$

Defining the logarithmic decrement (δ) of the damped motion as the natural logarithm of the ratio of amplitudes of vibration at the instants of time t and $t + T$, one gets

$$\delta = rT = \frac{2\pi r}{q} = \frac{\pi C}{mq} \quad \left(\because 2r = \frac{C}{m}\right) \tag{9.82}$$

It is clear that δ gives a method of evaluating C, the damping coefficient, since all other quantities can be determined either experimentally or by a displacement curve of the damped motion.

2. *Relaxtion Time or Modules of Decay*

The decay can be characterized by time constant T (also called the damping time), which is defined as the time required for the energy to drop to $e^{-1} = 0.368$ of its initial value.

Hence, $$2r\tau = 1$$

or $$\tau = \frac{1}{2r}$$

When damping in light, $2r \to 0$ and $\tau \to \infty$, the system continues to oscillate with undamped amplitude for a long time.

3. *Quality Factor of an Oscillator*

There is a dimensionless parameter, which is used to characterize the degree of damping of an oscillator. It is defined as

$$Q = \frac{\text{Energy stored in the system}}{\text{Energy dissipated per radian}} \tag{9.83}$$

The energy dissipated per radian is the energy lost by the oscillator during the time it oscillates through one radian. Since during one time period T, the system oscillates through 2π radians, the time required to oscillate through one radian is

$$\frac{T}{2\pi} = \frac{1}{q}$$

It is easy to calculate Q for a lightly damped oscillator. From Eq. (9.78), we get

$$\frac{dE}{dt} = -2rE_o e^{-2rt}$$

$$= -2rE$$

The energy lost in a small time interval Δt is given by

$$\Delta E \approx \left|\frac{dE}{dt}\right| \Delta t$$

$$= 2rE\Delta t$$

Since the time for oscillation through one radian is $\frac{1}{q}$, the energy dissipated is $\frac{2rE}{q}$.

Therefore the quality factor, $$Q = \frac{E}{2rE/q} = \frac{q}{2r} \approx \frac{\omega_0}{2r} \tag{9.84}$$

Q (also called figure of merit) for a lightly damped oscillator is >>1. A heavily damped oscillator loses its energy fast, so that its Q is very low. To give typical values, mention may be made that a tuning fork has $Q \approx 10^3$ and a superconducting microwave cavity has $Q \approx 10^7$.

EXAMPLE 9.7

A particle of mass 5 g moves along a straight line under the influence of two forces (i) a force of attraction towards the origin which in dynes is numerically equal to 40 times the instantaneous distance from the origin, and (ii) a damping force proportional to the instantaneous speed such that when the speed is 10 cm/s the damping

force is 200 dynes. Assuming that at $t = 0$, $x = 20$ cm and $dx/dt = 0$, set up the equation of motion and find the expression for displacement as a function of time. Further, find the amplitude, period and frequency of the damped oscillations.

Solution

If the displacement is denoted by x, then the force of attraction towards the origin is $-40x$ and the damping force is $R_m\, dx/dt$. When $dx/dt = 10$ cm/s,

$$10R_m = 200; \text{ therefore } R_m = 20.$$

The equation of motion is

$$5\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 40x = 0$$

or
$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 8x = 0 \qquad \text{(i)}$$

$x = e^{\alpha t}$ is a solution, provided

$$\alpha^2 + 4\alpha + 8 = 0$$

or
$$\alpha = \frac{-4 \pm \sqrt{16 - 32}}{2} = -2 \pm 2i$$

Thus, the solution of (i) is

$$x = e^{-2t}(A\cos 2t + B\sin 2t)$$

where A and B are undetermined constants.

These may be obtained from the initial conditions

$$x = 20 \quad \text{at} \quad t = 0$$

$\therefore$
$$A = 20$$

Further,
$$\frac{dx}{dt} = 0 \text{ at } t = 0$$

$$\frac{dx}{dt} = e^{-2t}(-40\sin 2t + 2B\cos 2t) + (-2e^{-2t})$$
$$(20\cos 2t + B\sin 2t)$$

At $t = 0$, one gets

$$0 = 2B - 40$$

or
$$B = 20$$

Hence the equation of motion becomes

$$x = e^{-2t}(20\cos 2t + 20\sin 2t)$$

Amplitude
$$= [\sqrt{(20)^2 + (20)^2}\,]\, e^{-2t} = 20\sqrt{2}\, e^{-2t}$$

$$\text{Period} = \frac{2\pi}{2} = \pi \text{ s}$$

$$\text{Frequency} = \frac{1}{\pi} \text{ Hz}$$

EXAMPLE 9.8

The natural frequency of a mass vibrating on a spring is 20 Hz while its frequency with damping is 16 Hz. Find the logarithmic decrement.

where C_1 and C_2 are constants to be determined from initial conditions.

At $t = 0$, let $q = q_o$ and $I = \frac{dq}{dt} = 0$, then $q_o = C_1 + C_2$

$$0 = -\frac{R}{4L}(C_1 + C_2) + C_1\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} - C_2\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Putting $C_1 = q_0 - C_2$ in the above equation, we get

$$\frac{R}{2L}q_o - (q_o - C_2)\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} + C_2\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = 0$$

or
$$2\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\,C_2 = q_0\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} - \frac{R}{2L}q_o$$

Thus,
$$C_2 = \frac{q_0}{2}\left[1 - \frac{1}{2\frac{L}{R}\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}}\right]$$

and
$$C_1 = \frac{q_0}{2}\left[1 + \frac{1}{2\frac{L}{R}\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}}\right]$$

Three cases arise depending upon the relative values of $\frac{R^2}{4L^2}$ and $\frac{1}{LC}$. These are the following:

Case I: $\frac{R^2}{4L^2} > \frac{1}{LC}$

When the resistance is high so that this condition is fulfilled, then from Eq. (9.87), it is obvious that the discharge is non-oscillatory and decays with time (Fig. 9.10 (i)).

Case II: *Critically damped motion*. When the values of the circuit elements are such that

$$\frac{R^2}{4L^2} = \frac{1}{LC}$$

the circuit is said to the critically damped. The charge on the condenser decays exponentially with time (Fig. 9.10 (ii)).

Case III *Oscillatory motion*. When the condition $\frac{R^2}{4L^2} < \frac{1}{LC}$ is fulfilled, then Eq. (9.87) becomes

The quality factor, $$Q = \frac{L\omega}{R}$$

$$= \frac{2.22 \times 10^3}{100} = 22.2$$

The amplitude of oscillations $$A = q_o e^{-\frac{R}{2L}t}$$

$$= q_o e^{-2.5 \times 10^2 t}$$

When amplitude becomes 5 per cent of the initial value then

$$\frac{q_0}{20} = q_o e^{-2.5 \times 10^2 t}$$

Taking logarithms $$-\log_e 20 = -2.5 \times 10^2 t$$

$$t = \frac{\log_e 20}{2.5 \times 10^2} = 11.98 \text{ ms}$$

9.5.2 Electromagnetic Damping in a Moving Coil Galvanometer

A moving coil galvanometer consists of a current-carrying rectangular coil on an axis in a uniform magnetic field. The magnetic field is provided by a permanent magnet, so shaped that the moving coil experiences the same magnitude of the field at all orientations. The steady current to be measured, produces a torque that is proportional to the current. The coil rotates under the electromagnetic torque and comes to an equilibrium position where the turning torque is counterbalanced by the restoring torque provided by the stiffness of the suspension.

When the suspended coil of a galvanometer rotates in a strong magnetic field, it is resisted in the open circuit by the following two causes:

(i) Viscous drag of air and mechanical friction. The damping force is approximately proportional to the angular velocity of the system and is usually negligibly small.

(ii) Induced currents in the neighbouring conductors. The open circuit damping couple is proportional to the angular speed $\left|\frac{d\theta}{dt}\right|$ according to the law of electromagnetic induction. This is represented by $-b_e \frac{d\theta}{dt}$ where b_e is the damping coefficient. However, when the circuit is closed, there results an additional damping due to the induced current in the coil. This is inversely proportional to the resistance of the circuit and is given by $\frac{-\psi}{R}\frac{d\theta}{dt}$, where ψ involves the coil constants such as area, magnetic flux and so on.

If θ denotes the angular displacement from the equilibrium position, the equation of motion is

$$I \frac{d^2\theta}{dt^2} = -C\theta - b_e \frac{d\theta}{dt} - \frac{\Psi}{R}\frac{d\theta}{dt} \tag{9.92}$$

where C is the restoring couple per unit twist of the suspension and I is the moment of inertia of the vibrating system. Thus, rewriting Eq. (9.92), we get

$$\frac{d^2\theta}{dt^2} + \frac{1}{I}\left(b_e + \frac{\psi}{R}\right)\frac{d\theta}{dt} + \omega_0^2\theta = 0 \tag{9.93}$$

where $\omega_o = \sqrt{\frac{C}{I}}$ and putting $\frac{1}{I}\left(b_e + \frac{\psi}{R}\right) = \gamma_e$, the solution of Eq. (9.93) in analogy with Eq. (9.63), is

$$\theta = C_1 \exp\left[\left(-\frac{\gamma_e}{2} + \sqrt{\frac{\gamma_e^2}{4} - \omega_0^2}\right)t\right] + C_2 \exp\left[\left(-\frac{\gamma_e}{2}\sqrt{\frac{\gamma_e^2}{4} - \omega_0^2}\right)t\right] \tag{9.94}$$

where C_1 and C_2 are undetermined constants to be evaluated from the initial conditions. Three cases arise.

Case I: *Dead-beat motion*

If the damping is high such that $\frac{\gamma_e^2}{4} > \omega_0^2$, then there will result two real roots of the Eq. (9.94). Calling these α and β, we get

$$\theta = C_1e^{-\alpha t} + C_2e^{-\beta t}$$

$$= C_1 \exp\left[\left(-\frac{\gamma_e}{2} + \sqrt{\frac{\gamma_e^2}{4} - \omega_0^2}\right)t\right] + C_2 \exp\left[\left(-\frac{\gamma_e}{2} - \sqrt{\frac{\gamma_e^2}{4}\omega_0^2}\right)t\right]$$

The displacement decays exponentially without any change of direction, as the motion is non-oscillatory. This is the case of dead beat motion.

Case II: *Critical Damping*

When $\frac{\gamma_e^2}{4} = \omega_0^2$ the galvanometer is said to be critically damped. The coil comes to rest in a minimum of time after deflection and the direction of motion never reverses.

Case III: *Light Damping: Ballistic Motion*

When $\gamma_0^2/4 < \omega_0^2$, both the roots α and β become imaginary and the solution becomes

$$\theta = \exp\left(-\frac{\gamma_e t}{2}\right)\left[C_1\exp\left[i\sqrt{\omega_0^2 - \frac{\gamma_e^2}{4}}\,t\right] + C_2 \exp\left[-i\sqrt{\omega_0^2\,\frac{\gamma_e^2}{4}}\,t\right]\right]$$

This equation can be recast in the form

$$\theta = C_o \exp\left(-\frac{\gamma_e t}{2}\right)\sin(qt + \phi_o) \tag{9.95}$$

$$= f_0 \frac{-\omega^2 \sin \omega t - 2r\omega \cos \omega t + \omega_0^2 \sin \omega t}{\left(\omega_0^2 - \omega^2\right)^2 + 4r^2\omega^2}$$

$$= f_0 \frac{\left(\omega_0^2 - \omega^2\right)\sin \omega t - 2r\omega \cos \omega t}{\left(\omega_0^2 - \omega^2\right)^2 + 4r^2\omega^2}$$

$$= b \sin \omega t \cos \psi - b \cos \omega t \sin \psi$$

$$= b \sin (\omega t - \psi) \qquad (9.102)$$

where the following substitutions have been made

$$b \cos \psi = \frac{f_0\left(\omega_0^2 - \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2 + 4r^2\omega^2}$$

$$b \sin \psi = \frac{2r\omega f_0}{\left(\omega_0^2 - \omega^2\right)^2 + 4r^2\omega^2}$$

so that
$$b = \frac{f_0}{\left[\left(\omega_0^2 - \omega^2\right)^2 + 4r^2\omega^2\right]^{1/2}}$$

and
$$\tan \psi = \frac{2r\omega}{\left(\omega_0^2 - \omega^2\right)}$$

The complete solution, therefore, is

$$x = C_0 e^{-rt} \sin (qt + \varphi_0) + b \sin (\omega t - \psi) \qquad (9.103)$$

The first term is the transient term since it dies away with time as e^{-rt}. During the transient state, the oscillator oscillates neither with its natural frequency nor the frequency of the impressed force. The second term is called the steady-state term and governs the motion of the oscillator after the transient term has ceased to be effective. During the steady state, the oscillator performs forced oscillations with the impressed force frequency.

Let us discuss the steady-state solution

$$x = b \sin (\omega t - \psi) \qquad (9.104)$$

Substituting the value of b, one gets

$$x = \frac{f_0 \sin (\omega t - \psi)}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

$$= \frac{F_0 \sin (\omega t - \psi)}{[(m\omega_0^2 - m\omega^2)^2 + 4r^2m^2\omega^2]^{1/2}}$$

$$= \frac{F_0 \sin (\omega t - \psi)}{\omega[R_m^2 + (m\omega - S/\omega)^2]^{1/2}}$$

Use has been made of the substitutions

$$R_m = 2\,rm$$
$$m\omega_0^2 = S$$

Therefore, $$b = \frac{F_0}{\omega[R_m^2 + (m\omega - S/\omega)^2]^{1/2}}$$

Defining Z_m, the mechanical impedance as

$$Z_m = \left[R_m^2 + \left(m\omega - \frac{S}{\omega}\right)^2\right]^{1/2} \quad (9.105a)$$

X_m, the mechanical reactance as

$$X_m = \left(m\omega - \frac{S}{\omega}\right) \quad (9.105b)$$

and R_m, the mechanical resistance as

$$R_m = 2rm \quad (9.105c)$$

one gets $$Z_m^2 = X_m^2 + R_m^2 \quad (9.106)$$

The displacement x [Eq. (9.104)] becomes

$$x = \frac{F_0}{\omega Z_m} \sin(\omega t - \psi) \quad (9.107)$$

The steady-state motion is completely specified by the amplitude b and phase angle ψ. Three eases arise dependipg upon the value of the driving frequency.

Case I: *Low driving frequency,* $\omega < \omega_0$

The phase angle

$$\psi = \arctan \frac{2r\omega}{(\omega_0^2 - \omega^2)}$$

and for this condition

$$\psi \to 0 \quad (9.108)$$

This shows that the driving force and the resulting displacement are in phase.

The amplitude

$$b = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

$$\approx \frac{F_0/m}{\omega_0^2} = \frac{F_0}{S} \quad (9.109)$$

Case II: *Resonance,* $\omega = \omega_0$

The amplitude at resonance becomes maximum and is given by

$$b_{max} = \frac{f_0}{2r\omega}$$

and the phase angle

$$\psi = \text{arc tan } \frac{2r\omega}{(\omega_0^2 - \omega^2)}$$

$$= \text{arc tan } \infty$$

$$= \frac{\pi}{2}$$

Thus the displacement lags the impressed force by $\frac{\pi}{2}$.

The value of b_{max} depends upon the damping coefficient r. In the absence of damping, i.e. when $r = 0$, b_{max} becomes infinite; however, in reality some frictional mechanism is always operative.

It may be remarked that resonance does not occur at $\omega = \omega_0$ but at a frequency slightly less. This may be easily shown as follows: Equating $db/d\omega = 0$, we get

$$\frac{db}{d\omega} = \frac{d}{d\omega}\left[\frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}\right]$$

$$= -f_0 \frac{-4\omega(\omega_0^2 - \omega^2) + 8r^2\omega}{2[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{3/2}}$$

$$= 0$$

which yields the condition

$$-4\omega(\omega_0^2 - \omega^2) + 8r^2\omega = 0$$

or

$$\omega_0^2 - \omega_2 - 2r^2 = 0$$

Solving it for $\omega = \omega_{max}$, one gets

$$\omega_{max} = \omega_0 \sqrt{1 - \frac{R_m^2}{2m^2\omega_0^2}} \tag{9.110}$$

Obviously the frequency at which the resonance occurs is slightly less than ω_0. However, lesser the damping, more near it is to the natural frequency.

Case III: *High driving frequency*, $\omega \gg \omega_0$

The amplitude becomes

$$b = \frac{f_0}{\sqrt{(\omega^4 + 4r^2\omega^2)}} \cong \frac{f_0}{\omega^2}$$

as r is a small quantity for light damping.

The phase angle

$$\psi = \text{arc tan } \frac{2r\omega}{(\omega_0^2 - \omega^2)}$$

$$= \text{arc tan } (-0)$$

$$= \pi$$

As the frequency ω of the impressed force is increased, the amplitude decreases and the phase tends towards π. The dependence of the amplitude and the phase angle upon ω is depicted in (Fig. 9.13) (a) and (b). The phase always lags behind the applied force.

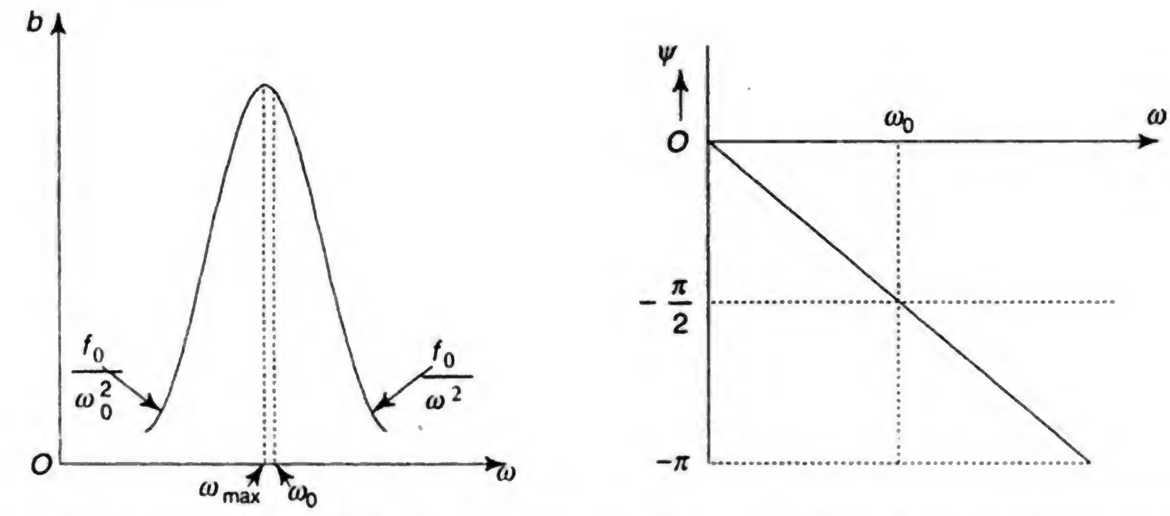


Fig. 9.13 (a) Dependence of the amplitude b upon the frequency ω of the driving force; (b) phase difference ψ as a function of the driving frequency ω

9.7 RESONANCE—QUALITY FACTOR OF A DRIVEN OSCILLATOR

Let us apply the considerations of energy to the case of the driven oscillator. The displacement for the steady state is

$$x = \frac{F_0}{\omega Z_m} \sin(\omega t - \psi) \tag{9.107}$$

The velocity is given by

$$\dot{x} = \frac{F_0}{Z_m} \cos(\omega t - \psi) \tag{9.111}$$

Now

$$\text{KE} = \frac{1}{2} m\dot{x}^2$$

$$= \frac{1}{2}\, m\, \frac{F_0^2}{Z_m^2} \cos^2(\omega t - \psi) \tag{9.112}$$

and

$$\text{PE} = \frac{1}{2}\, Sx^2$$

$$= \frac{1}{2}\, S\, \frac{F_0^2}{\omega^2 Z_m^2} \sin^2(\omega t - \psi) \tag{9.113}$$

The total energy

$$E = \text{KE} + \text{PE}$$

$$= \frac{1}{2} \frac{F_0^2}{Z_m^2} \left[m \cos^2(\omega t - \psi) + \frac{S}{\omega^2} \sin^2(\omega t - \psi)\right] \tag{9.114}$$

Since energy is time-dependent, so we concentrate on the time average values. Since

$\langle E(\omega)\rangle$ takes the following form in view of the approximation

$$\langle E(\omega)\rangle = \frac{1}{4}\frac{F_0^2}{m}\frac{2\omega_0^2}{4\omega_0^2(\omega-\omega_0)^2+(2r\omega_0)^2}$$

$$= \frac{1}{8}\frac{F_0^2}{m}\frac{1}{(\omega-\omega_0)^2+r^2} \tag{9.117}$$

The function $[(\omega - \omega_0)^2 + r^2]^{-1}$ contains the total frequency dependence of $\langle E(\omega)\rangle$ and is called a *resonance curve* or *Lorentzian* (Fig. 9.14). Its maximum height at resonance, i.e. $\omega = \omega_0$ is $1/r^2$ and it falls to one-half the maximum

when $$(\omega - \omega_0)^2 = r^2$$

or $$\omega - \omega_0 = \pm r$$

The full width at half the maximum value is called the resonance width $\Delta\omega$ (Fig. 9.15). If ω_- and ω_+ are the frequencies on the negative and positive sides of ω_0, where the amplitude falls to half, then

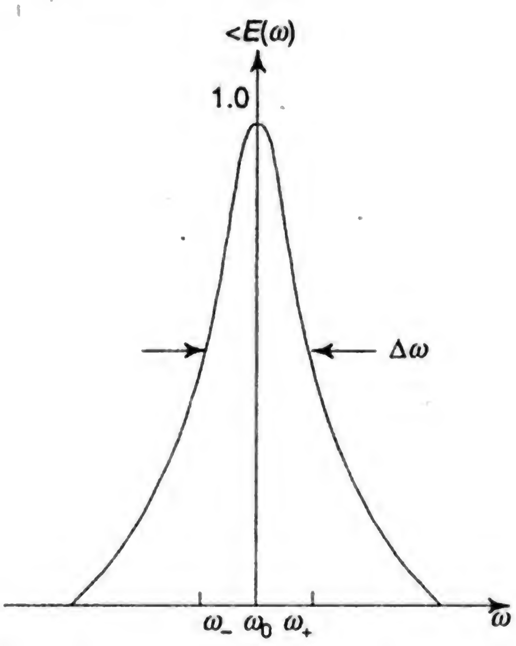


Fig 9.15 Resonance curve of < E (ω) > versus ω

$$\omega_+ - \omega_- = 2r$$

$$\Delta\omega = 2r$$

As the width of the resonance curve decreases, the curve becomes narrower and higher and the frequency range over which the oscillator responds, becomes smaller. The oscillator becomes progressively more selective.

The frequency-selectivity property of the oscillator is characterised quantitatively by the quality factor Q of the oscillator in this case also. It was shown in Eq. (9.84) that the quality factor for a lightly damped oscillator is

$$Q = \frac{\omega_0}{2r}$$

In the case of the driven oscillator, the width of the resonance curve $\Delta\omega = 2r$, therefore the quality factor becomes

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\text{Resonance frequency}}{\text{Frequency width of the resonance curve}} \tag{9.118}$$

The higher value of Q implies that the oscillator is more selective than the one with smaller value of Q. Certain atomic systems have $Q \sim 10^8$ and since the resonance frequency is determined by atomic constants, the frequency of oscillation is reasonably independent of the external conditions. Such atomic clocks serve as frequency standards, since in view of the stability and accuracy of the frequency, these far outstrip the astronomical standards of time.

The definition of Q in terms of energy absorbed by the driven oscillator will lead to the .same result [Eq. (9.118)]. [Refer to Example 9.13 for the derivation.]

9.8 ELECTRICAL RESONANCE

Whenever a system is acted upon by an external action, which is varying periodically in time, the response of the system as measured by its amplitude and phase, or the power absorbed, undergoes rapid changes as the frequency of the external field of force passes through a certain range of values. The response is characterised by two parameters—a frequency ω_0 and the natural width of the driven system—and the resonance condition is said to occur when the interaction between the driven and the driving systems has been maximised. The maximum amplitude occurs at or near ω_0 and the most marked changes occur over a range $\pm\Gamma$ wrt the maximum. It is proposed to examine the case of electrical resonance.

An electrical circuit consisting of an inductance L, a capacitance C, and a resistance R is connected to an external source of alternating voltage $V_o \sin \omega t$ (Fig. 9.16).

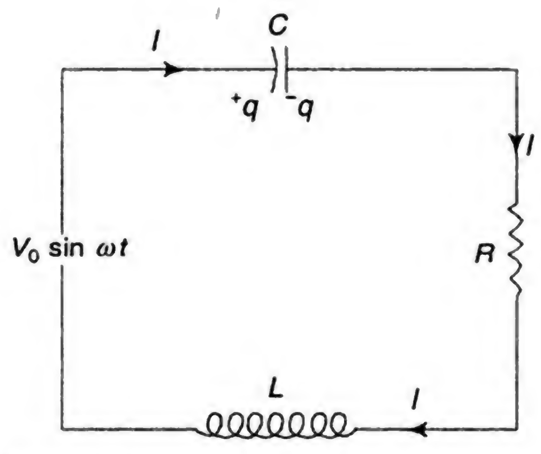


Fig. 9.16 An electrical circuit containing L, C and R in series driven by an external sinusoidal voltage

The circuit will work as a driven oscillator and the resistive loss in this case will be compensated by the supply of energy by the voltage source.

Thus
$$\left(\left(\frac{d^2}{dt^2}+2r\frac{d}{dt}+\omega_0^2\right)(x_1+x_2)\right)$$
$$=\left(\frac{d^2}{dt^2}+2r\frac{d}{dt}+\omega_0^2\right)x_1+\left(\frac{d^2}{dt^2}+2r\frac{d}{dt}+\omega_0^2\right)x^2$$
$$=F_1(t)+F_2(t)$$

It is due to the linearity of the equation of motion (because only the first power of x occurs) that the superposition principle holds. Had there been a quadratic term like x^2 in the equation of motion, its presence will mix and multiply two simultaneous driving frequencies ω_1 and ω_2 and produce a range of harmonic fequencies $2\omega_1$, $3\omega_1$...; $2\omega_2$, $3\omega_2$..., and combination or sideband frequencies $\omega_1 + \omega_2$, $\omega_1 + 2\omega_2$, $\omega_1 - 2\omega_2$; etc.

EXAMPLE 9.14

Show that in the presence of damping, the average power dissipated per cycle in the steady state of a forced harmonic oscillator is exactly equal to the average power supplied by the driving force to maintain the amplitude of oscillation.

Solution

The presence of a damping force causes the continual dissipation of energy in the case of a harmonic oscillator. The oscillator will be able to maintain its energy or amplitude provided the driving force supplies the energy regularly.

Let the driving force by $\mathbf{F} = \mathbf{F}_0 \sin \omega t$. Then the rate of work done by it is

$$P=\frac{dW}{dt}=\mathbf{F}\cdot\frac{d\mathbf{x}}{dt}$$
$$=F_0\sin\omega t\,\frac{F_0}{Z_m}\cos(\omega t-\psi)$$

since force and velocity are in the same phase, at the resonance.

Now

$$P_{\text{av}}=\frac{\text{Power supplied in one cycle}}{\text{Period of the cycle}}$$
$$=\frac{\int_0^T P\,dt}{T}$$
$$=\frac{F_0^2}{Z_mT}\int_0^T\sin\omega t\,(\cos\omega t\cos\psi+\sin\omega t\sin\psi)\,dt$$
$$=\frac{F_0^2}{Z_mT}\left[\int_0^T\frac{\sin 2\omega t}{2}\cos\psi\,dt+\int_0^T\left(\frac{1-\cos 2\omega t}{2}\right)\sin\psi\,dt\right]$$
$$=\frac{F_0^2}{2Z_m}\sin\psi$$

because $$\int_0^T \frac{\sin 2\omega t}{2} \cos \psi \, dt = 0,$$

Since $$\psi = \arc \tan \frac{2r\omega}{(\omega_0^2 - \omega^2)}$$

we have $$\sin \psi = \frac{b(2r\omega)}{f_0} = \frac{R_m}{Z_m}$$

Thus $$P_{av} = \frac{F_0^2 R_m}{2Z_m^2} \tag{i}$$

Since R_m, the mechanical resistance is the resistive force per unit velocity, the total resistive force is $R_m \dot{x}$ and the rate of work done by the resistive force is

$$(R_m \dot{x})\ \dot{x} = R_m \dot{x}^2 = R_m \frac{F_0^2}{Z_m^2} \cos^2 (\omega t - \psi)$$

or $$\langle R_m \dot{x}^2 \rangle = \frac{F_0^2 R_m}{2Z_m^2} \tag{ii}$$

which agrees with Eq. (i), proving thereby that the power supplied to maintain the oscillation, is equal to the power dissipated against the frictional force.

EXAMPLE 9.15

The amplitude of a forced vibration is given by

$$b = \frac{f_0}{\sqrt{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]}}$$

If the quality factor of the oscillator $Q = 50$, calculate the value of b/b_{max}, when $\omega/\omega_0 = 0.99$.

Solution

The amplitude is maximum at resonance, i.e. at $\omega = \omega_0$. Calling the maximum amplitude b_{max}, one gets

$$b_{max} = \frac{f_0}{2r\omega_0}$$

The quality factor, $Q = \omega_0/2r = 50$, therefore $2r = \omega_0/50$.
The expression for amplitude is

$$b = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

$$= \frac{f_0}{\omega_0^2 [(1 - \omega^2/\omega_0^2)^2 + 4r^2\omega^2/\omega_0^4]^{1/2}}$$

Putting the value $2r = \dfrac{\omega_0}{50}$, we get

$$b = \frac{f_0}{\omega_0^2[(1-\omega^2/\omega_0^2)^2 + 1/50^2\,\omega^2/\omega_0^2]^{1/2}}$$

$$= \frac{50\,f_0}{\omega_0^2[(2500\,(1-\omega^2/\omega_0^2)^2 + \omega^2/\omega_0^2]^{1/2}}$$

Putting $\dfrac{\omega}{\omega_0} = 0.99$, one gets

$$b = \frac{50\,f_0}{\omega_0^2[(2500\,(1-(0.99)^2)^2 + (0.99)^2]^{1/2}}$$

$$= \frac{50\,f_0}{1.4\omega_0^2}$$

Rewriting $b_{\text{max}} = \dfrac{f_0}{2r\omega_0} = \dfrac{50 f_0}{\omega_0^2}$

Therefore, one gets

$$\frac{b}{b_{\text{max}}} = \frac{1}{1.4} = 0.71$$

QUESTIONS

9.1 What is meant by periodic motion? Mention a few examples.

9.2 What do you understand by simple harmonic motion? Obtain the differential equation for simple harmonic motion and write down the formulae for angular frequency and time period.

9.3 Express the solution of simple harmonic motion in the exponential, complex exponential or trigonometric form. Each form contains two real constants whose values are found from the initial conditions. What are these conditions?

9.4 What is the requirement for linearity of oscillations?

9.5 Show that for a harmonic oscillator, the average potential energy is equal to the average kinetic energy and each is equal to half the toal energy.

9.6 Assuming damping to be proportional to velocity, write down the differential equation for a damped harmonic oscillator. Solve the differential equation so obtained and discuss in detail all the three cases. Find an expression for frequency in the case of oscillatory motion.

9.7 Derive expressions for the average total energy and average power dissipation in the case of a damped harmonic oscillator. Further, show that the rate of change of the total energy gives the rate of doing work against the frictional force.

9.8 Define the quality factor of a damped oscillator. Deduce an expression for it.

9.9 What are forced vibrations? Examine the effect of a periodic force on the motion of a damped oscillator. Discuss the 'transient' as well as the steady state terms in the complete solution.

9.10 The amplitude b of forced vibration in a mechanical system is given by

$$b = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

Show that for (i) $\omega \ll \omega_0$, the response is independent of the mass, (ii) for $\omega = \omega_0$, the amplitude at resonance depends inversely on the damping constant r and (iii) for $\omega \gg \omega_0$, the response is independent of the spring constant of the system.

9.11 Show that the amplitude resonance occurs at a frequency that is slightly less than ω_0. However, lighter the damping, nearer is this frequency to the natural frequency.

9.12 Define the quality factor Q for a driven oscillator in terms of (i) bandwidth and (ii) energy. Show that both definitions lead to the same result.

9.13 What is a Lorentzian or resonance curve? Show that its maximum height at resonance, i.e. $\omega = \omega_0$ is $1/r^2$.

9.14 What is the sharpness or frequency-selectivity of an oscillator? How is it characterized?

9.15 Why does an atomic clock serve better than an astronomical time standard?

9.16 State the superposition principle. Why does it hold for solution of the harmonic oscillator equation?

9.17 Define relaxation time or modulus of decay of a damped oscillating system. Is it in any way useful for determining the logarithmic decrement of the system?

9.18 What is meant by logarithmic decrement and damping factor? How is the damping factor determined experimentally?

9.19 Deduce the condition under which the discharge of a condenser through an inductance and a resistance is oscillatory. Deduce expressions for (a) frequency, (b) power dissipation, and (c) quality factor of the LCR circuit.

9.20 Discuss the case of electromagnetic damping and deduce the conditions when the galvanometer is (a) dead-beat, (b) oscillatory, and (c) ballistic. Mention the factors for making the motion ballistic.

9.21 Show that the energy of a damped simple harmonic oscillator falls to $1/e$ of its initial value in an interval of time equal to the relaxation time.

9.22 Prove that the amplitude of damped motion falls to $1/e$ of its initial value in an interval of time equal to two relaxation times.

9.23 The smaller the damping, larger will be the relaxation time and greater will be the quality factor. Is it so?

9.24 Show that damping has negligible effect on the frequency of a harmonic oscillator, if its quality factor (Q) is high.

9.25 Show that for a pure LC circuit, the quality factor $Q = \infty$

9.26 Is there any semblance between a mechanical and an electrical damped harmonic oscillator? If so, bring out a detailed parallelism.

9.27 Discuss the conditions for oscillatory discharge of a condenser through a circuit containing resistance and inductance. Does the presence of the resistance affect the amplitude and frequency of damped oscillations?

9.28 Show that the equations governing the free oscillations of the following systems are similar:

(i) Torsional pendulum

(ii) Simple pendulum

(iii) Helmholtz resonator

10

Frames of Reference

In order to discuss the motion of a mechanical system, one has to specify its position as a function of time, and it is only meaningful to give the position relative to a fixed point. For instance, the position of a flying aircraft is specified with respect to the coordinate system fixed on the earth; the motion of a charged particle in a particle accelerator is given relative to the accelerator. The system with respect to which the motion is discussed is called a frame of reference. The choice of a particular frame of reference is dictated by the convenience of the problem. The acceleration of a body can be caused by the interaction of other bodies or it can arise from some distinctive properties of the reference frame itself. A passenger in a train experiences a jolt on the sudden start or stop of the train. This is an evidence that the carriage is in nonuniform motion relative to the earth.

10.1 A FEW COMMON DEFINITIONS

For a clearer understanding of certain oft-repeated terms, such as particle, reference frame, coordinate system, clock, event, etc., we will digress a little to offer some comments on these basic notions.

Particle

A particle is a system that can, for all practical considerations be localized at a point. It is characterized by its mass (determining its response to the applied force) and charge (determining its interaction with the electrical charges in the rest of the world). Elementary particle physics presents us with a host of particles, such as electrons, protons, neutrons, pions, etc. which, in addition to these properties, have other attributes, such as spin (intrinsic angular momentum), magnetic dipole moment, etc. During its lifetime, a particle is a system specified by constant values of its parameters.

Rigid Body

As defined in Chapter 8, a body is said to be rigid if the distance between any two particles of which remains constant, under conditions of rest or motion. One can measure the distance by a measuring rod, itself a rigid body. As such, the definition

of rigidity is circular and we accept the rigid body as a basic concept*. Nonetheless, its existence is basic to our idea of the reference frame in terms of which the motion of a particle is specified.

Reference Frame

One can specify the position of one body only relative to another. A reference frame is the space determined by a rigid body regarded as a base. One can imagine the extension of the rigid body as far as desired by a lattice of measuring rods. A point is located in space by knowing its three coordinates with respect to the origin of the reference system.

The state of rest or motion of a body is specified with reference to a frame; the former refers to the situation when the body occupies the same position in the reference system, whereas the latter implies a change in the position.

Relativistically, one includes time-keeping in addition to position-measuring in the concept of a reference frame. For this purpose, one imagines identical clocks at the lattice points distributed throughout the surrounding space.

Clock

A clock is an entity that repeats itself regularly like a pendulum or an alternating electromagnetic field. In relativity, one has to synchronize the clocks according to a specified formula by a master clock and then distribute these throughout the space. However, in Newtonian mechanics, time is considered absolute and flowing uniformly for all bodies in the universe and so there is no need of any synchronization at all. The universal nature of time in Newtonian mechanics is not any supplementary hypothesis, since it is a direct outcome of the assumption of instantaneous action at a distance. In electrodynamics, where the signal velocity is finite, time is not absolute. Relativistically, each reference frame has its own array of clocks.

Event

An event is specified by the space coordinates of the point of its location as well as the time of its occcurrence. Thus an event is known completely if we know all its coordinates (x, y, z, t). Obviously the transformations that relate an event as observed by observers in two reference frames involve time as well as space coordinates.

10.2 INERTIAL REFERENCE FRAMES

Absolute space is an imagined framework in which bodies move, which, without any relation to anything external, is always similar and immovable. Since experiment only reveals relative motion, absolute space has no physical significance. At best one can locate one body only with respect to another say the earth in relation to the sun, the sun in relation of stars, the stars relative to the globular clusters and so on. Just as there is no absolute motion, there is no absolute space or absolute inertial frame of reference. Two frames of reference can be said to be inertial frames of

*The concept of a rigid body is untenable at relativistic speeds, as may be seen in textbooks on special theory of relativity.

reference with respect to one another when they are either at rest or in uniform relative motion with respect to one another.

For practical purposes, an inertial frame is that frame in which a body moves with constant velocity only if there is no net force on it. Newton's first law of motion is an affirmation of the existence of inertial frames. In these frames Newton's first law of motion always holds good.

Another property that can be utilized for defining inertial frames is the one according to which the equation of motion of a body takes on the simplest form, in the sense that it is free from certain additive terms which are characteristic of frames accelerated with respect to inertial frames*. The additional terms called inertial forces arise from the rotation of the frame or translational acceleration. The fixed star frame is an example of an inertial frame.

We define a local inertial frame as a reference frame in which a body, shielded from all external influences, has zero acceleration. In order to achieve these criteria, it is assumed that the local inertial frame is in free fall in the prevailing gravitational fields, the gravitational field due to the material content of the frame itself being zero. In addition, the frame has zero spin relative to the fixed-stars. An orbiting satellite without spin and free of drag, could idealise for the local inertial frame. Inside such a satellite, a bullet fired will move with constant velocity in a straight line relative to the cabin.

10.3 COORDINATE TRANSFORMATIONS WITHIN A REFERENCE FRAME

In a certain reference system, there are infinitely many possible choices of origin and direction of axes. If we change the given coordinate system to another as a basis, the different terms in the equation will transform to the new ones. When an equation preserves the same form under a transformation, it is said to be invariant under that transformation. However, if two quantities change according to the same law, these are called covariant to each other and the equation is said to be covariant when its two sides are covariant.

Whether the position of the origin of a coordinate system makes any change in the form of the equation depends on whether the experiments proceed in exactly the same way when carried out at different places. Independence from the choice of origin of a coordinate system implies the homogeneity of space. Indifference to direction of the axes implies that the space is isotropic.

Notes: In the framework of general relativity, the presence of a gravitational field (represented as a curvature of space-time) makes the space nonhomogeneous and anisotropic. However, in Newton's theory of gravitation, the space of a reference frame is considered homogeneous and isotropic. Only under the circumstance, when very intense gravitational fields are involved, that the predictions of general relativity differ significantly from Newton's theory of gravitation. In the present discussion, no such situation is envisaged.

Let us discuss homogeneity of space. All dynamical variables (except the position coordinates), such as velocity, momentum, mass, and so on, are independent of

*Such noninertial frames are discussed in Sec. 10.7.

the origin of the coordinate system, since these involve the change of position vectors.

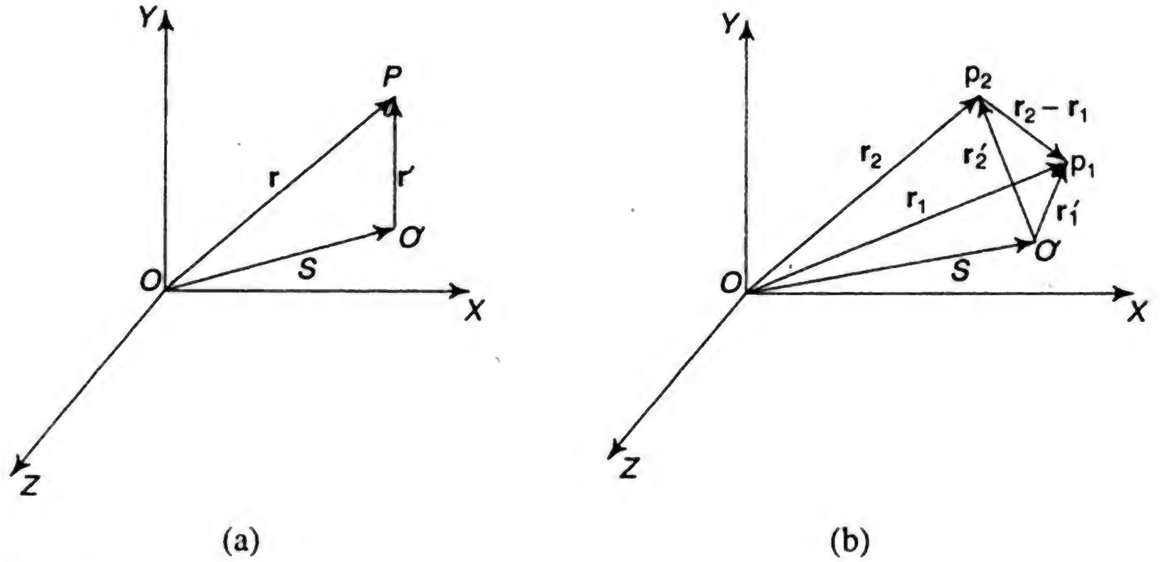


Fig. 10.1 (a) The position vector **r** of a point P with respect to the origin of the reference frame $OXYZ$; (b) The difference vector $(\mathbf{r}_2 - \mathbf{r}_1)$ is unchanged by the shift of the origin O to O'

Let **r** be the position vector of a point P defined with respect to the origin of the frame $OXYZ$, $\mathbf{r}'$, the position vector of P with respect to O' [Fig. 10.1 (a)]. Then

$$\mathbf{r}' = \mathbf{r} - \mathbf{S} \tag{10.1}$$

However, the difference of two vectors $(\mathbf{r}_2 - \mathbf{r}_1)$ is unchanged by the shift of origin [Fig. 10.1 (b)] since both the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are changed by the same vector such that

$$\mathbf{r}_2' - \mathbf{r}_1' = (\mathbf{r}_2 - \mathbf{S}) - (\mathbf{r}_1 - \mathbf{S}) = \mathbf{r}_2 - \mathbf{r}_1 \tag{10.2}$$

The velocity
$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$= \lim_{t_2 \to t_1} \frac{\mathbf{r}_2(t_2) - \mathbf{r}_1(t_1)}{t_2 - t_1} \tag{10.3}$$

is also invariant under the shift of origin. According to the requirement of homogeneity of space in the equation of motion of a particle $\mathbf{F} = d\mathbf{p}/dt$; the force **F** should involve **r** only through its difference with some other position vector. The force of interaction between particles may depend on the distance of their separation, rather than its position relative to the coordinate system. All known interactions fulfil this requirement. For example, the Newtonian gravitational force exerted by particle 2

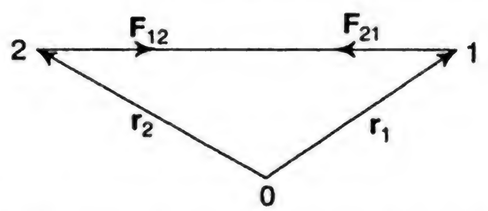


Fig. 10.2 $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of particles 1 and 2 with respect to the origin O

the other and in the opposite direction. The first law was stated clearly by Galileo and is known as the principle of inertia. In an inertial frame, if the mass of a body is constant, then according to the second law, we get

$$\mathbf{F} = \frac{d(m\mathbf{u})}{dt} = m\frac{d\mathbf{u}}{dt} = m\mathbf{a} \tag{10.6}$$

where **F** is the force acting on the body of mass m having velocity **u** and acceleration **a**. The third law implies that action and reaction are always exerted on different bodies. It may be realized that Newton's laws are formulated from experiments with big bodies, such as billiard balls which move at speeds much less than the speed of light c. However, when speeds of the order of or comparable to c are involved, basic modifications become essential in our concepts of space and time. The new concepts come under the realm of special theory of relativity which will be treated in the next chapter. However, the results of the special theory will lead to the Newtonian predictions in the limit of $u \ll c$ or what tantamounts to the same requirement that $c \to \infty$.

Principle of Relativity

Consider an ocean liner moving with uniform velocity on a calm day. The passengers can play their games inside the liner, just as if they were playing on land and can ignore the motion of the ship. However, on a stormy day, the sudden acceleration will surely affect their game and they will have to make allowance for that.

If one carries out experiments on large bodies inside a ship moving with uniform velocity and analyse them, then one would conclude that Newton's laws hold to a very good approximation inside the ship. Without looking out of the ship, it is impossible to infer on the basis of the experiments carried within that the ship is moving. However, if it is conveyed that the ship is in motion, then one cannot determine the speed without looking at something external to the liner. This is an example of the principle of relativity, according to which the laws of physics are the same in all inertial reference frames. The only way to find the relative velocity between two frames is by comparing the data different observers in the two frames take on the same event.

Furthermore, the absolute velocity of an inertial frame cannot be determined from mechanical experiments done in that frame; since the equation of motion is invariant under Galilean transformations [Eq. (10.7)]. No inertial frame is pre-eminent among a set of inertial frames in uniform motion with respect to each other, for the laws of mechanics are the same in all. The fact that one can talk of the relative velocity between two inertial frames and not of an absolute velocity of a frame is called Newtonian relativity, usually referred to as Galilean invariance.

10.5 GALILEAN TRANSFORMATIONS

An event is known from the coordinates of the place ot its location and the time of its occurrence. For example, the event may be the collision of two particles or turning on of an oscillator.

S and S' are two inertial frames, whose origins O and O' coincide at $t = t' = 0$; the axes OX and $O'X'$ being parallel to each other (Fig. 10.4). The frame S' is moving

EXAMPLE 10.2

The velocity of sound in still air at 25 °C is 358 m s^{-1}. Find the velocity measured by an observer moving with a velocity of 90 km h^{-1} (a) away from the source, (b) toward the source and (c) perpendicular to the direction of propagation in air. The source is at rest relative to the ground.

Solution

Let the frame S be fixed on the ground and thus be at rest relative to the air. Another frame S' with its axis $O'x'$, parallel to Ox, moves with relative velocity **v**. The sound source is at the origin O of system S. The velocity of the observer O' with respect to O is v = 90 km h^{-1} = 25 m s^{-1}. The velocity of sound in still air is V = 358 m s^{-1} and let it be denoted by V' as measured by the observer in S'. Then

(a) $V' = V - v$
$= 333 \text{ m s}^{-1}$

(b) $V' = V + v = 383 \text{ m s}^{-1}$

(c) $V' = \sqrt{V^2 + v^2}$
$= 358.9 \text{ m s}^{-1}$

and $$\tan \alpha' = \frac{V'_{y'}}{V'_{x'}} = \frac{V}{-v} = -14.32$$

$$\alpha' = 94°$$

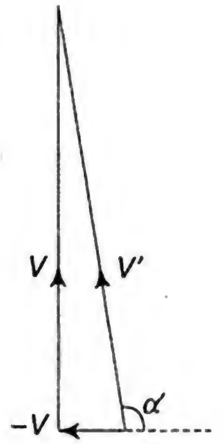


Fig. 10.5 Velocity of sound as measured by an observer moving perpendicular to direction of its propagation

EXAMPLE 10.3

The windows and doors of a car are closed and the car is standing still on a horizontal road. The string of a hydrogen balloon is tied to the floor of the car. The driver steps on the accelerator and causes 200 cm/s^2 uniform acceleration. Find the angle of the string of the balloon with the vertical after the steady state has reached and transients have died down.

Solution

As the car starts getting accelerated, everything inside the car is pushed backwards with acceleration **a**. This is equivalent to the additional uniform gravitational field. The resultant gravitation field is $\mathbf{g}' = \mathbf{g} + \mathbf{a}$. In such a situation, the hydrogen balloon will point opposite to the resultant gravitational field $\mathbf{g}'$. If $\mathbf{g}'$ makes an angle θ with the vertical, then

$$\tan \theta = \frac{|\mathbf{a}|}{|\mathbf{g}|} = \frac{200}{981}$$

or $$\theta = \tan^{-1}\frac{200}{981} = 11.3°$$

Thus the balloon will point upwards along a direction making an angle of 11.3° with the vertical.

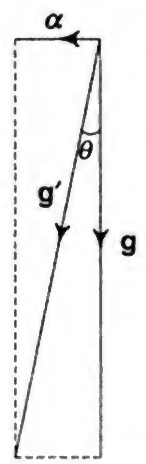


Fig. 10.6 Effective acceleration due to gravity acting on the balloon tied to an accelerated car

EXAMPLE 10.4

An ocean liner is moving with velocity v with respect to the earth. A ball is thrown downwards on the deck. Show that the trajectory of the ball is a straight line relative to the observer in the ship and appears parabolic to the one on the ground.

Solution

Let S denote the reference system which is at rest relative to the earth and S' reference system with respect to which the ship is at rest. Let the mass m be dropped from rest relative to the ship at $t = 0$, from point at a distance x_0 from O', the origin of S'. The origins O and O' coincide at $t = t' = 0$, and the x-axis moves along x'-axis parallel to $\mathbf{v}$, the velocity of the ship relative to the earth. S' moves with velocity $\mathbf{v}$ along the x-axis of S. The axes Oy and $O'y'$ remain parallel.

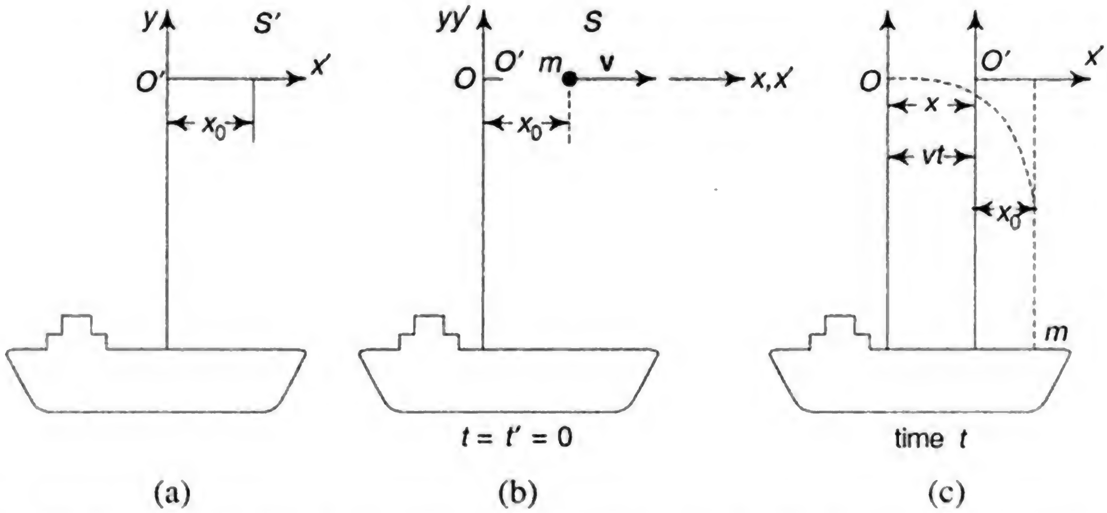


Fig. 10.7(a) The ship is at rest in S' and relative to S', the mass m falls vertically downwards under gravity

(b) S is the laboratory system, the origins O and O' coincide at $t = t' = 0$. Relative to the laboratory system S, the mass m has the velocity of the ship $\mathbf{v}$, as well as the vertical acceleration under gravity

(c) The mass m falls in a parabolic path relative to the laboratory system S

To the observer in S', the mass m is dropped from a point whose coordinates are:

$$x' = x_0$$
$$y' = 0$$
$$t' = 0 \tag{i}$$

According to Newton's law of universal gravitation, the force on the mass m is

$$|\mathbf{f}| = G\,\frac{mM}{r^2} \tag{ii}$$

where G = universal gravitation constant

M = mass of the earth

r = distance of m from the centre of the earth

Applying Newton's second law of motion, relative to S', the mass m acquires an acceleration a' given by

$$|\mathbf{f}| = m|\mathbf{a}'| \tag{iii}$$

of a force on a particle far removed from other particles, is a sure indication that its frame of reference is an accelerated one.

(b) Uniformly Rotating Frame: Coriolis Force and Centrifugal Force

Let $x_1x_2x_3$ be an inertial reference frame S fixed in space and $x'_1x'_2x'_3$ reference frame S' that is fixed in a rigid body and is uniformly rotating in space with respect to S with angular velocity $\boldsymbol{\omega}$ (Fig. 10.8). The unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ refer to the reference frame S and $\mathbf{i}'_1$, $\mathbf{i}'_2$, $\mathbf{i}'_3$, to the frame S'. The position vector $\mathbf{r}$ of the point P is given by

$$\begin{aligned} \mathbf{r} &= x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 \\ &= x'_1\mathbf{i}'_1 + x'_2\mathbf{i}'_2 + x'_3\mathbf{i}'_3 \end{aligned} \tag{10.23}$$

Owing to the rotational motion of the rigid body, the unit base vectors $\mathbf{i}'_1$, $\mathbf{i}'_2$ and $\mathbf{i}'_3$ are continually changing and in taking time derivatives, the unit vectors are treated as variables. Thus

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= \frac{d}{dt}(x'_1\mathbf{i}'_1 + x'_2\mathbf{i}'_2 + x'_3\mathbf{i}'_3) \\ &= \frac{dx'_1}{dt}\mathbf{i}'_1 + \frac{dx'_2}{dt}\mathbf{i}'_2 + \frac{dx'_3}{dt}\mathbf{i}'_3 + x'_1\frac{d\mathbf{i}'_1}{dt} + x'_2\frac{d\mathbf{i}'_2}{dt} + x'_3\frac{d\mathbf{i}'_3}{dt} \end{aligned}$$

or
$$\dot{\mathbf{r}} = \dot{x}'_1\mathbf{i}'_1 + \dot{x}'_2\mathbf{i}'_2 + \dot{x}'_3\mathbf{i}'_3 + x'_1\frac{d\mathbf{i}'_1}{dt} + x'_2\frac{d\mathbf{i}'_2}{dt} + x'_3\frac{d\mathbf{i}'_3}{dt} \tag{10.24}$$

The linear velocity $\mathbf{v}$ of a particle is expressed as $d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r}$ where $\boldsymbol{\omega}$ is its angular velocity Eq. (2.44). Therefore, we get

$$\begin{aligned} \frac{d\mathbf{i}'_1}{dt} &= \boldsymbol{\omega} \times \mathbf{i}'_1 \\ \frac{d\mathbf{i}'_2}{dt} &= \boldsymbol{\omega} \times \mathbf{i}'_2 \\ \frac{d\mathbf{i}'_3}{dt} &= \boldsymbol{\omega} \times \mathbf{i}'_3 \end{aligned} \tag{10.25}$$

where $\boldsymbol{\omega}$ is the rotational velocity of the frame. Rewriting Eq. (10.24) in the light of Eq. (10.25), we get

$$\begin{aligned} \dot{\mathbf{r}} = x'_1\mathbf{i}'_1 + x'_2\mathbf{i}'_2 + x'_3\mathbf{i}'_3 + x'_1(\boldsymbol{\omega} \times \mathbf{i}'_1) \\ + x'_2(\boldsymbol{\omega} \times \mathbf{i}'_2) + x'_3(\boldsymbol{\omega} \times \mathbf{i}'_3) \end{aligned} \tag{10.26}$$

Equation (10.26) can be written as follows:

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{space}} = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{r} \tag{10.27}$$

where $\left(\frac{d\mathbf{r}}{dt}\right)_{\text{space}}$ is the linear velocity of a particle with respect to S $(Ox_1x_2x_3)$ and $\left(\frac{d\mathbf{r}}{dt}\right)_{\text{body}}$ is its linear velocity in the rotating frame S' $(Ox'_1x'_2x'_3)$. This result is actually true for any vector and can be represented by the following operator equation:

$$\left(\frac{d}{dt}\right)_{\text{space}} = \left(\frac{d}{dt}\right)_{\text{body}} + (\boldsymbol{\omega} \times) \qquad (10.28)$$

Equation (10.27) may be written as

$$\mathbf{v}_{\text{space}} = \mathbf{v}_{\text{body}} + \boldsymbol{\omega} \times \mathbf{r} \qquad (10.29)$$

Applying the operator equation (10.28) to the vector $\mathbf{v}_{\text{space}}$ and using Eq. (10.29), we get

$$\frac{d\mathbf{v}_{\text{space}}}{dt} = \mathbf{a}_{\text{space}} = \left(\frac{d\mathbf{v}_{\text{space}}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{v}_{\text{space}}$$

$$= \frac{d}{dt}(\mathbf{v}_{\text{body}} + (\boldsymbol{\omega} \times \mathbf{r})_{\text{body}}) + \boldsymbol{\omega} \times (\mathbf{v}_{\text{body}} + \boldsymbol{\omega} \times \mathbf{r})$$

or

$$\mathbf{a}_{\text{space}} = \mathbf{a}_{\text{body}} + 2(\boldsymbol{\omega} \times \mathbf{v}_{\text{body}}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} \qquad (10.30)$$

The acceleration in the rotating frame S' is

$$\mathbf{a}_{\text{body}} = \mathbf{a}_{\text{space}} - 2(\boldsymbol{\omega} \times \mathbf{v}_{\text{body}}) - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} \qquad (10.31)$$

The equation of motion in the fixed space-axis is

$$\mathbf{F}_{\text{space}} = m\mathbf{a}_{\text{space}} \qquad (10.32)$$

Hence,

$$m\mathbf{a}_{\text{body}} = \mathbf{F}_{\text{space}} - 2m(\boldsymbol{\omega} \times \mathbf{v}_{\text{body}}) - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} \qquad (10.33)$$

Obviously, to an observer in the rotating frame, the body appears to be moving under an effective force $\mathbf{F}_{\text{eff}} = m\mathbf{a}_{\text{body}}$. Let us examine the terms in Eq. (10.33), one by one.

Centrifugal Force: The term $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is the ordinary centrifugal force directed away from the centre (due to negative sign). It is a fictitious force acting on a particle ($\mathbf{v} = 0$) in the rotating frame, is perpendicular to $\boldsymbol{\omega}$, and its magnitude is $m\omega^2 r \sin\theta$ (Fig. 10.9 (b)). On the surface of earth, it reduces the value of g, the acceleration due to gravity.

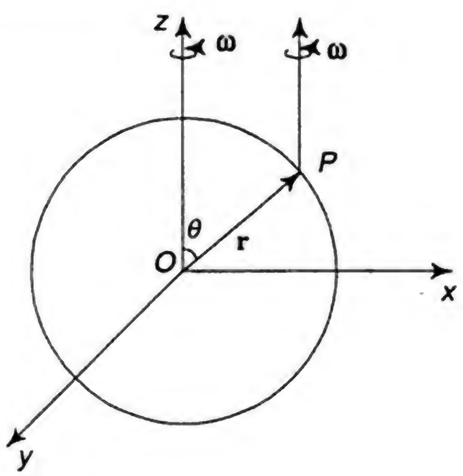


Fig. 10.9 (a) $\mathbf{r}$ is the radius vector of a point P on the circumference of a rotating frame, say, the earth

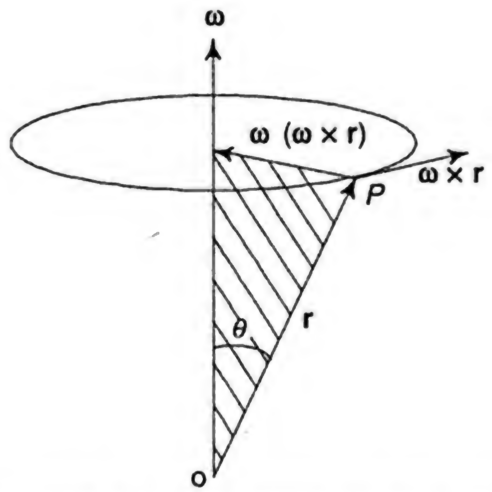


Fig. 10.9 (b) The centrifugal force $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ will be directed away from the centre due to the negative sign.

Further, $$\boldsymbol{\omega}\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 5 \\ \sin t & -\cos t & e^{-t} \end{vmatrix}$$

$$= \mathbf{i}(-3e^{-t} + 5\cos t) + \mathbf{j}(5\sin t - 2e^{-t}) + \mathbf{k}(-2\cos t + 3\sin t)$$

Thus, $$\left.\frac{d\mathbf{A}}{dt}\right|_F = (6\cos t - 3e^{-t})\mathbf{i} + (6\sin t - 2e^{-t})\mathbf{j} + (3\sin t - 2\cos t - e^{-t})\mathbf{k}$$

(b) $$\left.\frac{d\mathbf{A}}{dt}\right|_M = \frac{dA_1}{dt}\mathbf{i} + \frac{dA_2}{dt}\mathbf{j} + \frac{dA_3}{dt}\mathbf{k}$$

$$= \cos t\mathbf{i} + \sin t\mathbf{j} - e^{-t}\,\mathbf{k}$$

(c) The acceleration of the particle as seen by the observer in the fixed $X_1X_2X_3$ system is

$$\left.\frac{d^2\mathbf{A}}{dt^2}\right|_F = \left.\frac{d}{dt}\right|_F \left.\frac{d\mathbf{A}}{dt}\right|_F$$

$$= \left.\frac{d}{dt}\right|_F \left(\left.\frac{d\mathbf{A}}{dt}\right|_M + \boldsymbol{\omega}\times\mathbf{A}\right)$$

$$= \left(\left.\frac{d}{dt}\right|_M + \boldsymbol{\omega}\times\right)\left(\left.\frac{d\mathbf{A}}{dt}\right|_M + \boldsymbol{\omega}\times\mathbf{A}\right)$$

$$= \left.\frac{d^2\mathbf{A}}{dt^2}\right|_M + \frac{d}{dt}(\boldsymbol{\omega}\times\mathbf{r})|_M + \boldsymbol{\omega}\times\left.\frac{d\mathbf{A}}{dt}\right|_M + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})$$

$$= \left.\frac{d^2\mathbf{A}}{dt^2}\right|_M + \left.\frac{d\boldsymbol{\omega}}{dt}\right|_M \times\mathbf{A} + \boldsymbol{\omega}\times\left.\frac{d\mathbf{A}}{dt}\right|_M + \boldsymbol{\omega}\times\left.\frac{d\mathbf{A}}{dt}\right|_M + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})$$

$$= \left.\frac{d^2\mathbf{A}}{dt^2}\right|_M + \left.\frac{d\boldsymbol{\omega}}{dt}\right|_M \times\mathbf{A} + 2\boldsymbol{\omega}\times\left.\frac{d\mathbf{A}}{dt}\right|_M + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})$$

Let us calculate the expression term by term.

$$\left.\frac{d^2\mathbf{A}}{dt^2}\right|_M = \frac{d^2A_1}{dt^2}\mathbf{i} + \frac{d^2A_2}{dt^2}\mathbf{j} + \frac{d^2A_3}{dt^2}\mathbf{k}$$

$$= -\sin t\mathbf{i} + \cos t\mathbf{j} + e^{-t}\,\mathbf{k}$$

$$\left.\frac{d\boldsymbol{\omega}}{dt}\right|_M \times\mathbf{A} = 0$$

$$\boldsymbol{\omega}\times\left.\frac{d\mathbf{A}}{dt}\right|_M = (2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})\times(\cos t\mathbf{i} + \sin t\mathbf{j} - e^{-t}\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 5 \\ \cos t & \sin t & -e^{-t} \end{vmatrix}$$

$$= \mathbf{i}(3e^{-t} - 5\sin t) + \mathbf{j}(5\cos t + 2e^{-t}) + \mathbf{k}(2\sin t + 3\cos t)$$

Therefore, $$2\boldsymbol{\omega}\times\left.\frac{d\mathbf{A}}{dt}\right|_M = (6e^{-t} - 10\sin t)\mathbf{i} + (10\cos t + 4e^{-t})\mathbf{j} + (4\sin t + 6\cos t)\mathbf{k}$$

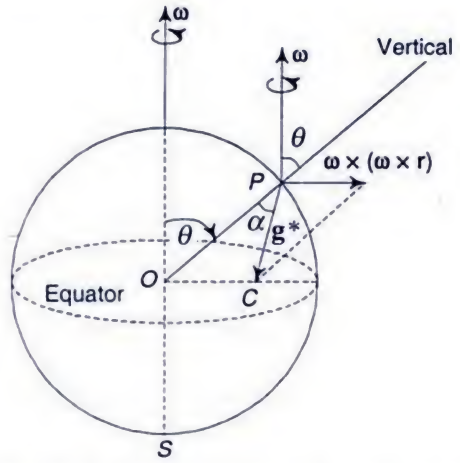


Fig 10.10 Vector addition of **g** and **ω** × (**ω** × **r**) resulting into **g***

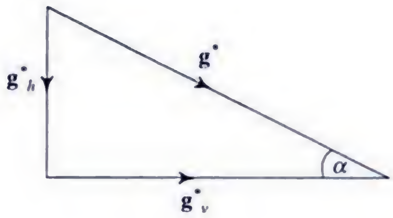


Fig 10.11 Apparent acceleration due to gravity resolved into rectangular components

Now magnitude of the centrifugal acceleration at the equator, is given by

$$\omega^2 r = \left(\frac{2\pi}{86,400}\right)^2 \times 6370 \times 10^5$$
$$= 3.4 \text{ cm s}^{-2}$$

Obviously $\omega^2 r \ll g$ so that $g_v^* \approx g$. Thus, if α is the angle between the apparent and true verticals, we get

$$\tan\alpha \approx \alpha \approx \frac{g_h^*}{g_v^*} \approx \frac{\omega^2 r}{g} \sin\theta\cos\theta = \frac{\omega^2 r}{2g}\sin 2\theta$$

It will have the maximum value at $\theta = 45°$.

Now inserting the values:

$$\omega = 7.28 \times 10^{-5} \text{ rad s}^{-1}$$

Mean radius of the earth, $r = 6370$ km

and $g = 981 \text{ cm s}^{-2}$, we get

$$\alpha \approx 0°6'$$
$$g_h^* = 0; \quad g_v^* = g$$

At the poles $\theta = 0$, therefore $g^* = g$ and at the equator $\theta = \pi/2$,

$$g_h^* = 0; \quad g_v^* = g - \omega^2 r$$

thus $g^* = g - \omega^2 r$. The value of acceleration due to gravity will be greater at the poles by 3.4 cm s^{-2}. However, the actual measured difference is 5.2 cm s^{-2}. This discrepancy is attributed to the fact that earth is not a perfect sphere, and is flattened at the poles. Thus the value of g itself is greater at the poles then at the equator, the centrifugal term disregarded.

(b) Effect of Coriolis Force

The additional fictitious force, the coriolis force $-2m\,\boldsymbol{\omega} \times \mathbf{v}$, is the velocity depen-

where the symbols have their usual meanings. For the case of earth rotating with constant angular velocity $\boldsymbol{\omega}$ about its axis, $\dot{\boldsymbol{\omega}} = 0$ and the equation becomes

$$m\frac{d^2\mathbf{r}}{dt^2} = \boldsymbol{F} - 2m(\boldsymbol{\omega} \times \mathbf{v}) - m(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}))$$

(a) The centrifugal force $= -m(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}))$

It is directed away from the centre and perpendicular to the angular velocity $\boldsymbol{\omega}$, Fig 10.9(b).

(b) Consider a particle at a point at colatitude λ (colatitude is defined as $\frac{\pi}{2}$ – latitude). Then

$$|m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})| = m\boldsymbol{\omega}|\boldsymbol{\omega} \times \mathbf{r}| = m\omega^2 r \sin \lambda$$

(c) It is maximum at the equator and

(d) It is minimum at the north and south poles.

EXAMPLE 10.8

Find the Coriolis force on a train of mass 10^6 kg, moving from north to south at a latitude of 60° north with a speed of 72 km/hr.

Solution

The Coriolis force acting on the train

$$\mathbf{F}_{cor} = -2m\boldsymbol{\omega} \times \mathbf{v}$$

Now, from the adjoining Fig. E10.8, we get

$$|\boldsymbol{\omega} \times \mathbf{v}| = \omega v \sin 120° = \omega v \cos 30°$$

$$F_{cor} = 2m\omega v \cos 30°$$

$$= 2 \times 10^6 \times \left(\frac{2\pi}{24 \times 60 \times 60}\right) \times \frac{72 \times 10^3}{3600} \times \frac{\sqrt{3}}{2}$$

$$= 2517.8 \text{ N towards west}$$

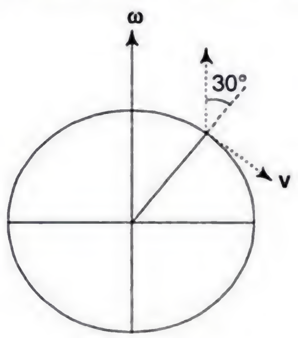


Fig. E10.8

EXAMPLE 10.9

A mass of 50 g is moving with linear velocity of 100 cm/s normal to the axis of rotation in a rotating frame of reference. The mass is at a distance of 10 cm from the axis of rotation. Calculate the Coriolis force experienced by the mass.

Solution

The linear velocity

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

Therefore, the angular speed of the frame

$$= \frac{v}{r} = 10 \text{ rad/s}$$

Coriolis acceleration of the mass $= |-2\boldsymbol{\omega} \times \mathbf{v}|$

$$= 2\omega v$$

$$= 2 \times 10 \times 100$$

$$= 20 \text{ m/s}$$

Thus, the Coriolis force acting on the mass = 1 N

EXAMPLE 10.10

(a) A particle is dropped from height h with zero velocity and falls freely under gravity. Calculate the horizontal displacement of the particle due to Coriolis force.

(b) Estimate the westward displacement of the particle where h = 100 metres at (i) poles (ii) latitude 30° and (iii) equator.

Solution

Assuming that the x-axis is towards east, the y-axis towards north and the z-axis vertically upwards Fig. E10.10 the angular velocity of earth at latitude λ is

$$\boldsymbol{\omega} = \omega \cos \lambda \mathbf{j} + \omega \sin \lambda \mathbf{k} \tag{1}$$

and for the vertical fall of the particle with velocity v

$$\mathbf{v} = -v\,\mathbf{k} \tag{2}$$

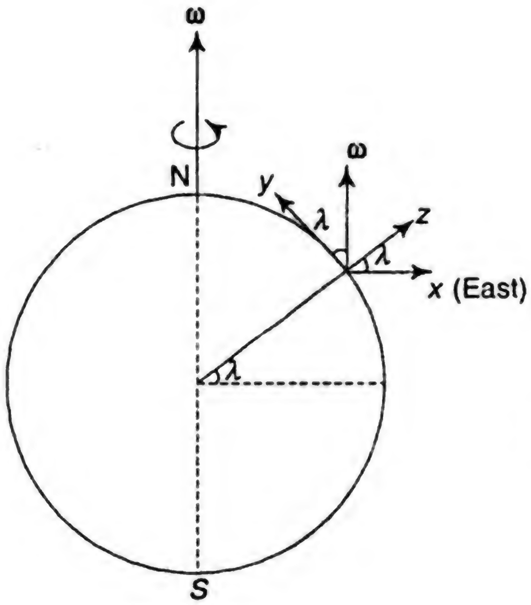


Fig. E10.10

The Coriolis acceleration

$$\mathbf{a}_c = -2\boldsymbol{\omega} \times \mathbf{v}$$

$$= -2\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ 0 & 0 & -v \end{vmatrix}$$

$$= 2\omega v \cos \lambda \mathbf{i} \tag{3}$$

The Coriolis acceleration is towards the east in the northern hemisphere.

This leads to the following equation of motion of the particle in the x-direction

$$\frac{d^2x}{dt^2} = 2\omega v \cos \lambda \tag{4}$$

Initially, the particle is falling vertically downwards, that is, v = gt, so Eq. (4) becomes

$$\frac{d^2x}{dt^2} = 2\omega g t \cos \lambda \tag{5}$$

Integrating it wrt t, we get

$$\frac{dx}{dt} = 2\omega g \cos \lambda \frac{t^2}{2} + C_1$$

where C_1 is a constant of integration.

When $t = 0$, $\frac{dx}{dt} = 0$, therefore, $C_1 = 0$, and the equation for velocity becomes

$$\frac{dx}{dt} = \omega g \cos \lambda t^2 \tag{6}$$

Integrating it again wrt t, we get

$$x = \omega g \cos \lambda \frac{t^3}{3} + C_2$$

where C_2 is constant of integration. When $t = 0$, $x = 0$, therefore, $C_2 = 0$.

Therefore, $$x = \frac{1}{3} \omega g \cos \lambda t^3 \tag{7}$$

Now, $$h = \frac{1}{2} gt^2$$

or $$t = \left(\frac{2h}{g}\right)^{\frac{1}{2}} \tag{8}$$

The displacement $$x = \frac{1}{3} \omega g \cos \lambda \left(\frac{2h}{g}\right)^{\frac{3}{2}}$$

$$= \left(\frac{8}{9g}\right)^{\frac{1}{2}} h^{\frac{3}{2}} \omega \cos \lambda \tag{9}$$

(b) $h = 100$ m

The displacement x at the poles ($\lambda = 90°$) = 0.

When $\lambda = 30°$, we get

$$\text{Displacement} = \left(\frac{8}{9 \times 980}\right)^{\frac{1}{2}} \times (10^4)^{\frac{3}{2}} \times \left(\frac{2\pi}{24 \times 60 \times 60}\right) \times \frac{\sqrt{3}}{2}$$

$= 1.89$ cm

At the equator $\lambda = 0°$. Therefore,

$$x = \left(\frac{8}{9 \times 980}\right)^{\frac{1}{2}} \times (10^4)^{\frac{3}{2}} \times \left(\frac{2\pi}{24 \times 60 \times 60}\right) \times 1$$

$= 2.19$ cm

In the calculation above, the effects of wind, viscosity, etc have been ignored.

EXAMPLE 10.11

Calculate the fictitious force and the total force on a body of mass 5 kg in a frame of reference moving (i) vertically upwards, and (ii) vertically downwards, with an acceleration of 5 m/s^2 (g = 9.8 m/s^2).

Solution

Weight of the body $= mg$

$= 5 \times (-9.8)$

$= -49$ N

$=$ 49 N downwards

(i) The fictitious force acting on it during the upward motion

$= - ma_0$

$= -5 \times 5 = -25$ N

$= 25$ N downwards

Hence the total force $= 49 + 25 = 74$ N downwards, i.e. the body appears to be heavier. The fictitious force acting on it during the downward motion

$= -ma_0$

$= 5[-(-5)]$

$= 25$ N upwards

The net force experienced by the body $= 49 - 25 = 24$ N dewnwards so that it seems to be lighter.

EXAMPLE 10.12

(i) Prove that for small ω, the observed acceleration due to gravity g^* for a point in colatitude θ is

$$g^* = g - \omega^2 r \sin^2 \theta$$

where g is the real value of acceleration due to gravity, ω the angular velocity of the earth and r the radius of the earth at the place in colatitude θ.

(ii) If the earth stops rotating suddenly, how will it affect the value of g at a place in colatitude 45°? (Radius of the earth $= 6.37 \times 10^8$ cm.)

Solution

The apparent acceleration of a particle at a place in colatitude θ is

$$\mathbf{g}^* = \mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad \text{(i)}$$

where $\mathbf{g}$ is the true acceleration due to gravity. Take the axes OY and OX along and perpendicular to $\boldsymbol{\omega}$, with unit vectors $\mathbf{j}$ and $\mathbf{i}$ along them respectively (Fig. 10.14). Then we have

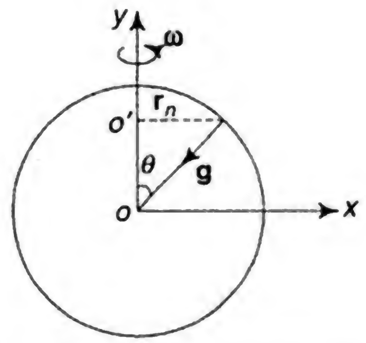


Fig 10.14 The rotating co-ordinate system

$$\mathbf{g} = -g (\sin \theta \mathbf{i} + \cos \theta \mathbf{j})$$

$$\boldsymbol{\omega} = \omega \mathbf{j}$$

$$\mathbf{r}_N = r \sin \theta \mathbf{i}$$

Substituting these values in the above Eq. (i), we get

$$\mathbf{g}^* = -g (\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) - \omega \mathbf{j} \times (\omega \mathbf{j} \times \mathbf{r} \sin \theta \mathbf{i})$$

$$= -g (\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) + \omega^2 r \sin \theta \mathbf{i}$$

Hence, the magnitude of the apparent acceleration is

$$g^* = [(g \sin \theta - \omega^2 r \sin \theta)^2 + g^2 \cos^2 \theta)]^{1/2}$$

Neglecting terms involving ω^4, we get

$$g^* = (g^2 - 2g\omega^2 r \sin^2 \theta)^{1/2}$$

$$= g\left(1 - \frac{2\omega^2 r \sin^2\theta}{g}\right)^{1/2}$$

$$= g\left(1 - \frac{r\omega^2 \sin^2\theta}{g}\right)$$

$$= g - \omega^2 r \sin^2\theta \qquad \text{(ii)}$$

(ii) When the rotation of earth ceases, $\omega = 0$, and g^* becomes g. Thus the increase in the value of acceleration due to gravity is

$$g - g^* = \omega^2 r \sin^2\theta$$

$$= \left(\frac{2\pi}{24 \times 60 \times 60}\right)^2 \times 6.37 \times 10^8 \times \frac{1}{2}$$

$$= 1.68 \text{ cm/s}^2$$

10.9 FOUCAULT'S PENDULUM

A Foucault's pendulum is an ordinary pendulum with a heavy bob carried by a very long suspension, free to swing in any direction and arranged to be perfectly symmetric so that its period of oscillation in any plane is precisely the same.

It is used to demonstrate the rotation of the earth about its own axis and was employed by Foucault at Paris for a public demonstration of the earth's rotation in 1851. When the pendulum is started, it swings in a definite vertical plane and the plane of oscillation is observed to precess around the vertical axis during a period of several hours.

The equation of motion of a uniformly rotating frame, when an external force $\mathbf{F}_{ext}$ ($F_x\mathbf{i}$, $F_y\mathbf{j}$, $F_z\mathbf{k}$) is acting, according to Eq. (10.33) (dropping the suffix body and space),

$$m\mathbf{a} = \mathbf{F} - 2m(\boldsymbol{\omega} \times \mathbf{v}) - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}$$

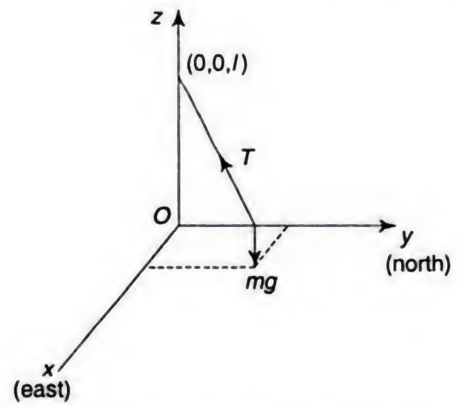


Fig. 10.15 Foucault's pendulum. T is the tension of the string which supports a heavy bob of mass m; the pendulum is free to swing in any direction

when $\boldsymbol{\omega}$ is small, the term $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ being proportional to ω^2 may be neglected. In addition when $\boldsymbol{\omega}$ is constant, the last term also vanishes. Therefore, the equation of motion becomes

$$m\mathbf{a} = \mathbf{F} - 2m\,(\boldsymbol{\omega} \times \mathbf{v}) \tag{10.38}$$

Expressing the velocities $\mathbf{v}$ and $\boldsymbol{\omega}$ in cartesian components, we get

$$\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$$
$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

Assuming the particle to be at a place with colatitude θ and defining the coordinate system as in Fig. 10.16, we get

$$\omega_x = 0$$
$$\omega_y = \omega \sin \theta$$
$$\omega_z = \omega \cos \theta \tag{10.39}$$

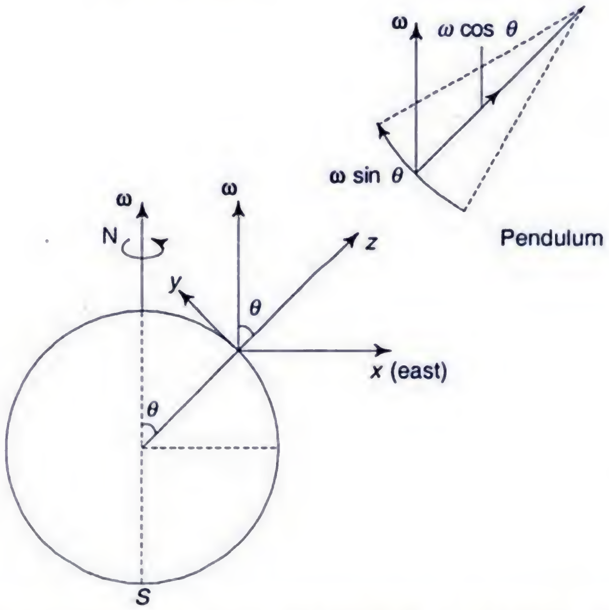


Fig. 10.16 The x-axis is towards the east and perpendicular to the plane of paper, y-axis is towards the north and z-axis is upwards, along the plumb line. The origin is at the equilibrium position of the bob

Hence
$$\boldsymbol{\omega} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_y & \omega_z \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}$$

$$= (\omega_y\dot{z} - \omega_x\dot{y})\,\mathbf{i} + \omega_z\dot{x}\mathbf{j} - \omega_y\dot{x}\mathbf{k}$$
$$= (\omega\sin\theta\dot{z} - \omega\cos\theta\dot{y})\,\mathbf{i} + \omega\cos\theta\dot{x}\mathbf{j} - \omega\sin\theta\dot{x}\mathbf{k} \tag{10.40}$$

The equations of motion in the component form are

$$m\ddot{x} = F_x - 2m\omega\,(\dot{z}\sin\theta - \dot{y}\cos\theta)$$
$$m\ddot{y} = F_y - 2m\omega\cos\theta\dot{x} \tag{10.41}$$

Calling $\frac{g}{l} = \omega_0^2$, we rewrite Eq. (10.47) as

$$\ddot{x} + \omega_0^2 x = 2\omega \dot{y} \cos\theta$$

$$\ddot{y} + \omega_0^2 y = -2\omega \dot{x} \cos\theta \tag{10.48}$$

Combining these equations by writing

$$u = x + iy$$

we get

$$\ddot{u} + 2i\omega \cos\theta \dot{u} + \omega_0^2 u = 0 \tag{10.49}$$

Calling $\Omega = \omega \cos\theta$, we get

$$\ddot{u} + 2i\Omega\dot{u} + \omega_0^2 u = 0 \tag{10.50}$$

In the operator form, this differential equation can be written as

$$(D^2 + 2i\Omega D + \omega_0^2)\, u = 0 \tag{10.51}$$

which gives $\quad D^2 + 2i\Omega D + \omega_0^2 = 0$

so that $\quad D = -\, i\Omega \pm i\omega_1$

where $\quad \omega_1^2 = \omega_0^2 + \Omega^2$

Hence the general solution of Eq. (10.50) is

$$u = Ae^{-i(\Omega-\omega_1)t} + Be^{-i(\Omega+\omega_1)t}$$

$$= (Ae^{i\omega_1 t} + Be^{-i\omega_1 t})\, e^{-i\omega t \cos\theta} \tag{10.52}$$

where A and B are undetermined constants.

Let us get the equation of the trajectory traced out by the bob. Denoting the complex amplitude of Eq. (10.52)

$$Ae^{i\omega_1 t} + Be^{-i\omega_1 t} = \zeta + i\lambda$$

we get

$$(A + B)\cos\omega_1 t = \zeta$$

$$(A - B)\sin\omega_1 t = \lambda$$

Eliminating t from the above equations, the resulting equation of the trajectory is

$$\frac{\zeta^2}{(A+B)^2} + \frac{\lambda^2}{(A-B)^2} = 1 \tag{10.53}$$

which is the equation of an ellipse with its centre at the origin. The other factor of Eq. (10.52) shows that the complex factor rotates through an angle $(\omega \cos\theta)t$. Then the ellipse rotates about the vertical axis with an angular velocity $\omega \cos\theta$ so that the period of rotation is $2\pi/\omega \cos\theta$.

The rotation of the plane of swinging of a pendulum as predicted by the above analysis and its verification provides a conclusive proof of the earth's rotation about its axis.

If the Foucault pendulum is set up at the north pole, it will oscillate as a simple pendulum in its inertial plane which remains fixed. However, at any other latitude, since the earth rotates from west to east with angular velocity ω, to an observer on earth the plane of oscillation of the pendulum will appear to rotate with an angular velocity $-\omega$ from east to west.

At the pole, $\theta = 0$ and the angular velocity of rotation is ω. Thus the plane of oscillation makes a complete revolution in 24 hours since the period is $T = 2\pi/\omega$. At any other latitude, the period is greater and is given by $T = 2\pi/\omega \cos\theta$. Obviously at the equator, $\theta = 90°$ and T becomes infinite.

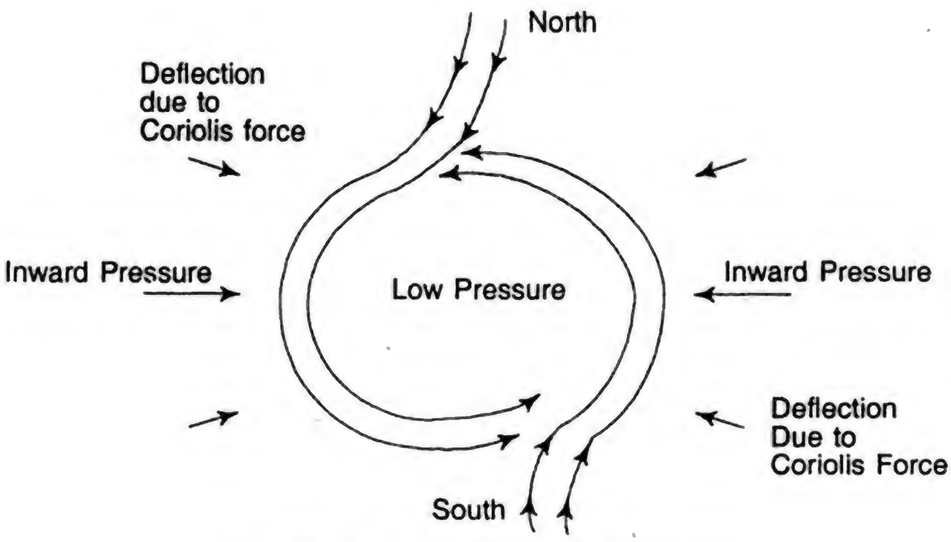


Fig. 10.18 The occurrence of cyclones

The direction of rotation of the cyclone is anticlockwise as seen from above in the northern hemisphere and clockwise in the southern hemisphere.

The effects due to Coriolis force also manifest in atomic physics. The rotational as well as vibrational motion may occur simultaneously in polyatomic molecules. The molecule rotates as a rigid whole and the constituent molecules vibrate about their equilibrium positions. The Coriolis force will be non-zero as the atoms are in motion with respect to the rotating coordinate system of the molecule and it will cause the atoms to move in a direction perpendicular to the original oscillations. Perturbations in molecular spectra due to Coriolis force appear in the interaction between the rotational and vibrational spectra.

QUESTIONS

10.1 How is the existence of rigid bodies essential for the concept of frame of reference?

10.2 Define a clock and comment on its importance in the specification of an event.

10.3 Define the term 'inertial frame of reference' and argue to show that Newton's first law of motion is an affirmation of its existence.

10.4 What does the term 'fixed stars' mean? How can these be used to define inertial frames.

10.5 Show that acceleration is invariant under the change of origin of the reference frame. How does this lead to homogeneity of space?

10.6 Discuss the principle of relativity.

10.7 Comment on the statement, 'All the three fundamental quantities of mechanics are invariant under Galilean transformations'.

10.8 Prove that a coordinate system which moves with constant velocity with respect to an inertial coordinate system is itself inertial.

10.9 The velocity of a particle is found to be $\mathbf{u}$ in frame S and $\mathbf{u}'$ in S', which itself is moving with velocity $\mathbf{v}$ with respect to S. If all the velocities are very small as compared to that of light, then show that $\mathbf{u} = \mathbf{u}' + \mathbf{v}$.

10.10 Show that the force under which a particle of mass m is moving will be observed to be the same by two observers having relative velocity $v \ll c$.

10.11 Prove that a frame rotating with a uniform angular velocity with respect to an inertial frame with coinciding origins is not inertial.

10.12 Comment on the statement, 'If a particle experiences a force even when it is far removed from other particles, then its frame of reference ought to be noninertial'.

10.13 Discuss the origin of fictitious forces in uniform rotational motion.

10.14 What is Coriolis force? Show that it owes its existence to the motion of a particle with respect to a rotating frame of reference.

10.15 What is centrifugal force? Show that the effect of the centrifugal force due to rotation of the earth on the acceleration due to gravity is maximum at the equator and minimum at the poles.

10.16 Show that the Coriolis force due to rotation of the earth deviates vertically falling particles towards east, and the displacement is proportional to $h^{3/2}$ for a given colatitude, where h is the height of fall.

10.17 State the assumptions made in the analysis of motion of a Foucault's pendulum.

10.18 Find an expression for the tension acting in the string of a Foucault's pendulum.

10.19 Prove that the trajectory of the bob of a Foucault's pendulum is elliptical.

10.20 Discuss the statement, 'Rotation of the plane of a long, swinging pendulum is a proof of the fact that the earth is rotating about its axis'.

10.21 How does the rotation of the earth affect the warm Gulf stream?

10.22 Discuss the effect of Coriolis force on the setting up of cyclones and trade winds.

10.23 Polyatomic molecules have rotational as well as vibrational motions. Will their interaction effect the spectra of polyatomic molecules?

10.24 Are fictitious or pseudo-forces real? If not, why do noninertial observers experience them? If so, why do different observers disagree about their presence or absence?

10.25 Prove that the Coriolis acceleration with respect to the inertial frame of a particle moving with an instantaneous velocity **v** with respect to a frame rotating with angular velocity ω is $2\,\boldsymbol{\omega} \times \mathbf{v}$.

10.26 Starting from Eq. (10.33), discuss the direction of pseudo-forces appearing in the expression.

10.27 Show that in the northern hemisphere, the cyclones (or water whirls) rotate anticlockwise when viewed from above.

10.28 At what points on the surface of the earth will the plane of oscillation of a Foucault pendulum rotate once in a day, once in two days and not at all?

10.29 Demonstrate that the motion of one projectile as seen by an observer in another projectile will always be a linear one.

10.30 'Earth cannot be used as an inertial frame'. Comment.

10.31 Distinguish between inertial mass and gravitational mass within the framework of Newtonian mechanics. Give experimental evidence for the proportionality between them.

10.32 Discuss the statement, 'Fictitious forces must be introduced into Newtonian mechanics when we choose as our reference rest frame a coordinate system rotating with respect to the fixed stars'.

10.33 Explain physically why a Foucault pendulum situated at the equator would not detect the rotation of the earth about its axis.

PROBLEMS

10.1 A train is moving uniformly and a particle is suspended from the roof of a carriage. The particle moves in a circular motion relative to the train. Use Galilean transformations to show that its path relative to the earth is a cycloid.

10.2 Momentum is conserved in a collision of two particles according to an observer in uniform motion, say a train. Show that the law of conservation holds for a ground observer.

10.3 Show that if a collision is elastic in one inertial frame, it is elastic in all inertial frames.

10.4 It is known that Galilean transformations break down for electromagnetic phenomena. Show that the e m. wave equation

$$\nabla^2 \Phi - \frac{1}{2}\frac{\partial^2 \Phi}{\partial t^2} = 0$$

is not invariant under Galilean transformations. Φ is the scalar potential.
[*Hint* : Use the chain rule, i.e. if $x = f(x', y', z', t')$, then

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial f}{\partial z'}\frac{\partial z'}{\partial x} + \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial x}\Bigg]$$

10.5 What will be the effective weight of a person carried
(i) vertically up in a rocket with an acceleration of 2 g and
(ii) vertcally down in a rocket with an accleration of 0.5 g
His weight on earth is 60 kg. *Ans*. 180 kg, 30 kg

10.6 Calculate ihe fictitious acceleration of the sun relative to a reference frame fixed on the earth. (Sun to Earth distance = 1.5×10^{12} cm.)
Ans. 7.8×10^4 cm/s^2 towards the centre of the earth

10.7 You are flying along the equator due east in a jet plane at 450 m s^{-1}. What is your Coriolis acceleration? *Ans*. 6.56×10^{-3} m s^{-2}

10.8 A pendulum is oscillating along the north-south direction at a place in latitude 30° N. What time must elaspse before the pendulum starts oscillating along NE–SW direction? *Ans*. 6 h

10.9 A car rounds a curve on a mountain road where radius of curvature is ρ. If the coefficient of friction is μ, prove that the greatest speed with which it can travel so as not to slip on this road is $\sqrt{\mu \rho g}$.

10.10 An xyz coordinate system rotates with angular velocity $\boldsymbol{\omega} = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$ with respect to a fixed XYZ frame having the same origin. If the position vector of a particle is given by $\mathbf{r} = \sin t\mathbf{i} - \cos t\mathbf{j} + t\mathbf{k}$, find (a) the apparent velocity and (b) the true velocity at any time. *Ans*. [(a) $\cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$
(b) $(t \sin t + 2 \cos t)\mathbf{i} + (2 \sin t - t \cos t)\mathbf{j}$]

[*Hint*: Apparent velocity = $\frac{d\mathbf{r}}{dt}$. True velocity = $\frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r}$]

10.11 A rocket is moving upwards with acceleration 5 g. What is the effective weight of an astronaut whose actual weight is 100 kg? *Ans*. 600 kg

10.12 How long would it take the plane of oscillation of Foucault pendulum to make one complete revolution if the pendulum is located at (a) north pole, (b) colatitude 45°, and (c) colatitude 85° ? *Ans*. 24 hr; 33.9 hr; 274.1 hr

10.13 If the earth were to cease rotating about its axis, what will be the change in the value of g at a place of latitude 60°? Assume the earth to be a sphere of radius 6.37×10^8cm. *Ans*. 0.84 cm/s^2

10.14 Show that the angular deviation of a plumb line from the true vertical at a point on earth's surface at a latitude λ is

$$\frac{r_o \omega^2 \sin \lambda \cos \lambda}{g - r_o \omega^2 \cos^2 \lambda}$$

where r_o is the radius of earth.

10.15 Prove that due to earth's rotation about its axis, winds in the northern hemisphere traveling from a high pressure to a low pressure area are rotated in a counterclockwise sense when viewed from the earth's surface. What happens to the winds in the southern hemisphere?

10.16 Prove that the centrifugal force acting on a particle of mass m on the earth's surface is a vector (a) directed away from the earth and perpendicular to the angular velocity ω and (b) of magnitude $m\omega^2 R \sin \lambda$, where λ is the colatitude. Where would the centrifugal force be (c) a maximum, (d) a minimum?

Ans. [(c) at the equator, (d) at the north and south poles]

11

Lorentz Transformations and their Relativistic Consequences

11.1 ORIGIN AND SIGNIFICANCE OF THE SPECIAL THEORY OF RELATIVITY

The special theory of relativity constitutes one of the most beautiful chapters of twentieth-century physics. Its origin provides a particularly interesting example of the overthrow of the notion of measure connected with space, time and motion, hitherto regarded as fundamental. The departure from classical concepts was forced by the discovery of new facts which were not in accord with earlier theories. The paradoxial properties of a theoretical entity called ether, which was construed to be at rest (in the early eighteenth century, James Bradley interpreted the aberration of light, which in terms of the wave theory of light implied that ether is stationary) and in motion (Michelson-Morley had inferred that ether is in motion with the earth), both at the same time. These contradictory facts required the reevaluation of the physical theory.

The classical concepts were closely associated with the successes of Newtonian mechanics. The principles of the classical view of physics may be briefly summarized as follows:

1. A physical phenomenon is considered to be completely understood only when a mechanical model of it has been constructed.
2. Classical mechanics is completely deterministic in describing the behaviour of a dynamical system, at any subsequent time, if the forces that are acting and the initial conditions are specified. This is the essence of a dynamical law and the only possible form of a physical law is the dynamical law of classical mechanics.
3. All physical processes take place in space and time. As discussed earlier in Chapter 5, space was attributed to have the properties of isotropy (the equivalence of all directions), homogeneity (the equivalence of all points of space) and Euclidean nature (the general expression for the distance between two points has the simplest possible form when cartesian coordinates are used). Further, it was assumed that although bodies move in space, their motion in no way affects the properties of space. The concept of absolute time of classi-

cal mechanics implied that unique universal time flows uniformly and equally, independent of the state of motion of physical bodies.

The postulalion of the special theory of relativity in 1905 (and the general theory in 1916) led to a radical revision of the accepted concepts of space and time. It denied the need and possibility of the mechanistic view of nature that one can construct mechanical models for all physical phenomena. It gave great fillip to the further development of contemporary physics, in particular atomic and nuclear physics. This role consisted not only of the use of important relations of the theory of relativity but also in showing that classical concepts obtained from everyday life turn out to be inadequate in dealing with new fields. To that extent, the theory of relativity envisaged the beginning of the development of a new, nonclassical physics.

All natural phenomena are described in terms of particles and fields. Nonrelativistic classical mechanics governs mostly the behaviour of matter in bulk under terrestrial conditions and as such is the correct mechanics for a vast range of phenomena; the range in which the speed of light can be considered infinitely large and the value of Planck's constant, h can be assumed to be zero. Special relativity gives us a formalism applicable to the dynamics of particles at all speeds up to and including the speed of light. Nonrelativistic classical mechanics follows as a low-velocity-limiting approximation from relativistic mechanics. Apart from this quantitative continuity between relativistic and pre-relativistic mechanics, there is a conceptual jump between the two. The space and time intervals are no longer independent of reference frames and turn out to be greatly different in frames moving with large velocities relative to one another. The simultaneity of events has to be qualified with additional specifications. In addition, there are other stranger features of the theory. Our commonsense notions are based on experiences and impressions gained in our infancy. The technology extended the range in which to test the theories. The special theory of relativity is strictly in accord with the observed facts and we are forced to stretch our minds to accept it.

The special theory of relativity is classified as classical, since there is one-to-one correspondence between the variables of the theory and the numbers measured in the experiment. The dynamical variables of position, velocity, acceleration and momentum can be measured to any degree of precision and accuracy. In contradiction to it, in quantum physics, the dynamical variables are represented by operators in the abstract space and we calculate the probability distribution functions to be compared with the numbers obtained from experimental measurements. In this particular context, special theory of relativity constitutes a branch of classical physics. The value of a dynamical variable averaged over a large number of identical experiments is governed by the classical equation of motion (Ehrenfest's theorem). It was relativity that predicted that energy can have negative sign, but it was the develpment of quantum mechanices in the hands of P. A. M. Dirac, that could interpret it in terms of negative energy states.

11.2 SEARCH OF A UNIVERSAL FRAME OF REFERENCE

Maxwell showed in 1864 that the velocity of electromagnetic waves or any disturbance in the electromagnetic fields will propogate with velocity c, which is the ratio

of the electrostatic to the electromagnetic unit of electrical charge. The value of c as found experimentally is 2.99792×10^{10} cm/s which agrees with the velocity of light in vacuo. This agreement with the value of velocity of light suggests the identity of light with electromagnetic radiation. In the traditions of nineteenth century physics, it was indeed natural to ascribe a medium to act as a vehicle for electromagnetic waves like sound waves which require the medium of air for their propagation. It seemed inevitable to postulate such a medium, called aether even though it was attrributed unusual properties like large shear modulus to account for the large velocity of light signals, zero density and perfect transparency to account for its undetectability. In Maxwell's theory this unobserved, all pervasive medium—aether— acts as a carrier for electromagnetic disturbances and c is the velocity of light measured by an observer who is at rest relative to it. However, the velocity of light c measured by an observer moving through aether with velocity v will be given by $c' = c + v$. It was this sequel that was to be tested by an experiment. Furthermore, it was conjectured that aether was the absolute space or the fundamental frame of reference Newton was looking for (or a frame of reference fixed relative to it), wherein Newton's laws of motion would hold perfectly.

The spinning and rotating earth should be moving through aether and an observer stationary with respect to the earth should experience an aether wind whose velocity is v with respect to the earth; much in analogy with the air wind felt by a running person. The earth has the orbital velocity of 30 km/s and $v/c \approx 10^{-4}$ and optical experiments which were accurate enough to detect the first-order effects in v/c were not able to detect the relative motion of the earth through aether. However, the subsequent attempt to interpret the negative results of the first-order experiments by Fitzgerald and Lorentz in 1892 was fraught with difficulties. It was however, agreed that the test of the aether hypothesis will necessitate a precision experiment capable of detecting a second-order effect of the order of $(v/c)^2 \approx 10^{-8}$, i.e. one part in hundred million.

Michelson-Morley experiment

Around that time, A A Michelson had invented the optical interferometer which had the best-known sensitivity at that time. In 1881 firstly alone and then in 1887 in collaboration with Morley, he carried out the now famous Michelson-Morley experiment. Its null result not only sounded the death knell of the aether hypothesis but also provided the experimental evidence for laying the foundation of the special theory of relativity.

This experiment consisted in sending light signals simultaneously in two directions at right angles to each other and measuring the difference in the times they take to come back to their starting point, after these had been reflected at optically equal distances.

The schematic diagram of the Michelson-Morley experiment is depicted in Fig. 11.1. The light rays from monochromatic source S fall on a half-silvered mirror, M, which divides the incident beam of half intensity each. The beams travel equal optical paths back and forth from M_1 and M_2 and after reflection from there, interfere in the field of view of the observer at O.

One of the mirrors was mounted on a micrometer screw and it was ensured that the motion of the movable mirror by only a fraction of a wavelength of light could be detected. The whole instrument was mounted on a stone slab floating in mercury and had the facility of interchanging the paths of light by rotating it through 90° without disturbing the adjustment. Were it so that one of the beams was at a disadvantage because of the motion of the earth through aether, there would result a shift in the interference pattern.

Suppose that **v** is the velocity of the earth through aether and that earth is moving from west to east. If d is the length of the optical path, then the time t_1 for the horizontal beam is

$$t_1 = \frac{d}{c-v} + \frac{d}{c+v} = \frac{2d}{c} \frac{1}{1-\frac{v^2}{c^2}} \tag{11.1}$$

As is clear from Fig. 11.1(b), the beam proceeding vertically must proceed along MA, so that its resultant with **v**, the aether drift, could be along MM_2. Thus, we get for t_2, the time for the vertical motion

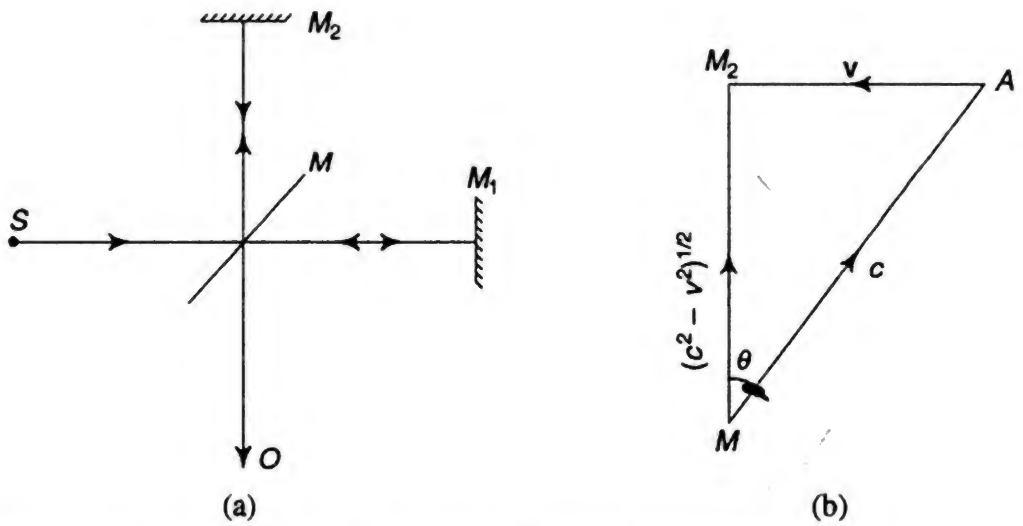


Fig. 11.1 (a) Schematic diagram of the Michelson-Morley experiment
(b) The velocity diagram showing the orientation of the light beam so that it can travel along MM_2 upwards in the presence of the orbital velocity v from west to east

$$t_2 = \frac{2d}{c} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{11.2}$$

When the arrangement is rotated through 90° and calling the new times $\bar{t}_1$ and $\bar{t}_2$ one gets

$$\bar{t}_1 = \frac{2d}{c} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{11.3}$$

The expected shift for $v = 3 \times 10^6$ cm/s, $c = 3 \times 10^{10}$ cm/s, $d = 11$ m (by repeated reflections) and $\lambda = 6 \times 10^{-5}$cm is

$$n = \frac{2 \times 1100 \times 9 \times 10^{12}}{9 \times 10^{20} \times 6 \times 10^{-5}}$$

$$= 0.37$$

$$\approx 0.4$$

The arrangement was capable of measuring one-hundredth of a fringe and as such the above shift was capable of accurate measurability. The real shift was too small, almost negligible indicating that the velocity of the earth relative to the aether frame can at best be equal to 10^3 cm/s which was the estimated accuracy of the measurement.

The null result of the Michelson-Morley experiment was such a deadly blow to the aether hypothesis that it was repeated by many groups in the subsequent 50-year period. In 1958, Cedarholm *et al.* repeated the Michelson-Morley experiment with microwaves and showed that if there is aether, and the earth is moving through it, then the earth's speed with respect to it has to be less than 10^{-3} of the earth's orbital velocity. The latest version of the Michelson-Morley experiment carried out by Jaseja, Javan, Murray and Townes (1964) utilises two lasers at right angles on a rotating frame; the frequency stability of lasers can be of the order of 20 Hz or so. It was possible to observe a frequency shift even 10^5 times smaller than the expected shift. As a result, it was demonstrated that the actual frequency shift is less than one-thousandth of the effect predicted on the assumption that light has a fixed velocity with respect to aether.

The null results of the Michelson-Morley experiment are understandable if the postulate of aether is rejected. It was Einstein who provided the theory of relativity in 1905 which was a major reconstruction of the description of physical phenomena. He concluded that the velocity of light is always the same in all directions and is independent of the relative uniform motion of the observer, medium and source.

11.3 POSTULATES OF THE SPECIAL THEORY OF RELATIVITY

The negative results of the Michelson-Morley type of experiments and also those about expected effects of relative velocity between two reference systems on wave propagation, led to the conclusion that the electromagnetic laws hold in all inertial systems with the same universal value of the velocity of light. Poincáre, a famous French mathematician, suggested that the motion referred to the supposed stationary aether (or the universal reference frame) cannot be observed and stated the principle of relativity that the laws of physics are the same for two observers moving with constant velocity relative to each other. The combination of the Poincáre principle of relativity with the finiteness of the velocity of propagation of interactions, called Einstein's principle of relativity was put forth by Einstein in 1905. The two postulates of the special theory of relativity are stated as follows:

(a) The Principle of Relativity

It is impossible to trace any distinction between any two inertial frames which are in uniform relative motion to each other by any physical measurement. In other words,

the laws of physics are the same in all inertial systems so that there is no preferred inertial frame and all the inertial frames are equivalent.

It is implied that in a gravity-free laboratory that is moving at constant velocity relative to another gravity-free laboratory, all experiments proceed in the same manner leading to the same results. The laws or equations describing physical phenomena have the same form in such reference frames. Thus, there is no such thing as absolute rest; there is no physical reasoning to prefer one inertial frame over the other. However, the numerical values of physical quantities are different relative to the two frames.

(b) The Postulate of Constancy of Velocity of Light

It states that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the source, intervening medium or observer.

Pictorially, the constancy of the velocity of light is represented in Fig. 11.2(a). It illustrates the inadequency of the Galilean velocity transformation. A test of the second postulate was carried out by Alväger, Farby, Kjellman and Walling at CERN (Centre European de Researches Nucleaires, Geneva) in 1964. The π^0 mesons were produced by protons of energies of about 3×10^{10} eV from the CERN proton synchroton. The π^0 mesons in the laboratory frame had velocity equal to $0.99975c$. The measured speed of the gamma-rays resulting from the decay of π^0 relative to the laboratory was $(2.9977 \pm 0.0004) \times 10^8$ m/s. This agreed with the accepted value of 2.9979×10^8m/s for the speed of light emitted from a stationary source. These experiments demonstrate clearly that the speed of light quanta (photons) emitted by a moving source is always equal to c, the velocity of light in empty space.

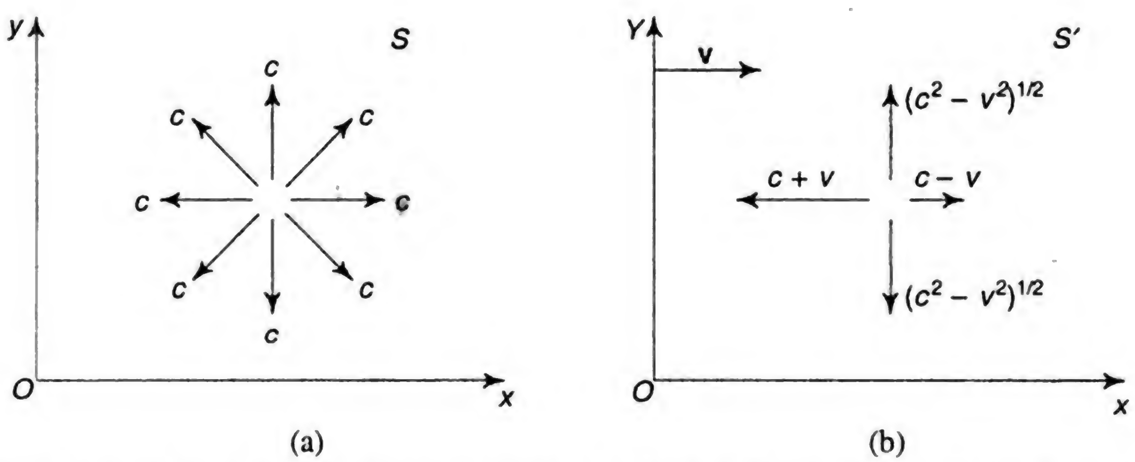


Fig. 11.2 (a) S is an absolute frame in which the speed of light in empty space is the same in all directions
(b) The velocity of light in empty space in the frame S' which moves with velocity v with respect to S, according to Galilean transformation

11.3.1 Consequences of Einstein's Postulates (Qualitative)

As stated earlier in Secs 11.2 and 11.3, the negative results of the Michelson-Morley experiment forced Einstein to conclude that the electromagnetic laws hold

in all inertial systems, with the value of the velocity of light, which is the same in all directions and is independent of the relative motion of the observer, medium and source. This invariance of the velocity of light c is embodied in the relationship as

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

where (x, y, z, t) refer to the termini of the light path in the unprimed system S and (x', y', z', t') to the termini in the primed system S' which is moving with velocity $\mathbf{v}$ relative to S. This relation is at the basis of the required transformation of coordinates (Lorentz transformations) and when x' is different from x, on account of relative motion in that direction, it will inevitably imply that t is different from t'. Thus under the new transformations, time will be no longer considered absolute for all observers in relative motion. The enunciation of the postulates was in accordance with the observed experimental facts that were at variance with the predictions of the Newtonian law of addition of velocities. Thus a new law of addition of velocities will be required which could explain the negative results of optical experiments adequately. Therefore, in the wake of the conclusion that each moving inertial frame has its own time t', the idea of an aether frame linked with absolute space becomes redundant and superfluous. The relationship implies that the ends of the light paths in all directions and in each of the relatively moving frames S and S' is a sphere about the origins O and O' [Fig. 11.2(a)]. To both the observers the propagating light spheres are reached simultaneously in their cases as measured by their respective clocks. This aspect is beyond our comprehension according to our everyday conceptions. However, it can be reconciled by the statement that simultaneity is not an absolute concept. Two events which are simultaneous in one frame, will not be simultaneous when observed from a moving inertial frame. Thus simultaneity is a relatively applicable concept and not a generally applicable one as it was thought according to Galilean transformations.

In addition, we will expect a difference in the measured times t and t', from the instant when O and O' separate from each other. This means that time intervals as measured by the observers in S and S' will be different. It will be shown in Sec. 11.5 that the time measured by a moving clock will always appear to be smaller than the corresponding interval measured in a system at rest. In other words, a moving clock runs slower than a clock at rest. Primarily it was Einstein's preoccupation with the nature of time that led him to his theory.

Another consequence of Einstein's postulates is the contraction of length along the line of relative motion (treated in Sec. 11.5). The modifications of our usual notion about measures of length, time and simultaneity on the basis of the special theory of relativity, are contrary to our notions based on commonsense. It was in this context that the theory was dubbed by physicists as well as the public at large as an attempt that mocked at our commonsense.

However, now the acceptability of the special theory has reached a level that all new theories are required to be consistent with it. So far, not a single experimental fact has been discovered which contradicts it. As such, it is an article of faith with the physicists.

11.4 LORENTZ TRANSFORMATIONS—DERIVATION

An event is defined by three coordinates of space and the time of its occurrence and Lorentz transformations relate the space and time coordinates of an event as observed from two inertial frames in relative motion.

S and S' are two inertial frames which are in relative motion with respect to each other with velocity $\mathbf{v}$. An event as observed by the observer in S is characterised by its location and time, by specifying the coordinates x, y, z, t. The same event as observed by the observer in S' is specified by the space-time coordinates x', y', z', t'. The problem is to find the functional relationships between the primed and unprimed coordinates such as $x' = x'(x, y, z, t)$, $y' = y'(x, y, z, t)$, $z' = z(x, y, z, t)$ and $t' = t'(x, y, z, t)$, i.e. the requisite transformation equations will relate the space-time coordinates of an event as observed by the observers in S and S'.

These will be the Lorentz transformation equations.The derivation of the Lorentz transformation equations will be based on the fundamental postulates of the theory of relativity and the assumption that space and time are homogeneous. The homogeneity of space and time implies that the measurement of a length or time interval of a particular event should be independent of its location or occurrence in the reference frame.

Consider the two reference systems S and S' (Fig. 11.3) moving with velocity $\mathbf{v}$ relative to each other. The axes OX and $O'X'$ are along the common axis and their origins O and O' coincide at $t = t' = 0$. The planes $x = 0$ and $x' = 0$ are kept parallel.

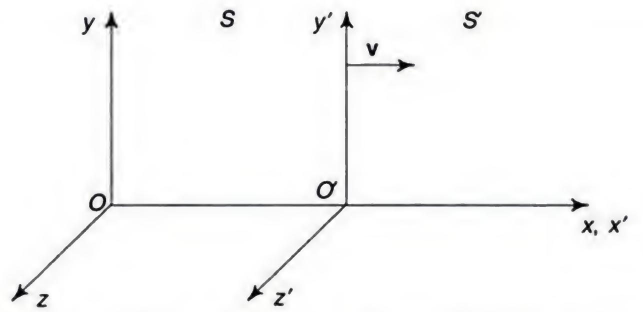


Fig. 11.3 S and S' are two inertial frames which are in relative motion with respect to each other with velocity $\mathbf{v}$

This assumption is not restrictive on our results, since due to isotropy of space, all directions in space are considered equivalent. The requisite transformations should conform to the following general requirements:

1. The postulate of the equivalence of all inertial frames requires that the direct (from the unprimed to the primed) and the reverse (from the primed to the unprimed) transformations should be symmetrical with respect to each other. One should be derivable from the other by changing $\mathbf{v}$ into $-\mathbf{v}$ and the primed quantities to unprimed ones and vice versa.
2. The finite points of one system should convert to the finite points of the other.
3. In the limit when $v \to 0$, the Lorentz transformations should reduce to the identity transformations, i.e.

$$x' = x; \quad y' = y; \quad z' = z \quad \text{and} \quad t' = t$$

4. In the limit when $c \to \infty$ or $:/c \ll 1$, the Lorentz transformations should reduce to the Galilean transformations, i.e.

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t \tag{10.7}$$

5. The law of addition of velocities as derived from the Lorentz transformations should leave the velocity of light c invariant, i.e. $c' = c$.

The transformation functions cannot be quadratic or of a degree higher than the first, since then the inversion will lead to irrationality. However, a linear fractional transformation may be inverted and still retain the same form, such as

$$x' = \frac{ax + b}{cx + d}$$

which gives $$x = \frac{b - dx'}{cx' - a}$$

This function for $x' = a/c$ will make x infinite and thereby violate the condition (2). Thus the only acceptable form of transformation is the linear one. The most general form of these transformations is

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$
$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t$$
$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \tag{11.7}$$

Here the coefficients a_{ij}, $i, j = 1, 2, 3, 4$ are undetermined constants. These will be determined from the above mentioned requirements, (l) – (5). The x-axis is coincident with the x'-axis, which will be tenable provided for $y = 0$, $z = 0$, i.e. for points lying on the x-axis, we get $y' = 0$, $z' = 0$. This leads to the following formulae for y' and z'

$$y' = a_{22}y + a_{23}z$$
$$z' = a_{32}y + a_{33}z \tag{11.8}$$

This implies that the coefficients a_{21}, a_{24}, a_{31} and a_{34} must be zero. The xy plane (corresponding to $z = 0$ plane) should transform to the $x'y'$ plane (corresponding to the $z' = 0$ plane). Analogously, the xz plane (corresponding to the $y = 0$ plane) should transform to the $x'z'$ plane ($y' = 0$). This argument requires that a_{23} and a_{32} should be zero and Eq. (11.8) reduce to

$$y' = a_{22}y$$
$$z' = a_{33}z \tag{11.9}$$

We will determine the coefficients a_{22} and a_{33} from the postulate of equivalence of inertial frames. Let a rod of unit length be at rest along the y-axis. Its length as measured by the observer S' will be a_{22}. Now suppose that the same unit rod lies along the y'-axis and its length as measured by S will be $1/a_{22}$. These inertial frames will be equivalent provided these measurements, which are of a reciprocal character,

are identical. This is, however, possible only if $a_{22} = 1/a_{22} = 1$. Analogously, one can show that $a_{33} = 1$. Thus Eq. (11.9) becomes

$$y' = y$$
$$z' = z \tag{11.10}$$

Further, we deal with the remaining equations for x' and t', namely

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \tag{11.11}$$

Regarding the t'-equation, the isotropy of space requires that t' should not depend on y and z; since if it were true, the clocks placed symmetrically in the yz-plane, for example at $+y$, $-y$ or $+z$, $-z$ about the x-axis would appear to disagree as observed by the observer in S'. This leads to the requirement that $a_{42} = a_{43} = 0$.

Let us apply the x'-equation to the origin O'. Since O' moves with velocity v, its coordinates as observed by the observer in S are given by the equation $x = vt$. Therefore, we anticipate that the correct transformation equation will be $x' = a_{11}(x - vt)$, since $x = vt$ will always reduce to the required equation $x' = 0$. Therefore, our equations assume the form

$$x' = a_{11}(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = a_{41}x + a_{44}t \tag{11.12}$$

The coefficients a_{11}, a_{41} and a_{44} are determined from the postulate of the constancy of the velocity of light. Let a light signal be emitted at $t = t' = 0$, when the origins O and O' are coinciding. The wave propagates with a speed in the form of a spherical wavefront in all directions. Its equations in the frames S and S' are

$$x^2 + y^2 + z^2 = c^2t^2 \tag{11.13}$$
$$x'^2 + y'^2 + z'^2 = c^2t'^2 \tag{11.14}$$

Substituting Eq. (11.12) into Eq. (11.14), we get

$$a_{11}^2 (x - vt)^2 + y^2 + z^2 = c^2(a_{41}x + a_{44}t)^2$$

After rearrangement of terms, one gets

$$(a_{11}^2 - c^2a_{41}^2)x^2 + y^2 + z^2 - 2(va_{11}^2 + c^2a_{41}a_{44})x\,t = (c^2a_{44}^2 - v^2a_{11}^2)t^2 \tag{11.15}$$

Since Eq. (11.15) has to be identical to Eq. (11.13), we get by comparing the coefficients

$$c^2a_{44}^2 - v^2a_{11}^2 = c^2$$

$$a_{11}^2 - c^2a_{41}^2 = 1$$

$$va_{11}^2 + c^2a_{41}a_{44} = 0$$

By solving the three equations simultaneously (see Example 11.1) one gets the three unknown coefficients a_{11}, a_{41} and a_{44} as

$$a_{44} = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

[redacted]

Applying Eq. (11.12),

$$x' = a_{11}(x - vt)$$

We retain the positive sign and thus get

$$a_{11} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{xii}$$

From Eq. (i), substituting the value of a_{11} from Eq. (xii), one gets

$$c^2 a_{44}^2 - v^2 \frac{1}{\left(1 - \frac{v^2}{c^2}\right)} = c^2$$

$$c^2 a_{44}^2 - \frac{v^2 c^2}{c^2 - v^2} = c^2$$

or

$$a_{44}^2 = 1 + \frac{v^2}{c^2 - v^2}$$

$$= \frac{1}{1 - \frac{v^2}{c^2}}$$

Thus

$$a_{44} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{xiii}$$

where we again choose the positive sign of the square root. Lastly, from Eqs (vii) and (viii), we get

$$a_{41} = -\frac{v}{c^2} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{xiv}$$

Thus the coefficients a_{11}, a_{41} and a_{44} have been determined.

EXAMPLE 11.2

(a) If the transformation between x and ct is required to be both symmetrical and unimodular, prove that these lead to the Lorentz transformation (unimodular means that the determinant of the transformation is unity).

(b) Derive the Lorentz transformations for an arbitrary direction of the velocity $\mathbf{v}$ relative to a coordinate system.

Solution

(a) The symmetry between x and ct requires Eq. (11.12) to be written as

$$x' = a_{11}\left(x - \frac{v}{c} ct\right)$$

and

$$ct' = a_{44}\left(ct - \frac{v}{c} x\right)$$

Comparing the above equation for ct' with the following equation from the set (11.12)

$$ct' = a_{44}\left(ct + \frac{a_{41}}{a_{44}}cx\right)$$

we have
$$\frac{a_{41}}{a_{44}}c = -\frac{v}{c}$$

or
$$a_{41} = -a_{44}\frac{v}{c^2} = -a_{11}\frac{v}{c^2} \qquad (\because a_{44} = a_{11})$$

The unimodularity of the transformation implies that

$$\begin{vmatrix} a_{11} & -a_{11}\dfrac{v}{c} \\ -a_{11}\dfrac{v}{c} & a_{11} \end{vmatrix} = a_{11}^2\left(1 - \frac{v^2}{c^2}\right) = 1$$

i.e.
$$a_{11} = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

These lead to the transformation equations

$$x' = \frac{x - vt}{\sqrt{1 - \dfrac{v^2}{c^2}}}; \qquad t' = \frac{t - \dfrac{v}{c^2}x}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

which are the Lorentz transformations.

(b) The special Lorentz transfromation equations are

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - \dfrac{v^2}{c^2}}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - \dfrac{v}{c^2}x}{\sqrt{1 - \dfrac{v^2}{c^2}}} \end{aligned} \tag{11.17}$$

Here
$$x = \frac{\mathbf{r}\cdot\mathbf{v}}{v}$$

$$x' = \frac{\mathbf{r}'\cdot\mathbf{v}}{v}$$

The component transverse to the velocity vector is obtained by subtracting the longitudinal component from the total as

Subtracting, we get

$$x_2 - x_1 = \frac{x_2' - x_1'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or
$$l = \frac{l'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Therefore
$$l' = l\sqrt{1 - \frac{v^2}{c^2}} \tag{11.20}$$

The length of an object in its rest frame is called its proper length, so that the proper length is always the greatest and to any other observer who is moving with velocity v, the rod appears to be contracted in the ratio $(1 - v^2/c^2)^{1/2}$: 1. Obviously this effect is reciprocal in character, since it depends on the square of the relative velocity.

According to Eq. (11.18), the transverse dimensions do not change because of motion, i.e.

$$y' = y$$
$$z' = z$$

The volume V' of a body as measured by the observer in S' is given by

$$V' = V\sqrt{\left(1 - \frac{v^2}{c^2}\right)} \tag{11.21}$$

where V is the proper volume of the body.

EXAMPLE 11.3

The earth will appear shortened along its diameter due to its orbital motion around the sun to an observer at rest relative to the sun. Calculate the change in length in the diameter. The orbital velocity of the earth is 30 km/s and the radius of the earth is 6371 km.

Solution

The apparent length is given by

$$l' = l\sqrt{1 - \frac{v^2}{c^2}}$$
$$= l\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)$$

where l is the proper length.

Therefore,
$$\Delta l = l - l'$$
$$= \frac{1}{2}l\left(\frac{v^2}{c^2}\right)$$
$$= \frac{1}{2} \times 2 \times 6371 \times 10^3 \left(\frac{30 \times 10^3 \times 10^2}{3 \times 10^{10}}\right)^2$$

$$= \frac{1}{2} \times 2 \times 6371 \times 10^3 \times \left(\frac{30\times10^3}{3\times10^8}\right)^2$$

$$= 6.37 \times 10^{-2} \text{ m}$$

11.5.2 Time Dilation or Apparent Slowing of Moving Clocks

Let an observer in the system S' send light signals from the point $(x', 0,0)$ at t'_1, and at a subsequent time t'_2. The interval $(t'_2 - t'_1)$ as observed by this observer will appear like an interval $(t_2 - t_1)$ to an observer in S.

From Eq. (11.18), one gets the corresponding times t_1 and t_2 as

$$t_1 = \frac{t'_1 + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t_2 = \frac{t'_2 + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Therefore

$$(t_2 - t_1) = \frac{(t'_2 - t'_1)}{\sqrt{1-\frac{v^2}{c^2}}} \tag{11.22}$$

Denoting $(t_2 - t_1) = \tau$ and $(t'_2 - t'_1) = \tau'$, we get from Eq. (11.22) that

$$\tau = \frac{\tau'}{\sqrt{1-\frac{v^2}{c^2}}} \tag{11.23}$$

The proper time (the time read by a clock moving with a given object) of a moving object is always less than the corresponding interval measured in a system at rest. It implies that a moving clock runs slower and this kinematical effect of relativity is called time dilation (or dilatation).

The effect of time dilation becomes important for high energy particles. If the particle is unstable, e.g. $\pi^{\pm}$, π^0, $\mu^{\pm}$ etc. its life time in flight is always considerably greater than τ_0, its life time measured in a decay at rest.

The Twin Paradox

The twin paradox (or clock paradox) has been a subject of controversy during 1957-59, after the initial skirmishes beginning in 1939. According to it, if one clock remains at rest in an inertial frame, and another (which has been synchronised with the first one) is taken off to a distant planet on any sort of path and finally brought back to the starting point, the time elapsed by the moving clock will be less than the

$$\frac{v^2}{c^2} = 1 - \frac{1}{60^4} = \frac{60^4 - 1}{60^4} = \frac{(60^2+1)(60^2-1)}{60^4}$$

or
$$\frac{v}{c} = \frac{(3601)(61)(59)}{60 \times 60 \times 60 \times 60} = 0.999$$

Hence
$$v = 0.999\ c$$

EXAMPLE 11.6

The mean life time of a μ-meson when it is at rest is 2.2×10^{-6} s. Calculate the average distance it will travel in vacuo before decay, if its velocity is 0.9 c.

Solution

If the proper mean life of μ mesons is T_0, then in the laboratory frame with respect to which these have velocity v, the mean life will be γT_0 and they travel an average distance $\gamma v\ T_0$ before decaying.

According to the data in the problem,

$$\gamma = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = \frac{10}{\sqrt{19}} = \frac{10}{4.36}$$

Average distance travelled before the decay

$$= \gamma v T_0$$

$$= \frac{10}{4.36} \times \frac{9 \times 3 \times 10^{10}}{10} \times 2.2 \times 10^{-6}\ \text{cm}$$

$$= 1.36\ \text{km}$$

EXAMPLE 11.7

Show that the electromagnetic wave equation

$$\frac{\partial^2 \phi}{dx^2} + \frac{\partial^2 \phi}{dy^2} + \frac{\partial^2 \phi}{dz^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{dt^2} = 0$$

is invariant under Lorentz transformations. ϕ stands for either a Cartesian component of vector potential **A** or scalar potential.

Solution

The equation will be invariant if it retains the same form when expressed in terms of new variables x', y', z', t'. The Lorentz transformation equations are

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

To express the wave equation in terms of the primed variables, we first find from the Lorentz transformations that

$$\frac{\partial x'}{\partial x} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}};\quad \frac{\partial x'}{\partial t} = -\frac{v}{\sqrt{1-\frac{v^2}{c^2}}};\quad \frac{\partial t'}{\partial x} = -\frac{v/c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\frac{\partial t'}{\partial t} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}};\frac{\partial y'}{\partial y} = \frac{\partial z'}{\partial z} = 1;\frac{\partial x'}{\partial y} = \frac{\partial x'}{\partial z} = \frac{\partial y'}{\partial x} = \ldots = 0$$

From the chain rule, and using the above results, we have

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial \phi}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial \phi}{\partial z'}\frac{\partial z'}{\partial x} + \frac{\partial \phi}{\partial t'}\frac{\partial t'}{\partial x}$$

$$= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\partial \phi}{\partial x'} + \frac{-v/c^2}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\partial \phi}{\partial t'}$$

Differentiating again wrt x, we have

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{1-\frac{v^2}{c^2}}\left(\frac{\partial^2 \phi}{\partial^2 x'} + \frac{v^4}{c^4}\frac{\partial^2 \phi}{\partial t'^2}\right) - \frac{2v}{c^2 - v^2}\frac{\partial^2 \phi}{\partial x \partial t'}$$

Similarly, we have $\dfrac{\partial \phi}{\partial t} = \dfrac{-v}{\sqrt{1-\frac{v^2}{c^2}}}\dfrac{\partial \phi}{\partial x'} + \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}\dfrac{\partial \phi}{\partial t'}$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{1}{1-\frac{v^2}{c^2}}\left(v^2\frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial t'^2}\right) - \frac{2vc^2}{c^2 - v^2}\frac{\partial^2 \phi}{\partial x' \partial t'}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial y'}$$

Substituting these in the wave equation, we obtain

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial y'^2} + \frac{\partial^2 \phi}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t'^2}$$

Therefore, the wave equation is invariant under Lorentz transformations.

EXAMPLE 11.8

Show that the four-dimensional volume element *dxdydzdt* is invariant under Lorentz transformations.

$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

where $$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

We get $$\frac{\partial}{\partial x'} = \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)$$

$$\frac{\partial}{\partial t'} = \gamma\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)$$

so $$\frac{\partial^2}{\partial x'^2} = \gamma^2\left(\frac{\partial^2}{\partial x^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x\partial t} + \frac{v^2}{c^4}\frac{\partial^2}{\partial t^2}\right)$$

$$\frac{\partial^2}{\partial y'^2} = \frac{\partial^2}{\partial y^2}, \quad \frac{\partial^2}{\partial z'^2} = \frac{\partial^2}{\partial z^2}$$

$$\frac{\partial^2}{\partial t^2} = \gamma^2\left(v^2\frac{\partial^2}{\partial x^2} + 2v\frac{\partial^2}{\partial x\partial t} + \frac{\partial^2}{\partial t^2}\right)$$

so $$\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

EXAMPLE 11.11

The length of a moving rod measured by an inertial observer is $3\sqrt{2}$ cm. The rod is moving making an angle of 45° with the direction of motion and with velocity 0.8 c. Find the proper length of the rod.

Solution

Assuming that the velocity 0.8 c is the velocity of frame S'. Resolving the length of the moving rod along the direction of motion, say x-axis, then

$$l'_x = 3\sqrt{2}\cos 45° = 3 \text{ cm}$$
$$l'_y = 3\sqrt{2}\sin 45° = 3 \text{ cm}$$

Lorentz contraction factor $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$

$$= (1 - 0.64)^{-\frac{1}{2}}$$

$$= \frac{5}{3}$$

Then $$u_x = \frac{dx}{dt}$$

and $$u'_x = \frac{dx'}{dt'}$$

Now from Lorentz transformation [Eq. (11.17)] taking the differentials of the coordinates and time, we have

$$\begin{aligned}
dx' &= \frac{dx - vdt}{\sqrt{1 - \dfrac{v^2}{c^2}}} \\
dy' &= dy \\
dz' &= dz \\
dt' &= \frac{dt - \dfrac{vdx}{c^2}}{\sqrt{1 - \dfrac{v^2}{c^2}}}
\end{aligned} \tag{11.24}$$

Thus $$\frac{dx'}{dt'} = \frac{dx - vdt}{dt - \dfrac{vdx}{c^2}} = \frac{\dfrac{dx}{dt} - v}{1 - \dfrac{v}{c^2}\dfrac{dx}{dt}}$$

$$\frac{dy'}{dt'} = \frac{dy\sqrt{1 - \dfrac{v^2}{c^2}}}{dt - \dfrac{v}{c^2}dx} = \frac{\dfrac{dy}{dt}\sqrt{1 - \dfrac{v^2}{c^2}}}{1 - \dfrac{v}{c^2}\dfrac{dx}{dt}}$$

$$\frac{dz'}{dt'} = \frac{dz\sqrt{1 - \dfrac{v^2}{c^2}}}{dt - \dfrac{v}{c^2}dx} = \frac{\dfrac{dz}{dt}\sqrt{1 - \dfrac{v^2}{c^2}}}{1 - \dfrac{v}{c^2}\dfrac{dx}{dt}}$$

or

$$\begin{aligned}
u'_x &= \frac{u_x - v}{1 - \dfrac{v}{c^2}u_x} \\
u'_y &= \frac{u_y\sqrt{1 - \dfrac{v^2}{c^2}}}{1 - \dfrac{v}{c^2}u_x} \\
u'_x &= \frac{u_z\sqrt{1 - \dfrac{v^2}{c^2}}}{1 - \dfrac{vu_x}{c^2}}
\end{aligned} \tag{11.25}$$

The inverse transformations are

$$u_x = \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x}$$

$$u_y = \frac{u'_y\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v}{c^2}u'_x} \tag{11.26}$$

$$u_z = \frac{u'_z\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v}{c^2}u'_x}$$

In the nonrelativistic approximation, i.e. when $v/c \ll 1$, the above equations reduce to

$$u_x = u'_x + v$$

$$u_y = u'_y$$

$$u_z = u'_z$$

which are the results of Newtonian, mechanics. Let us consider the case of the motion of a particle along the x-axis, then

$$u_x = u, \quad u_y = u_z = 0$$

and we get from Eq. (11.26),

$$u = \frac{u' + v}{1 + \frac{v}{c^2}u'}$$

or $$u' = \frac{u - v}{1 - \frac{vu}{c^2}}$$

When $u = c$,

$$u' = \frac{c - v}{1 - \frac{v}{c}} = c$$

Thus when a particle is moving with the velocity c with respect to S (which is possible only for a zero-rest mass particle like a photon), its velocity as observed from S' is still c. This illustrates that velocity transformations are consistent with the principle of constancy of the speed of light, as they should be, since Lorentz transformations are based on the principle of constancy of the speed of light.

EXAMPLE 11.16

If u and u' are the velocities of a particle in the frames S and S' which are moving with velocity v relative to each other, prove that

$$\sqrt{1-\frac{u'^2}{c^2}} = \frac{\sqrt{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u^2}{c^2}\right)}}{\left(1-\frac{vu_x}{c^2}\right)}$$

Also, give the corresponding inverse transformation.

Solution

The velocity of a particle relative to S' is given by

$$u'^2 = u_x'^2 + u_y'^2 + u_z'^2$$

$$= \frac{(u_x - v)^2 + \left(1-\frac{v^2}{c^2}\right)u_y^2 + \left(1-\frac{v^2}{c^2}\right)u_x^2}{\left(1-\frac{vu_x}{c^2}\right)^2}$$

$$= \frac{(u_x - v)^2 + \left(u^2 - u_x^2\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{vu_x}{c^2}\right)^2}$$

Thus $$1-\frac{u'_2}{c^2} = 1 - \frac{\left(\frac{u_x}{c}-\frac{v}{c}\right)^2 + \left(\frac{u^2}{c^2}-\frac{u_x^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{u_x v}{c^2}\right)^2}$$

$$= \frac{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}+\frac{v^2u^2}{c^4}}{\left(1-\frac{vu_x}{c^2}\right)^2}$$

$$= \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u^2}{c^2}\right)}{\left(1-\frac{vu_x}{c^2}\right)^2}$$

$$= \frac{2.0 \times 10^{10}}{\left[1 + \frac{(2.25)(2.0)}{9}\right] \times 1.34}$$

$$= \frac{2.0 \times 10^{10}}{3.72} = 0.538 \times 10^{10} \text{ cm/s}$$

The resultant velocity

$$u = \sqrt{u_x^2 + u_y^2} = 1.21 \times 10^{10} \text{ cm/s}$$

and $\qquad \theta = \tan^{-1} \dfrac{u_y}{u_x} = \tan^{-1} .498 = 26.47°$ with velocity 2×10^{10} cm/s

11.5.4 Velocity of Light in a Moving Fluid—Fizeau Experiment

In 1851 Fizeau showed that the speed of light in a moving fluid depended on the speed of the fluid relative to the laboratory. Fizeau employed an interferometer to measure the velocity of light in liquids flowing in a pipe, both in the direction of and opposed to the propagation of light. The Einstein velocity addition theorem gives the simplest explanation of the dependence of the speed of light in a flowing fluid on its refractive index.

Let S be the laboratory system in which water is moving with uniform velocity v in the positive x-direction, as depicted in Fig. 11.5. The direction of light is also in the positive x-direction. The frame S' is moving with velocity v with respect to S and the water is at rest in it. Let n' denote the refractive index of stationary water in S'. The velocity of light u' (u'_x, u'_y, u'_z) in S' has the components,

$$u'_z = c/n'; \; u'_y = u'_z = 0 \tag{11.27}$$

Applying Eq. (11.26), we get

$$u_y = u_z = 0$$

$$u_x = \frac{u'_x + v}{\left(1 + \frac{v u'_x}{c^2}\right)} = \frac{\left(\frac{c}{n'} + v\right)}{\left(1 + \frac{vc}{n'c^2}\right)}$$

$$= \frac{c}{n'}\left(1 + \frac{n'v}{c}\right)\left(1 + \frac{v}{n'c}\right)^{-1}$$

Expanding $\left(1 + \dfrac{v}{n'c}\right)^{-1}$ by the Binomial theorem and neglecting terms of the order of v^2/c^2, one gets

$$u_x = \frac{c}{n'}\left(1 + \frac{n'v}{c}\right)\left(1 - \frac{v}{n'c}\right)$$

Thus a signal can never be propagated at a velocity larger than the velocity of light. In case any signal propagates at a velocity higher than c, the principle of causality will break down under that situation, the effect will precede the cause in its time sequence.

11.5.7 Relativistic Optical Effects: Aberration of Light, Doppler Effect, Spectral Red-Shift

Imagine a light source at O', the origin of S' emitting a train of plane electromagnetic waves of unit amplitude. The rays are chosen to lie in the plane $x'y'$ and make an angle θ' with the x'-axis. Its propagation is described by the equation*

$$\cos 2\pi\left[\frac{x'\cos\theta' + y'\sin\theta'}{\lambda'} - \nu' t'\right] \tag{i}$$

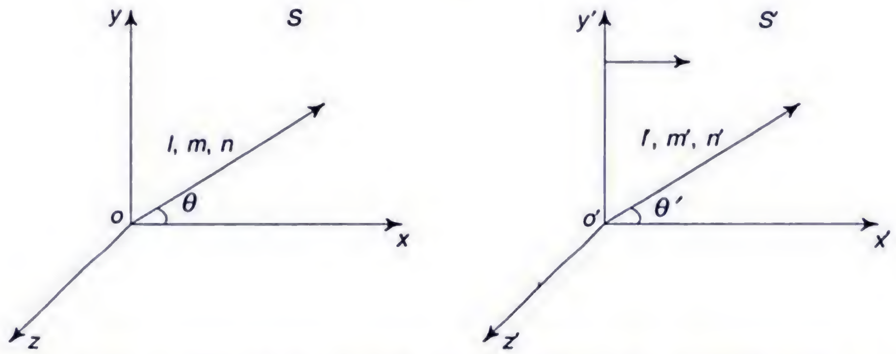


Fig. 11.6 A ray (or wave normal) of plane electromagnetic wave is emitted from O', the origin of S' making an angle θ' with the x'-axis

In the S frame of reference, these wave-fronts will also be planes, since the Lorentz transformations being linear, transform a plane into a plane. Therefore, in the frame S, the equation becomes

$$\cos 2\pi\left[\frac{x\cos\theta + y\sin\theta}{\lambda} - \nu t\right] \tag{ii}$$

Here the unprimed quantities x, y, λ, ν and θ refer to the unprimed frame S. Let us express Eq. (i) in terms of undashed quantities through Lorentz tranformations. Rewriting Eq. (11.17), we get

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned} \tag{11.17}$$

*A plane wave of amplitude a frequency ν and propagating in a direction having direction cosines (l, m, n) is represented by

$$a\cos 2\pi\left(\frac{lx + my + nz}{\lambda} - \nu t\right)$$

The velocity components in the frame S of the star are

$$u_x = 0$$

$$u_y = -c$$

The velocity components in the earth's frame S' are obtained from Eq. (11.25) and are given as

$$u'_x = \frac{u_x - v}{1 - \dfrac{vu_x}{c^2}} = -v$$

$$u'_y = \frac{u_y\sqrt{1 - \dfrac{v^2}{c^2}}}{1 - \dfrac{vu_x}{c^2}} = -c\sqrt{1 - \frac{v^2}{c^2}}$$

The anlge of incidence to the normal θ' is given by

$$\tan\theta' = \frac{u'_x}{u'_y} = \frac{-v}{-c\sqrt{1 - \dfrac{v^2}{c^2}}} \approx \frac{v}{c}$$

The orbital velocity of earth is 30 km/s and the apparent tilt from a star at the zenith is

$$\tan\theta' \approx \frac{v}{c}$$

$$= \frac{3 \times 10^6}{3 \times 10^{10}} = 10^{-4} \text{ rad}$$

$$= 20.5''$$

This is in accord with the measured value of the tilt due to aberration of light from the stars.

Equation (vi) connecting the frequencies of the light wave as observed from S and S' is rewritten as

$$\nu = \frac{\nu'\left(1 + \dfrac{v}{c}\cos\theta'\right)}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

This is the relativistic equation for Doppler effect. Its inverse equation is

$$\nu' = \frac{\nu\left(1 - \dfrac{v}{c}\cos\theta\right)}{\sqrt{1 - \dfrac{v^2}{c^2}}} \tag{11.30}$$

The relativistic formula reduces to the classical result in the limit $v/c \ll 1$. Rewriting Eq. (11.30),

Equation (11.30) predicts a more striking result of transverse Doppler effect, since it has no classical analogue. For $\theta = \pi/2$, one gets from Eq. (11.30).

$$\nu = \nu' \sqrt{1 - \frac{v^2}{c^2}}$$

The frequency observed in a perpendicular direction is lower than the proper frequency. This effect was verified experimentally first by Ives and Stilwell in 1938 and 1941 and subsequently by Walter Kündig in 1963. The treatment illustrates that by considering the invariance of phase of a wave, the Lorentz transformation applied to optical phenomena gives three effects, namely aberration, longitudinal and transverse Doppler effects.

EXAMPLE 11.18

A physicist is arrested for driving through the red lights at a traffic junction. At the trial the physicist claims that he was driving so fast that the red light appeared green to him. How fast must he be driving? The wavelength of red and green light are 6300 Å and 5400 Å respectively.

Solution

Let ν_g and ν_r denote the frequencies of green and red light respectively and λ_g and λ_r the corresponding wavelengths. Employing the relativistic Doppler formula, we have

$$\frac{\nu_g}{\nu_r} = \frac{\lambda_r}{\lambda_g} = \frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{7}{6}$$

or
$$1 + \frac{v^2}{c^2} + \frac{2v}{c} = \frac{49}{36} - \frac{49}{36}\left(\frac{v^2}{c^2}\right)$$

or
$$85\left(\frac{v}{c}\right)^2 + 72\left(\frac{v}{c}\right) - 13 = 0$$

Therefore
$$\frac{v}{c} = \frac{-72 \pm \sqrt{(72)^2 + 52 \times 85}}{2 \times 85}$$

$$= 0.153$$

which gives
$$v = 0.153\ c$$

EXAMPLE 11.19

A spaceship coasting in interstellar space counters an alien space probe ,which has a radio transmitter. As the probe approaches, the frequency initially received by the ship is 130 MHz. As the probe recedes into the distance, the frequency eventually drops to 60 MHz. What is the intrinsic frequency of the probes' transmitter? What is the relative speed of the two ships?

Solution

Let the intrinsic frequency of the probes' transmitter be ν_o and v be the relative velocity of the two ships. The spaceship receives signals at frequency

11.3 What does the term 'universal frame of reference' convey? Give brief arguments to bring out the fact that Aether failed to be recognized as a universal frame.

11.4 State the principle of the Michelson-Morley experiment and describe it to prove that light has a fixed velocity with respect to the Aether,

11.5 Discuss the statement: 'The Michelson-Morley experiment gave sufficient evidence that the concept of Aether is redundant'.

11.6 Bring out the impact of negative results of the Michelson-Morley experiment on contemporary physics.

11.7 State Einstein's principle of relativity and discuss its implications.

11.8 What is the principle of constancy of velocity of light. Comment on its experi- mental verification.

11.9 State Einstein's postulates of special relativity and discuss their impact on the then prevailing concept of physics.

11.10 List the requirements which any transformation from one coordinate system to another moving with uniform velocity relative to it must fulfil.

11.11 Give arguments to show that functions defining transformations from one inertial frame to another can neither be linear fractional nor have degree more than one.

11.12 What are Lorentz transformations? Obtain an expression for these. Complete the steps required to obtain Eq. (11.16).

11.13 Show that the converse of a Lorentz transformation is also a Lorentz transformation.

11.14 What is proper length and show that this is the maximum value a length can have?

11.15 What is length contraction and show that this effect is reciprocal in nature?

11.16 Explain the meaning of the term time dilation and cite two examples where such an effect has been observed.

11.17 What is twin (or clock) paradox? Give arguments leading to its resolution.

11.18 Obtain an expression for relativistic transformation of velocities and hence show that velocity of light is not affected by the velocity of the emitting source with respect to the observer.

11.19 State Einstein's velocity addition theorem and prove that this is in conformity with the principle of constancy of speed of light.

11.20 Comment on the results of Fizeau's experimental on velocity of light in flowing liquids in the light of the special theory of relativity.

11.21 In the derivation of Eq. (11.28) it is assumed that water and light are travelling along the $+x$ direction. Derive a similar expression for the case when light is traversing in the $+x$ direction, whereas the water is flowing along the $-x$ direction.

11.22 What does the term 'simultaneity of relativity' mean? How does it fit into the concept of causality?

11.23 Prove that simultaniety has only a relative and not an absolute meaning.

11.24 Show that the principle of causality imposes limit on the maximum velocity with which a signal can be transmitted.

11.25 If at some stage it becomes possible to transmit a signal at a speed greater than that of light, then either the theory of relativity will have to be modified or the concept of causality abandoned'. Discuss this statement.

PROBLEMS

11.1 Show that two successive Lorentz transformations in the same direction commute and are equivalent to one Lorentz transformation.

11.2 Two electron beams travel along the same straight line but in opposite directions with velocities $v = 0.9c$ relative to the laboratory system. Find the relative velocity v of the electrons according to Newtonian mechanics. What will be the velocity measured by an observer moving with one of the electron beams?

Ans. (a) $v = 2 \times 0.9\ c = 1.8\ c$, (b) $v = 0.994\ c$

11.3 Two rulers, each of which has a length l_o in its own rest frame, move towards each other with equal velocities v relative to a given reference system. Find the length l of each of the rulers in the reference frame in which the other ruler is at rest.

$$Ans.\ l = l_o \frac{1 - \frac{v^2}{c^2}}{1 + \frac{v^2}{c^2}}$$

11.4 An unstable particle has the mean proper lifetime of 2 μs. What will be its lifetime when it is travelling with a speed of 0.9 c. *Ans.* 4.58 μs.

11.5 Show that $x^2 + y^2 + z^2 - c^2t^2$ is Lorentz invariant.

11.6 The average lifetime of a neutron as a free particle at rest is 15 min. It disintegrates spontaneously into an electron, a proton and a neutrino. Calculate the average minimum velocity with which it must leave the sun in order to reach the earth without decay. The sun is at a distance of 11×10^{10} m from earth. *Ans.* 1.13×10^8 m/s

11.7 The spectral line of wavelength 4000 °A in the spectrum of light from a star is found to be shifted toward the red end of the spectrum by 1 °A. Calculate the recessional velocity of the star. *Ans.* 7.5×10^6 cm/s

11.8 Frame S' is moving with constant velocity 2×10^8 m/s wrt S along x-axis. An electron has velocity u'_x relative to S' the components of which are:

$u'_x = 6 \times 10^7$ m/s ; $u'_y = 4 \times 10^7$ m/s ; $u'_z = 3 \times 10^7$ m/s Find the velocity components in frame S. What is the magnitude of u. *Ans.* [$u_x = 2.29 \times 10^8$ m/s; $u_y = 2.63 \times 10^7$ m/s; $u_z = 1.97 \times 10^7$ m/s; $u = 2.32 \times 10^8$ m/s].

11.9 A certain transition in potassium produces light of frequency 8.0×10^{14} Hz. When this transition occurs in a distant galaxy, the light reaching the earth has the frequency 5.0×10^{14} (a red shift). Determine the radial motion of the galaxy with respect to the earth. *Ans.* [The galaxy is receding at speed 0.438 c]

11.10 The length of the side of a square as measured by an observer in a stationary frame of reference S is l. What will be its apparent area as observed by an observer in a reference frame S' moving with velocity v along one of the sides of the square?

$$Ans. \left(1^2 \sqrt{1 - \frac{v^2}{c^2}}\right)$$

11.11 The proper mean lifetime of π^+ meson is 2.5×10^{-8}s. Deduce

(1) Mean lifetime of π^+ meson moving with velocity 2.4×10^{10} cm/s

(2) The distance traveled by the meson during one mean life

(3) The distance traveled without relativistic effects

Ans. [4.166×10^{-8} s; 10m; 600 cm]

11.12 What is the velocity of nuclear particles whose mean lifetime is observed to be 2.5×10^{-7}s.The proper lifetime is 2.5×10^{-8}s. *Ans.* (0.99 c)

11.13 Two oppositely directed spaceships move with identical velocity of 0.7 c as measured by an observer on earth. What is the velocity of one spaceship as observed from the other? *Ans.* (0.94 c)

11.14 A photon is observed from a spaceship moving with a speed of 0.9 c. What is the speed of the photon with respect to the spaceship? *Ans.* (c)

12

Relativistic Energy and Momentum: Four-Vectors

In Chapter 11 we derived Lorentz transformation equations which, in turn, were used to deduce their kinematical consequences. It is proposed to extend the treatment to the implications of special relativity to dynamics. One approach can be the formal formulation of the dynamical equation of motion in accordance with the postulates of the special theory of relativity. This will require the concepts of four-vectors and relativistic invariance. However, there is another approach that is relatively easy and less formal, since it uses physical arguments to show the relation between the dynamical variables of Newtonian mechanics and their relativistic counterparts. This will be based on the law of conservation of momentum and its inclusion in relativistic mechanics. This will necessarily modify our measure of mass and bring out its dependence on velocity. The measures of other derived concepts of momentum, energy and force will be modified accordingly to conform to relativistic transformations.

12.1 VARIATION OF MASS WITH VELOCITY

Let two identical and perfectly elastic particles of masses m_1' and m_2' moving with velocities $+\mathbf{u}'$ and $-\mathbf{u}'$ parallel to the x'-axis in the system undergo a head-on collision. The particles will be brought to rest momentarily and then rebound under the elastic forces and move back with velocities $-\mathbf{u}'$ and $+\mathbf{u}'$ respectively, relative to the system S'.

We will like to view the same collision from the system S, which is moving with velocity $-\mathbf{v}$ relative to S' along the x-axis. Let the particles of masses m_1 and m_2 have velocities $\mathbf{u}_1$ and $\mathbf{u}_2$, before collision with respect to the system S. At the instant of collision, the colliding particles come to rest relative to each other and let the sum of the masses be M, measured in S, when the particles are instantaneously at rest during the course of the collision. At the instant of collision, the colliding particles are at rest relative to S', but move with velocity + $\mathbf{v}$ relative to S.

It is postulated that the conservation of mass and linear momentum of the particles hold during the collision such that

$$m_1 + m_2 = M \tag{12.1}$$

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = M\mathbf{v} \tag{12.2}$$

From Eq. (11.26), we get

$$u_x = \frac{u_x' + v}{1 + \dfrac{vu_x'}{c^2}}$$

and applying it to particles 1 and 2, we get

$$u_1 = \frac{u' + v}{1 + \dfrac{vu'}{c^2}} \tag{12.3}$$

$$u_2 = \frac{-u' + v}{1 - \dfrac{vu'}{c^2}} \tag{12.4}$$

The direction of u_2 will depend on the relative magnitudes of u' and v in the Fig. 12.1. Rewriting Eq. (12.2), after substitutions for u_1 and u_2 from Eqs (12.3) and (12.4) respectively and for M from Eq. (12.1), one gets

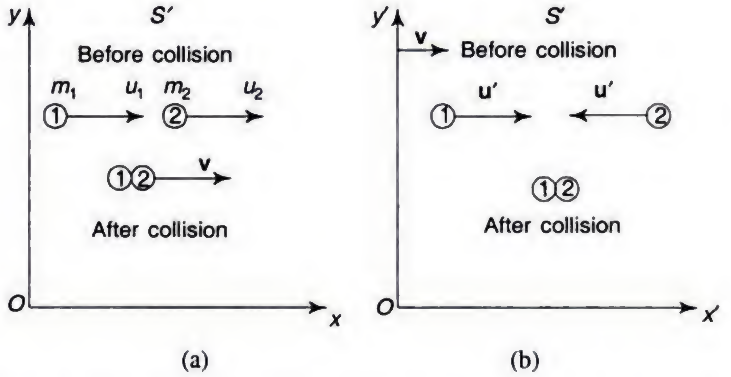


Fig. 12.1(a) During collision balls 1 and 2 are brought to rest with respect to each other, but move with velocity **v** with respect to system S

(b) During collision the balls are brought to rest with respect to each other and also with respect to S'

$$m_1\left[\frac{u' + v}{1 + \dfrac{vu'}{c^2}}\right] + m_2\left[\frac{-u' + v}{1 - \dfrac{vu'}{c^2}}\right] = (m_1 + m_2)\, v$$

or

$$m_1\left[\frac{u' + v}{1 + \dfrac{vu'}{c^2}} - v\right] = m_2\left[v + \frac{u' - v}{1 - \dfrac{vu'}{c^2}}\right]$$

Therefore,

$$\frac{m_1}{\left(1 + \dfrac{vu'}{c^2}\right)}\left[u' + v - v - \frac{v^2u'}{c^2}\right]$$

$$= \frac{m_2}{\left(1 - \frac{vu'}{c^2}\right)}\left[v - \frac{v^2u'}{c^2} + u' - v\right]$$

or
$$\frac{m_1}{m_2} = \frac{1 + \frac{vu'}{c^2}}{1 - \frac{vu'}{c^2}} \tag{12.5}$$

Making use of the result of Example 11.16, we get

$$\left(1 + \frac{vu'_x}{c^2}\right) = \left[\frac{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u'^2}{c^2}\right)}{1 - \frac{u^2}{c^2}}\right]^{1/2}$$

For particle 1, one has

in S': $u'_x = u'$; and $u' = u'$

and in S: $u = u_1$

Substituting these values in the above result, one gets

$$1 + \frac{vu'}{c^2} = \left[\frac{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u'^2}{c^2}\right)}{1 - u_1^2/c^2}\right]^{1/2} \tag{12.6}$$

For particle 2, one has

in S': $u'_x = -u'$

$u' = u'$

and in S: $u = u_2$

Substituting these values in the same result again, one has

$$1 - \frac{vu'}{c^2} = \left[\frac{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u'^2}{c^2}\right)}{1 - \frac{u_2^2}{c^2}}\right]^{1/2} \tag{12.7}$$

Dividing Eq. (12.6) by Eq. (12.7) one gets

$$\frac{1 + \frac{vu'}{c^2}}{1 - \frac{vu'}{c^2}} = \frac{\sqrt{1 - u_2^2/c^2}}{\sqrt{1 - u_1^2/c^2}} \tag{12.8}$$

Substituting in Eq. (12.5), we get

$$m_1\sqrt{1 - \frac{u_1^2}{c^2}} = m_2\sqrt{1 - \frac{u_2^2}{c^2}} \tag{12.9}$$

If $m_1 = m_0$ when $u_1 = 0$ and $m_2 = m_0$ when $u_2 = 0$, then Eq. (12.9) is satisfied if

$$m_1 = \frac{m_0}{\sqrt{1 - \dfrac{u_1^2}{c^2}}}$$

and
$$m_2 = \frac{m_0}{\sqrt{1 - \dfrac{u_2^2}{c^2}}}$$

Thus for the conservation of both mass and momentum to hold good during the collision, the mass of a particle moving with velocity u relative to S is given by

$$m = \frac{m_0}{\sqrt{1 - \dfrac{u^2}{c^2}}} \tag{12.10}$$

where m_0 is the proper mass of the particle and m is called the relativistic mass. Obviously, the proper mass of a particle is the smallest.

The first experimental confirmation of the variation of mass with velocity came from Bucherer in 1909, when he carried out a series of measurements of e/m ratio of high velocity electrons of radioactive origin and showed that the value of the ratio was smaller for fast moving electrons. The charge on the electron e being a physical reality is Lorentz invariant and the mass m will have higher value for a faster electron.

Bucherer employed naturally occurring radioactive elements and selected electrons of a certain velocity by the velocity selector arrangement of orthogonal electric and magnetic fields (Section 13.6.1). These electrons were deflected by magnetic field and e/m determined from the deflection. The results obtained were in excellent agreement with Eq. (12.10).

EXAMPLE 12.1

A rocket propels itself rectilinearly through empty space by emitting radiation, whose recoil provides the necessary thrust. If v is the final velocity relative to its initial rest frame, prove that the ratio of the initial and final rest mass of the rocket is

$$\frac{m_i}{m_f} = \left[\frac{c + v}{c - v}\right]^{1/2}$$

Solution

According to the laws of conservation of energy and momentum, we get

$$m_i c^2 = \frac{m_f c^2}{\sqrt{1 - \dfrac{v^2}{c^2}}} + h\nu \tag{i}$$

$$\frac{h\nu}{c} = \frac{m_f v}{\sqrt{1 - \dfrac{v^2}{c^2}}} \tag{ii}$$

Now
$$u^2 = u_x^2 + u_y^2 + u_z^2$$
$$u'^2 = u_x'^2 + u_y'^2 + u_z'^2$$

where
$$u_x' = \frac{u_x - v}{1 - \dfrac{vu_x}{c^2}}$$

$$u_y' = \frac{u_y\sqrt{1 - \dfrac{v^2}{c^2}}}{1 - \dfrac{vu_x}{c^2}}$$

$$u_z' = \frac{u_z\sqrt{1 - \dfrac{v^2}{c^2}}}{1 - \dfrac{vu_x}{c^2}}$$

Thus
$$1 - \frac{u'^2}{c^2} = 1 - \frac{u_x'^2 + u_y'^2 + u_z'^2}{c^2}$$

$$= \frac{\left(1 - \dfrac{v^2}{c^2}\right)\left(1 - \dfrac{u^2}{c^2}\right)}{\left(1 - \dfrac{vu_x}{c^2}\right)^2}$$

Hence
$$m' = m\frac{\left(1 - \dfrac{vu_x}{c^2}\right)}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

It may be remarked that it is the x-component of u which occurs in the transformation formula. Thus if two bodies of equal rest mass are moving with the same speed in S but in different directions, the masses will not be the same in S'. This also leads to the known result, i.e. when the body is at rest in S, i.e. $u_x = 0$, then

$$m_i' = \frac{m}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

12.2 MASS-ENERGY EQUIVALENCE

Let a force $\mathbf{F}$ be acting on a particle which gets displaced by a distance $d\mathbf{l}$ in the direction of the force, then the work done dW is given by the scalar product of $\mathbf{F}$ and $d\mathbf{l}$, i.e.

$$dW = \mathbf{F} \cdot d\mathbf{l}$$

two broad principles of classical physics, namely the law of conservation of energy and law of conservation of mass are fused together into this single comprehensive law of conservation of total relativistic energy. This law is an article of faith with the physicists, since discoveries of new facts and particles have been based on the analysis involving this law.

Rewriting Eq. (12.17), we get

$$T = E - m_0c^2$$
$$= c^2 (m - m_0)$$

or
$$\Delta E = \Delta mc^2 \tag{12.18}$$

where Δm is the change in mass corresponding to the change in energy ΔE. According to Einstein, this is the most important result of the special theory of relativity.

EXAMPLE 12.3

The earliest proof of the validity of the Einstein mass energy relationship was provided by Cockroft and Walton in 1932, when they bombarded ${}_3Li^7$ with ${}_1H^1$ accelerated to energies of 0.1 to 0.7 MeV. The cloud chamber pictures showed that two α-particles so produced, leave the point of disintegration and proceed with equal energies in the opposite directions.

The reaction may be represented as

$${}_3L^7i + {}_1H^1 \rightarrow [{}_4Be^8] \rightarrow {}_2He^4 + {}_2He^4 + Q$$

where Q is the energy balance. $[{}_4Be^8]$ is the intermediate nucleus that being in the excited state, decays subsequently into two α's. Determine the value of Q.

Solution

The masses are
$$M({}_3Li^7) = 7.01818 \text{ a.m.u.}$$
$$M({}_1H^1) = 1.008142 \text{ a.m.u.}$$
$$M({}_2He^4) = 4.003860 \text{ a.m.u.}$$

Mass defect,
$$\Delta M = 0.01860 \text{ a.m.u.}$$

and
$$Q = \Delta Mc^2$$
$$= 17.32 \text{ MeV}$$

The experimental value of Q obtained from the energies of the incident protons and outgoing α-particles is 17.33 MeV. The agreement between the observed and the calculated values of Q, lends our faith in the correctness of the Einstein mass-energy relation.

EXAMPLE 12.4

The sun radiates energy continually and the solar energy reaching the top of the Earth's atmosphere does so at the rate of 1.35×10^3 watt/m^2. Calculate the decrease in the mass of sun per second.

Solution

The earth is at a distance of 1.5×10^{11} m from the sun. Thus the total energy radiated by the sun

$$\Delta E = 4\pi (1.5 \times 10^{11})^2 \times 1.35 \times 10^3$$
$$\simeq 4 \times 10^{26} \text{ J}$$

$$p_y = \frac{m_0 u_y}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{12.19}$$

$$p_e = \frac{m_0 u_z}{\sqrt{1 - \frac{u^2}{c^2}}}$$

and the total energy is given by

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Identically the corresponding quantities in S' are

$$p'_x = m' u'_x$$

$$= \frac{m_0 u'_x}{\sqrt{1 - \frac{u'^2}{c^2}}}$$

$$p'_y = \frac{m_0 u'_y}{\sqrt{1 - \frac{u'^2}{c^2}}}$$

$$p'_z = \frac{m_0 u'_z}{\sqrt{1 - \frac{u'^2}{c^2}}} \tag{12.20}$$

and $$E' = m'c^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{u'^2}{c^2}}}$$

In order to seek the proper transformation relations between the momentum components, let us work with the first equation of the set, Eq. (12.20). Rewriting it, we get

$$p'_x = \frac{m_0 u'_x}{\sqrt{1 - \frac{u'^2}{c^2}}}$$

Substituting for the primed quantities in terms of unprimed ones from the relations

$$\frac{1}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{1 - \frac{v u_x}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}\sqrt{1 - \frac{u^2}{c^2}}} \tag{Ex 11.9}$$

and
$$u'_x = \frac{u_x - v}{1 - \dfrac{vu_x}{c^2}} \tag{11.25}$$

we get
$$p'_x = \frac{m_0(u_x - v)}{\left(1 - \dfrac{vu_x}{c^2}\right)} \times \frac{\left(1 - \dfrac{vu_x}{c^2}\right)}{\sqrt{1 - \dfrac{v^2}{c^2}}\sqrt{1 - \dfrac{u^2}{c^2}}}$$

$$= \frac{m_0}{\sqrt{1 - \dfrac{u^2}{c^2}}} \frac{u_x - v}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

$$= \gamma(mu_x - mv)$$

$$= \gamma\left(p_x - \frac{vE}{c^2}\right) \tag{12.21}$$

Similarly substituting for

$$\frac{1}{\sqrt{1 - \dfrac{u'^2}{c^2}}} \text{ and } u'_y = u_y \frac{\sqrt{1 - \dfrac{v^2}{c^2}}}{1 - \dfrac{vu_x}{c^2}}$$

into the expression for p'_y, we get

$$p'_y = \frac{m_0}{\sqrt{1 - \dfrac{u'^2}{c^2}}} u'_y$$

$$= \frac{m_0(1 - vu_x/c^2)}{\sqrt{(1 - v^2/c^2)}\sqrt{(1 - u^2/c^2)}} \times \frac{u_y\sqrt{1 - v^2/c^2}}{1 - vu_x/c^2}$$

$$= \frac{m_0 u_y}{\sqrt{(1 - u^2/c^2)}}$$

$$= p_y$$

Similarly $\quad p'_z = p_z$

Lastly,
$$E' = m'c^2 = \frac{m_0 c^2}{\sqrt{1 - u'^2/c^2)}}$$

$$= \frac{m_0(1 - vu_x/c^2)c^2}{\sqrt{1 - v^2/c^2}\sqrt{1 - u^2/c^2}}$$

$$= \gamma mc^2(1 - vu_x/c^2)$$

$$= \gamma(E - vp_x)$$

One can easily write the inverse transformations

$$p_x = \gamma(p'_x + vE'/c^2)$$
$$p_y = p'_y$$
$$p_z = p'_z$$
$$E = \gamma(E' + vp'_x) \qquad (12.22)$$

Let us further show that the quantity $(p^2 - E^2/c^2)$ is an invariant. By direct substitution, we have

$$\begin{aligned} p'^2 - E'^2/c^2 &= p'^2_x + p'^2_y + p'^2_z - E'^2/c^2 \\ &= \gamma^2(p_x - vE/c^2)^2 + p_y^2 + p_z^2 - \gamma^2(E/c - vp_x/c)^2 \\ &= p_x^2 + p_y^2 + p_z^2 - E^2/c^2 \\ &= p^2 - E^2/c^2 \end{aligned}$$

If S' is the rest frame of the particle, then the left-hand side is equal to $-m_0^2c^2$. Hence

$$c^2p^2 - E^2 = -m_0^2c^4$$

or
$$E = \pm\sqrt{c^2p^2 + m_0^2c^4} \qquad (12.23)$$

Thus that the sign of the energy may be positive or negative is a consequence of relativity. However, it was P.A.M. Dirac who ascribed a physical meaning to the negative energy. It is an important relation which is frequently used in particle and nuclear physics to calculate the energy of a particle when its momentum is given or vice versa.

EXAMPLE 12.7

Prove by direct substitution that

$$E = c\sqrt{p^2 + m_0^2c^2}$$

Solution

Now
$$p = \frac{m_0 u_x}{\sqrt{1 - u^2/c^2}}$$

Hence
$$\begin{aligned} p^2c^2 + m_0^2c^4 &= \frac{m_0^2u^2c^2}{(1 - u^2/c^2)} + m_0^2c^4 \\ &= \frac{m_0^2u^2c^2 + m_0^2c^4 - m_0^2u^2c^2}{(1 - u^2/c^2)} \\ &= \frac{m_0^2c^4}{1 - u^2/c^2} = E^2 \end{aligned}$$

i.e.
$$E^2 = p^2c^2 + m_0^2c^4$$

or
$$E = c\sqrt{p^2 + m_0^2c^2}$$

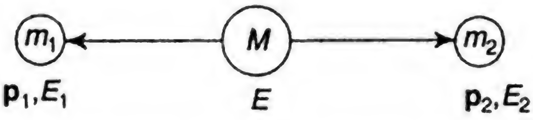
EXAMPLE 12.8

A particle of rest mass M decays at rest into two particles of rest masse m_1 and m_2; energies E_1 and E_2, momenta $\mathbf{p}_1$, and $\mathbf{p}_2$ respectively. Derive the expressions for the

energies E_1 and E_2. Apply the formalism to the decay of a π-meson into a μ-meson and a neutrino and get the total and kinetic energy of the μ-meson. $m_\pi c^2 = 139.6$ MeV; $m_\mu c^2 = 105.7$ MeV and the neutrino has the rest mass approximately zero.

Solution

The decay of a particle of mass M may be depicted as follows:



The particles of masses m_1 and m_2 must go off in opposite directions if linear momentum is to be conserved, such that

$$p_1 = p_2 \tag{i}$$

According to the law of conservation of energy, one gets

$$E_1 + E_2 = Mc^2 \tag{ii}$$

From Eq. (i), we obtain

$$c^2p_1^2 = c^2p_2^2$$

Now using Eq. (12.23),

$$c^2p_1^2 = E_1^2 - m_1^2c^4$$

and

$$c^2p_2^2 = E_2^2 - m_2^2c^4$$

Hence

$$E_1^2 - m_1^2c^4 = E_2^2 - m_2^2c^4$$

or

$$(E_1 + E_2)(E_1 - E_2) = (m_1^2 - m_2^2)\,c^4 \tag{iii}$$

Dividing Eq. (iii) by Eq. (ii), we get

$$E_1 - E_2 = \frac{(m_1^2 - m_2^2)c^2}{M} \tag{iv}$$

Manipulating Eqs (ii) and (iv), we obtain

$$E_1 = \frac{(M^2 + m_1^2 - m_2^2)c^2}{2M}$$

and

$$E_2 = \frac{(M^2 + m_2^2 - m_1^2)c^2}{2M}$$

The momenta of both the particles are given by

$$cp_1 = cp_2 = \sqrt{E_1^2 - m_1^2c^4} = \sqrt{E_2^2 - m_2^2c^4} \tag{v}$$

The formalism developed can be applied to radioactive decay, photon emission and meson decay. For example, π-meson decay is given by the equation

$$\pi^\pm \to \mu^\pm + \nu$$

We get the value of the total energy of μ-meson as

$$E_\mu = \frac{(139.6)^2 + (105.7)^2 - (0)^2}{2 \times 139.6}$$

$$= 109.8 \text{ MeV}$$

The kinetic energy of the μ-meson is

$$T_\mu = E_\mu - m_\mu c^2$$
$$= 4.1 \text{ MeV}$$

This prediction is confirmed by measurements. In view of the fundamental nature of this decay process, it serves as a standard for calibration in nuclear emulsion measurements. Powell and coworkers in 1947 were led to the discovery of the π-meson from the decay of the π-meson at rest in nuclear emulsions due to the unique energy of 4.1 MeV of the μ-meson.

EXAMPLE 12.9

A stationary particle, e.g. an atom or nucleus of rest mass M_0 is struck by a photon of energy $h\nu$ which is completely absorbed. Calculate the recoil velocity v of the combined system.

Solution

According to the conservation laws of energy and linear momentum, we get

$$E = M_0c^2 + h\nu = M'c^2 \tag{i}$$
$$p = h\nu/c = M'v \tag{ii}$$

where M' is the mass of the combined system. Thus

$$v/c = h\nu/M'c^2$$
$$= h\nu/(M_0c^2 + h\nu) \tag{iii}$$

When $h\nu \ll M_0c^2$, we have the fractional recoil velocity

$$v/c = h\nu/M_0c^2$$

which is expected from Newtonian calculation, since a body of invariant mass M_0, on being struck by a photon is given an impulse $h\nu/M_0c^2$.

EXAMPLE 12.10

The phenomenon of nuclear resonant scattering was demonstrated by P.B. Moon first in 1951 alone and then in 1953 with W.G. Davey, employing a radioactive source of ^{198}Hg mounted on the tip of a high-speed rotor. Gamma rays of energy 412 keV emitted from the moving source fall upon a stationary target of mercury, which scattered them. Experimentally, the scattering was maximum when the energy supplied to the gamma-rays by the Doppler shift was just able to raise the struck nucleus into a higher state, and this condition was reached at a source velocity of 700 m/s for ^{198}Hg. Justify the experimental finding by calculating the requisite velocity on the basis of the kinematical theory of Doppler effect.

Solution

Let a stationary atom of mass M_0 emit a photon of energy $h\nu$ and momentum $h\nu/c$. The emitter undergoes a recoil and acquires a velocity v. Let the mass of the recoiling atom be M'. Then according to the laws of conservation of energy and momentum, we have

$$E = M_0c^2 = M'c^2 + h\nu = E' + h\nu \tag{i}$$
$$P = 0 = M'v - h\nu/c = P' - h\nu/c \tag{ii}$$

Therefore

$$E' = M_0c^2 - h\nu \tag{iii}$$
$$cP' = h\nu \tag{iv}$$

$$v_x = \frac{\partial E}{\partial p_x}, v_y = \frac{\partial E}{\partial p_y}, v_z = \frac{\partial E}{\partial p_z}$$

both in the relativistic and Newtonian domains.

Solution

The energy is given in terms of momentum by

$$E = \sqrt{c^2 p^2 + (m_o c^2)^2}$$

$$= \sqrt{c^2 (p_x^2 + p_y^2 + p_z^2) + (m_o c^2)^2}$$

Differentiating wrt p, we get

$$\frac{\partial E}{\partial p_x} = \frac{1}{2} \frac{2c^2 p_x}{\sqrt{c^2 (p_x^2 + p_y^2 + p_z^2) + (m_o c^2)^2}}$$

$$= \frac{c^2 p_x}{E} = \frac{c^2 \gamma m_o v_x}{\gamma m_o c^2} = v_x$$

Analogously, $\dfrac{\partial E}{\partial p_y} = v_y$

and $\dfrac{\partial E}{\partial p_z} = v_z$

Since the Newtonian result is just the low-velocity limit of the relativistic result, the above results hold in Newtonian domain.

One could explicitly derive these results from

$$E = \frac{p^2}{2m_o} = \frac{p_x^2}{2m_o} + \frac{p_y^2}{2m_o} + \frac{p_z^2}{2m_o}$$

Now, $\dfrac{\partial E}{\partial p_x} = \dfrac{p_x}{m_o} = v_x$

Similarly, $\dfrac{\partial E}{\partial p_y} = v_y$ and $\dfrac{\partial E}{\partial p_z} = v_z$

12.4 FORCE TRANSFORMATIONS—ACTION AND REACTION

We develop the necessary transformation formulae for force $\mathbf{F}$ and $\mathbf{F}'$ between two inertial frames S and S' respectively through the definitions

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$\mathbf{F}' = \frac{d\mathbf{p}'}{dt'} \tag{12.24}$$

Thus $F_x' = \dfrac{dp_x'}{dt'}$

$$= \frac{\dfrac{dp_x'}{dt}}{\dfrac{dt'}{dt}} = \frac{\gamma\left(\dfrac{dp_x}{dt} - \dfrac{v}{c^2}\dfrac{dE}{dt}\right)}{\gamma\left(1 - \dfrac{v}{c^2}\dfrac{dx}{dt}\right)}$$

$$= \frac{F_x - \dfrac{v}{c^2}\dfrac{dE}{dt}}{1 - \dfrac{vu_x}{c^2}} \tag{12.25}$$

where we made use of the following transformations

$$p_x' = \gamma\left(p_x - \frac{vE}{c^2}\right) \tag{12.20}$$

$$t' = \gamma\left(1 - \frac{vx}{c^2}\right) \tag{11.17}$$

dE/dt is the rate of change of the particles' energy as measured in S. Let us show that this is the quantity $\mathbf{F} \cdot \mathbf{u}$. We have

$$E^2 = c^2p^2 + m_0^2c^4$$

$$= c^2(\mathbf{p}\cdot\mathbf{p}) + m_0c^4 \tag{12.23}$$

Thus $$E\frac{dE}{dt} = c^2\mathbf{p}\cdot\left(\frac{d\mathbf{p}}{dt}\right) \tag{12.24}$$

$$= c^2\mathbf{p}\cdot\mathbf{F} \tag{12.25}$$

But $$E = mc^2$$

Therefore $$\frac{dE}{dt} = \frac{1}{m}\mathbf{p}\cdot\mathbf{F}$$

$$= \mathbf{F}\cdot\mathbf{u} \tag{12.26}$$

Inserting it in Eq. (12.25), we get the transformation for the force component parallel to the direction of relative motion of two inertial frames as

$$F_x' = \frac{F_x - \dfrac{v}{c^2}(\mathbf{F}\cdot\mathbf{u})}{1 - \dfrac{v_x}{c^2}} \tag{12.27}$$

Similarly, we get $$F_y' = \frac{dp_y'}{dt'}$$

$$= \frac{dp_y'/dt}{dt'/dt} = \frac{\dfrac{dp_y}{dt}}{\gamma\left(1 - \dfrac{v}{c^2}\dfrac{dx}{dt}\right)}$$

then the force on these charges will depend both on their position as well as the velocity. Magnetic effects of moving charges follow as a natural consequence of force transformations from one frame to another, or in other words, electric and magnetic fields are intimately related to each other.

It can also be seen that in the realm of large velocities, the acceleration and force are not parallel in general, unlike the nonrelativistic dynamical case. Expressing force as

$$\mathbf{f} = d/dt\,(m\mathbf{u})$$

$$= m d\mathbf{u}/dt + \mathbf{u}\, dm/dt \tag{12.30a}$$

Now $$m = E/c^2$$

Therefore $$\frac{dm}{dt} = \frac{1}{c^2}\frac{dE}{dt} = \frac{1}{c^2}\frac{d(T + m_0 c^2)}{dt}$$

$$= \frac{1}{c^2}\frac{dT}{dt}$$

$$= \frac{1}{c^2}(\mathbf{f}\cdot\mathbf{u}) \qquad \left(\because \frac{dT}{dt} = \mathbf{f}\cdot\mathbf{u}\right)$$

Reverting back to Eq. (12.30a)

$$\mathbf{f} = m\,\frac{d\mathbf{u}}{dt} + \frac{u(\mathbf{f}\cdot\mathbf{u})}{c^2}$$

or $$\mathbf{a} = d\mathbf{u}/dt = \mathbf{f}/m - \mathbf{u}/mc^2\,(\mathbf{f}\cdot\mathbf{u}) \tag{12.31}$$

The first term is in the direction of force $\mathbf{f}$ whereas the second term is in the direction of $\mathbf{u}$ (which may not be parallel to $\mathbf{f}$) and hence the statement that in general the acceleration is not in the direction of the force. However, when $\mathbf{f}$ is perpendicular to $\mathbf{u}$, then $\mathbf{f}\cdot\mathbf{u} = 0$ and the above equation becomes

$$\mathbf{a} = \mathbf{f}/m = \mathbf{f}/m_0\sqrt{1 - u^2/c^2} \tag{12.32}$$

This latter case is exemplified by the motion of a charged particle in a magnetic field.

Example 12.12

In Newtonian mechanics, the relation $\frac{dE}{dt} = \mathbf{f}\cdot\mathbf{v}$ is valid, where E is the total energy of the particle that is moving with velocity v and is acted on by a force f. Show that this relation is also valid in relativistic mechanics.

Solution

The particle's instantaneous momentum $\mathbf{p}(t)$ and instantaneous energy $E(t)$ are related by

$$E^2 = c^2\,\mathbf{p}\cdot\mathbf{p} + (m_0 c^2)^2 \tag{1}$$

Differentiating Eq. (1) wrt time we get

$$2E\,\frac{dE}{dt} = 2c^2\mathbf{p}\cdot\frac{d\mathbf{p}}{dt}$$

or $$\frac{dE}{dt} = \frac{c^2 p}{E}\cdot\frac{dp}{dt} \tag{2}$$

the case of photons. These ideas were employed by Compton to interpret the Compton effect, which refers to the change in wavelength of photons on being scattered from light elements.

The principal massless particle is the photon, which is a quantum of electromagnetic field and due to its strong interaction with charged particles such as electrons, positrons, etc., it is easily detectable with the help of a photographic film, phototube or the eye. There is another particle called the neutrino which is associated with weak forces of radioactive beta decay. It is believed to be massless, since its mass has been shown to be less than 1/2000 the rest mass of the electron. Owing to its extremely weak interaction with matter (and consequently great capacity to transverse heavenly bodies like sun and stars without much interaction), its detection is not easy. Similarly, it is believed that there is another massless particle, called the graviton, which is associated with the gravitational force. However, due to its very weak interaction with matter, it has not been detected at all.

12.6 TACHYONS

According to the special theory of relativity, the relativistic mass, momentum and total energy of a particle are given by

$$m = \gamma m_0$$

$$\mathbf{p} = \gamma m_0 \mathbf{u}$$

$$E = \gamma m_0 c^2$$

where m_0 is the mass measured in an inertial frame with respect to which the particle is at rest and γ, the contraction factor $= 1/\sqrt{1 - u^2/c^2}$. An infinite amount of energy is necessitated for accelerating a particle up to the velocity of light since γ then becomes infinite. Einstein had implied that the velocity of propagation of interactions—electromagnetic or gravitational—is c, the velocity of light in vacuum. Some physicists had suggested that particles can indeed travel faster than light. According to them there is no reason why a particle cannot exist which is already moving at a velocity $\mathbf{u} > c$. Tachyons, from the Greek word *tachys* (meaning swift) is the name given to such particles. They assume that their rest mass is imaginary since it is not observable and their energy and momentum are real. It may be remarked that their existence has been postulated in accordance with the special theory.

Furthermore it is implied (without proof) that a tachyon, on losing energy gets speeded up until it is travelling infinitely fast and then it has no energy at all. This property is shown diagrammatically in Fig. 12.2.

According to the special theory, as u increases, $\mathbf{p}$, E increase but never reach the asymptote, $u = c$. The normal particles lie to the left of the line $u = c$ and the tachyons to the right.

Some attempts have been made to produce tachyons in the laboratory. At Princeton, T Alväger and M N Kriesier expected to produce a pair of equally and oppositely charged tachyons, T_+ and T_- by surrounding a source of γ-rays, Cs-134 with some materials. However, since a tachyon is expected to lose its energy in a

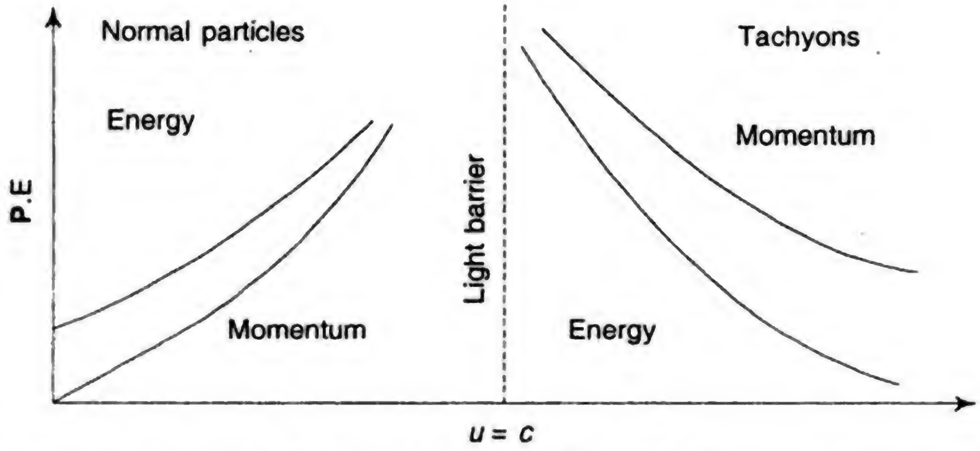


Fig 12.2 The momentum **P** and energy *E*, plotted as a function of velocity of normal particles $\mathbf{u} < c$ and tachyons (for which $\mathbf{u} > c$). For tachyons as energy decreases, the velocity increases and for normal particles, the velocity increases with increasing energy

distance of 10^{-3} cm or so, no successful detection could be done. It is conjectured that Einstein's special theory may not hold in certain regions of outer space, such as inside a quaser where the matter is in a highly compressed state. This leads to the evidence of the existence of tachyons inside quasers, but this is yet to be established. However, if the existence of tachyons is established in future, this will lead to the modifications of the current theories.

12.7 FOUR-VECTORS AND THEIR TRANSFORMATIONS

A four-vector is defined as a mathematical entity of four components which transform in a similar way to (x_1, x_2, x_3, x_4) or what amounts to like (dx_1, dx_2, dx_3, dx_4).

Rewriting Lorentz transformation [Eq. (10.17)], we get

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(1 - \frac{v}{c^2}x\right) \end{aligned} \tag{10.17}$$

Putting

$$\begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= z \\ x_4 &= ict \end{aligned}$$

Equation (10.17) becomes

$$\begin{aligned} x_1' &= \gamma\left(x + i\frac{v}{c}x_4\right) \\ x_2' &= x_2 \\ x_3' &= x_3 \\ x_4' &= \gamma\left(x_4 - i\frac{v}{c}x_1\right) \end{aligned} \tag{12.33}$$

The general formulae for the transformation of a four-vector A (A_1, A_2, A_3, A_4) are from analogy with Eq. (12.33)

$$\begin{aligned} A_1' &= \gamma\left(A_1 + i\frac{v}{c}A_4\right) \\ A_2' &= A_2 \\ A_3' &= A_3 \\ A_4' &= \gamma\left(A_4 - i\frac{v}{c}A_1\right) \end{aligned} \tag{12.34}$$

The length of a four vector is unchanged under rotation of axes, i.e. under Lorentz transformation. It is shown in books on special theory of relativity that Lorentz transformation is a rotation in the four-space, i.e. the space-time continuum. Thus

$$A_1'^2 + A_2'^2\ A_3'^2 + A_4'^2 = \gamma^2\left(A_1 + i\frac{v}{c}A_4\right)^2 + A_2^2 + A_3^2$$

$$+ \gamma^2\left(A_4 - i\frac{v}{c}A_1\right)^2 = A_1^2 + A_2^2 + A_3^2 + A_4^2 \tag{12.35}$$

If the squares of the length of a four-vector is positive, it is a space-like vector; whereas if the square of its length is negative, it is time-like vector. Examples of four-vectors are:

Position four-vector: $(\mathbf{r}, ct)$

Four-velocity: $\left(\dfrac{dx_1}{ds}, \dfrac{dx_2}{ds}, \dfrac{dx_3}{ds}, \dfrac{dx_4}{ds}\right)$

Four-acceleration: $\left(\dfrac{d^2\mathbf{r}}{ds^2}, \dfrac{d^2ct}{ds^2}\right)$

Four-current density: $(\mathbf{J}, c\rho)$

Four-momentum: $(\mathbf{p}, E/c)$

Wave four-vector: $(\mathbf{k}, \omega/c)$

where $\mathbf{k}$ is a wave vector and ω the frequency.

EXAMPLE 12.13

Prove on the basis of the invariance of the scalar product of two four-vectors that $E^2 = p^2c^2 + m_0^2c^4$.

Solution

Now $(\mathbf{p}, E/c)$ is a momentum four-vector in S. Since the length-of a four-vector is invariant under Lorentz transformation, therefore

$$p^2 + \left(i\frac{E}{c}\right)^2 = \text{const}$$

$$p^2 - \frac{E^2}{c^2} = \text{const}$$

When $p = 0$, $E = m_0c^2$ so that the constant is equal to $-m_0^2c^2$. Hence $p^2 - E^2/c^2 = -m_0^2c^2$ or $E^2 = p^2c^2 + m_0^2c^4$.

Alternatively, if a particle has rest energy $E_0 = m_0c^2$, then its energy and momentum measured in any other frame can be combined to form an invariant quantity: E^2

the lower bound of the j-particle lifetime. Given the half-width of the resonance curve at half the height is 2 Mev.

In another experiment, the J-meson decay products were detected. Find the mass of the J-meson decaying into an electron and a positron, if it is known that their energies are identical ($E_1 = E_2 = 3.1$ Gev) and the divergence angle between them is $\varphi = 60°$.

Solution

The particle energy in this experiment is E_{cm}, which corresponds to the position of the maximum. The half-width of the resonance curve at half maximum is $\Delta E = 2$ Mev.

According to uncertainty principle

$$\Delta E \times \tau \approx \hbar$$

where τ is the lifetime. Thus,

$$\tau \approx \frac{\hbar}{\Delta E} \approx \frac{6.628 \times 10^{-27}}{2 \times 3.14 \times 2 \times 1.59 \times 10^{-6}}$$

$$\approx 3 \times 10^{-22}\ \text{s}$$

To find the mass in another experiment, let us use the relativistic invariance of the scalar product of four-momentum

$$m^2c^4 = (E_1 + E_2)^2 - (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2 c^2$$

Since $E_1 = E_2 = E$, and consequently, $\boldsymbol{p}_1 = \boldsymbol{p}_2 = \boldsymbol{p}$; $\cos(\boldsymbol{p}_1, \boldsymbol{p}_2) = \cos\varphi = \frac{1}{2}$, we get

$$m^2c^4 = 4E^2 - 2p^2c^2 - 2p^2c^2 \cos^2\varphi$$

$$= 4E^2 - 3p^2c^2$$

Since e^+ and e^- are ultra relativistic particles ($v \approx c$), then $E \approx pc$ and

$$mc^2 \approx 3.1\ \text{Gev}.$$

EXAMPLE 12.16

At what energy of a proton incident on a resting proton in the reaction

$$p + p \to d + \pi^+$$

may the kinetic energy of the pion vanish in the lab frame? Given that the deuteron mass $m_dc^2 = 2m_pc^2 = 2 \times 0.94$ Gev and that of the pion $m_\pi c^2 = 0.14$ Gev.

Solution

A common problem in high energy physics is the production of two or more particles by collision between a particle (particle 1) with mass m_1, momentum $\boldsymbol{p}_1 = \boldsymbol{p}$ and energy E_1, impinging on the target (particle 2) of mass m_2 at rest in the laboratory.

We will relate the incident energy and momentum in the laboratory to the centre of mass (C M) variables, on the basis of the invariance of the scalar product of two four-vectors. Thus,

$$p^2 - \frac{E^2}{c^2} = p'^2 - \frac{E'^2}{c^2}$$

The LHS refers to the laboratory where $\boldsymbol{p} = \boldsymbol{p}_1'$ and the RHS refer to the (CM) system, where

$$\boldsymbol{p}' = \boldsymbol{p}_1' + \boldsymbol{p}_2' = 0$$

where p_i and E_i are the momenta and energies of the protons in the laboratory system prior to the reaction. Thus, in our case

$$\beta_{\text{CM}} = \frac{pc}{\sqrt{p^2c^2 + m_p^2c^4} + m_pc^2} = 0.83$$

The energy of each of the colliding protons in the CM system (the energies are equal since the momenta and masses of colliding particles are equal) is

$$E = \frac{m_pc^2}{\sqrt{1-(\beta_{\text{cm}})^2}} = \gamma m_pc^2 = 1.68 \text{ Gev}$$

The total energy in the centre of mass system that can be expended to produce new particles is

$$2E - 2m_pc^2 = 1.48 \text{ Gev}$$

since both protons (or other baryons) remain after the reaction due to the conservation of baryon number. Most pions are produced when all these particles are at rest in the CM system while nucleons remain to be baryons. Therefore, the number of pions generated

$$n = \frac{2E - 2m_pc^2}{m_\pi c^2} = \frac{1.48 \times 10^9}{140 \times 10^6} \approx 10$$

EXAMPLE 12.18

An empty box of total mass M with perfectly reflecting walls is at rest in the lab frame. The electromagnetic standing waves are introduced along the x-direction, consisting of N photons, each of frequency ν_o.

(a) State, what the rest mass of the system (box + photons) will be when the photons are present

(b) Show that the answer can be obtained by considering the momentum/or energy of the box-plus photon system in any inertial frame moving along the x-axis.

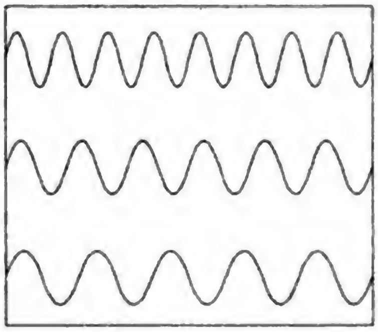


Fig. E12.18

Solution

(a) Consider the initial state of the system. Write the 4-momentum of the box and photons as p^μ_{box} and p^μ_{ph} respectively.

$$p^\mu_{\text{box}} = (M_0c, 0) \qquad (1)$$

$$p^\mu_{\text{ph}} = \left(\frac{Nh\nu_o}{c}, 0\right)$$

where we have used the fact that since a traveling wave can be represented as the sum of traveling waves with opposite momenta, the total momentum is obviously zero. Thus the 4-momentum of the system is

$$p^\mu = p^\mu_{\text{box}} + p^\mu_{\text{ph}}$$

Solution

(a) From momentum and energy conservation, we can write

$$\boldsymbol{p} = \tilde{\boldsymbol{p}} + \boldsymbol{p}_e \tag{1}$$

$$E + E_e = \tilde{E} + \tilde{E}_e \tag{2}$$

where $\boldsymbol{p}$, $\tilde{\boldsymbol{p}}$, E and $\tilde{E}$ are the momenta and energies of the photon before and after the scattering respectively. $\boldsymbol{p}_e$, $\tilde{E}_e$ are the final momentum and energy of the electron and E_e its initial energy.

We have for the electron

$$\tilde{E}_e = \sqrt{p_e^2 c^2 + m^2 c^4}, \qquad E_e = mc^2$$

and for the photon

$$E = pc \qquad \tilde{E} = \tilde{p} c$$

Rewriting the Eqs (1) and (2) as

$$\boldsymbol{p} - \tilde{\boldsymbol{p}} = \boldsymbol{p}_e \tag{3}$$

$$pc + mc^2 = \tilde{p} c + \sqrt{p_e^2 c^2 + m^2 c^4} \tag{4}$$

(b) To solve these equations, we can express the momentum of the recoil electron $\boldsymbol{p}_e$ in two ways

$$\boldsymbol{p}_e^2 = (\boldsymbol{p} - \tilde{\boldsymbol{p}})^2 \tag{5}$$

$$\boldsymbol{p}_e^2 = (p - \tilde{p})^2 + 2mc\,(p - \tilde{p}) \tag{6}$$

So $$p\tilde{p}\,(1 - \cos\theta) = mc\,(p - \tilde{p})$$

And for a special case $\theta = \dfrac{\pi}{2}$, $\cos\theta = 0$, we get

$$p\tilde{p} = mc\,(p - \tilde{p}) \tag{7}$$

Dividing the equation by $p\tilde{p}$, we get

$$1 = mc\left(\frac{1}{\tilde{p}} - \frac{1}{p}\right)$$

Putting $$p = \frac{h}{\lambda}$$

$$\lambda' - \lambda = \frac{h}{mc}$$

EXAMPLE 12.20

Mössbuar Effect An atom in its ground state has mass m. It is initially at rest in an excited state of excitation energy ΔE. It then makes a transition to the ground state by emitting one photon. Find the frequency of the photon taking into account the relativistic recoil of the atom. Express your answer also in terms of the mass M of the excited atom. Discuss the result for the case of a crystalline lattice.

Solution

Writing the energy and momentum conservation laws, we get

$$p + p_{\text{ph}} = 0 \tag{1}$$

$$mc^2 + \Delta E = \hbar\omega + \sqrt{p^2 c^2 + m^2 c^4} \tag{2}$$

where p is the momentum of the atom after emitting the photon $p_{\text{ph}} = \dfrac{\hbar\omega}{c}$ is the momentum of the photons, and ω is the photon frequency. Substituting $p = -\dfrac{\hbar\omega}{c}$ from Eq. (1) into Eq. (2), and rewriting it in the form

$$mc^2 + \Delta E - \hbar\omega = \sqrt{\hbar^2\omega^2 + m^2c^4}$$

we get
$$\omega = \frac{\Delta E}{2\hbar}\left(\frac{\Delta E + 2mc^2}{\Delta E + mc^2}\right) \tag{3}$$

Now, taking into account that

$\Delta E + mc^2 = Mc^2$, we rewrite Eq. (3) as

$$\omega = \frac{\Delta E}{\hbar}\left(1 - \frac{\Delta E}{2Mc^2}\right) \tag{4}$$

The photon frequency ω is smaller by the amount $\dfrac{\Delta E}{2Mc^2\hbar}$ than its value without relativistic effects.

In the crystalline lattice (Mössbauer effect), the atoms are strongly coupled to the lattice and have an effective mass $M_o >> M$. From Eq. (4), we see that in this case the atom practically does not absorb energy, all of which goes into the energy of the photons, and therefore, there is no frequency shift due to this effect.

12.8 RELATIVITY AND NEWTONIAN MECHANICS

The implications of the postulates of special theory of relativity enunciated by Einstein were examined qualitatively in Sec. 11.3. These related to the modifications of the dynamical concepts of length, time, nonabsoluteness of simultaneity, the dependence of the mass of a particle on its velocity and the equivalence of mass and energy through the mass-energy equation. Their mathematical derivations have been examined in Secs. 11.5, 12.1 and 12.2. This development seems to amount to the refutation of Newtonian mechanics, which is valid in the realm of small velocities.This however, is an erroneous feeling since the special theory is the correct formulation of dynamics of particles at all speeds up to and including the velocity of light. All predictions of Newtonian mechanics are contained in the relativistic theory as a special case, which is valid at sufficiently small velocities. The relationship between the relativistic and Newtonian (non-relativistic) mechanics is an illustration of a general principle, called correspondence principle. According to this principle, a new theory may supercede the old established theory in its ability to explain hitherto unexplained facts. However, the new theory should be able to successfully account for all those results in any areas in which the old theory was valid experimentally. Thus according to the correspondence principle, the relativistic theory at small velocities ($v \ll c$) should lead to the results of the Newtonian mechanics and so it actually does as shown throughout the treatment. All the behaviour of macroscopic bodies and familiar mechanical systems are the cases which are perfectly well-understood in terms of nonrelativistic mechanics.

certain respects, an approximation to the experimental fact, or of limited validity for other reasons.

There is a growing opinion that the theory of relativity (special and general) may prove to be wrong when applied to the domain of very small distances (much less than the presumed size of elementary particles). Furthermore, there are reasons to suspect that relativity may not hold when applied to extremely large distances of the order of the presumed size of the universe (out to where the red-shift becomes appreciable). The theory may break down in yet other ways. A stage may come that may warrant the replacement of the theory by a more nearly correct theory, which may be as radically different from relativity as the latter is from Newtonian mechanics. That is the process of evolution of physical thought and the theory of relativity is no exception.

QUESTIONS

12.1 Derive an expression for the dependence of relativistic mass on its velocity.

12.2 What is proper mass of a particle? Give arguments to show that a particle with finite proper mass can never attain velocity equal to that of light.

12.3 State and prove the law of equivalence of mass and energy.

12.4 The unification of conservation of energy and mass is said to be the greatest achievment of special theory of relativity. Discuss this statement citing necessary examples.

12.5 Derive an expression for relativistiv kinetic energy of a moving particle and hence show that the classical expression is a special case of this in the limit when $v/c \ll 1$.

12.6 Obtain the equations governing relativistic transformation of momentum four-vector.

12.7 Prove that the total energy E' of a particle as observed in primed frame is related to that in the unprimed frame through $E' = \gamma(E - p_x v)$ where symbols have their usual meaning.

12.8 Starting from the transformation laws tor momentum and energy, show that $p^2 - E^2/c^2$ is an invariant quantity.

12.9 Discuss the physical meaning of negative total energy.

12.10 Derive expressions for the transformation formulae obeyed by the components of force.

12.11 Prove that acceleration due to the force acting on a body moving with large velocity need not be parallel to the direction in which force is applied.

12.12 'For relativistic systems, action and reaction are generally different'. Discuss.

12.13 How does the concept of photons fit into the framework of the special theory of relativity?

12.14 Justify the statement, 'We cannot choose an inertial frame in which x-ray photons are at rest'.

12.15 It is postulated that the quanta of gravitation, gravitons travel with the speed of light. Discuss the nature of these quanta.

12.16 The emission of β-radiations by nuclei was explained by assuming that a particle with nearly zero rest mass, called antineutrino, is also emitted. Can this particle travel with a velocity equal to that of light? Justify your answer.

12.17 What are tachyons? How do these differ from normal particles?

12.18 Discuss the present limitations in lhe detection of tachyons?

12.19 'The concept of tachyons is not in contradiction with the theory of relativity.' Discuss this statement.

12.20 Define a four-vector and give three examples of these.

12.21 Demonstrate that the scalar product of two four-vectors is invariant under Lorentz transformation.

12.22 Volume element of the space-time four-vector is defined by $dx\, dy\, dz\, dt$. Show that it is invariant under Lorentz transformation.

12.23 State the correspondence principle and use it to establish the relationship between relativistic and Newtonian mechanics.

PROBLEMS

12.1 Find the approximate relationship between the energy of a slow particle and its momentum upto terms proportional to $(p^2/m^2c^2)^2$. For slow particles, $p^2 \ll m^2c^2$.

$$Ans.\ E = mc^2 + \frac{p^2}{2m} + 3/8\ \frac{p^4}{m^3c^2}$$

12.2 Determine the relationship between the frequency of a photon scattered by a stationary free electron and the scattering angle (the Compton effect).

$$Ans.\ \nu = \frac{\nu_0}{1+\dfrac{h\nu_0}{mc^2}(1-\cos\theta)}$$

12.3 A stationary atom of mass m has an excitation energy E. Find the frequency of the photon emitted when the atom is in the excited state.

$$Ans.\ \nu = \frac{\Delta E}{h}\left(1-\frac{\Delta E}{2mc^2}\right)$$

12.4 Do you expect the performance of the cyclotron lo be affected by the relativistic variation of mass? How is it eliminated in the actual design?

12.5 (a) Prove that 1 amu = 931.5MeV/c^2 (1 amu = 1.66×10^{-27} kg)

(b) Find the energy equivalent to the rest mass of the electron and to the rest mass of the proton. *Ans.* 0.51 MeV (electron); 936.2 MeV (proton)

12.6 Calculate the binding energy of a deuteron, given that the mass of a proton = 1.6725×10^{-24} g, mass of a neutron = 1.6748×10^{-24} g and mass of a deuteron = 3.3433×10^{-24} g. *Ans.* 2.25 MeV

12.7 Compute the effective mass for a photon of wavelength 5000 Å (visible region) and for a photon of wavelength 1.0 Å (x-ray region).

Ans. (a) 4.4×10^{-36} kg (b) 2.206×10^{-32} kg

12.8 Show that the following processes are dynamically impossible:

(a) A single photon strikes a stationary electron and gives up all its energy to the electron.

(b) A single photon in empty space is transformed into an electron and a positron.

(c) A fast positron and a stationary electron producing only one photon.

12.9 Show that the rest mass of a particle is given by

$$m_0 = \frac{p^2c^2 - T^2}{2Tc^2}$$

where p is its momentum and T its kinetic energy. Calculate the rest mass of a particle if its momentum is 130 MeV/c when its kinetic energy is 50 MeV.

[*Hint:*
$$E^2 = p^2c^2 + m_0^2c^4$$
$$= (T + m_0c^2)^2$$

$$= T^2 + 2m_0c^2\,T + m_0^2c^4$$

Thus $$m_0 = \frac{p^2c^2 - T^2}{2Tc^2}$$

Ans. 144 MeV/c^2 or 282.35 m_e

12.10 How much energy is made available when 1g of uranium is completely converted into energy. *Ans.* 5.618 × 10^{22} eV

12.11 The value of solar constant is 2 cal/min. Calculate the annual gain in the mass of the earth. The radius of the earth = 6.4 × 10^3 km. *Ans.* 2.52 × 10^8 kg

12.12 Calculate the relative increase in momentum with increase in energy.
[*Hint:* $E^2 = m^2c^4 + p^2c^2$
Take log of both sides and differentiate.]

Ans. $$\frac{dE}{E} = \frac{p^2}{m^2c^4 + p^2}\frac{dp}{p}$$

12.13 A high energy photon (γ-ray) strikes a proton at rest and produces a π^0 according to the reaction $\gamma + P \rightarrow P + \pi^0$. Calculate the minimum energy that the γ-ray must have for the reaction to occur. The rest mass of the proton is 938 MeV and that of π^0 is 135 MeV. *Ans.* 145 MeV

12.14 An unstable particle of mass M and momentum $\boldsymbol{p}$ decays into two particles of masses m_1 and m_2, whose momenta and total energies are $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ and E_1; E_2, respectively. If θ is the opening angle between the paths of the generated particles, show that

$$M^2c^4 = m_1^2c^4 + m_2^2c^4 + 2E_1E_2 - 2p_1p_2c^2\cos\theta$$

12.15 C^{12} nucleus consists of six protons and six neutrons held together by nuclear forces. Find the binding energy of a C^{12} nucleus. Given $M_{c^{12}}$ = 12.0000 a.m.u., m_p = 1.007825 a.m.u, m_n = 1.008665 a.m.u. *Ans.* 92.17 MeV

12.16 Deduce the minimum energy and momentum of a gamma ray photon that can cause electron-positron pair production. *Ans.* 1.02 MeV; 5.46 × 10^{-17} gm cm s^{-1}

12.17 A body is initially at rest. Fifty per cent of its rest mass is destroyed and given as kinetic energy to the other half. What is the resulting velocity? *Ans.* 0.866 c

12.18 Determine the mass and speed of an electron having kinetic energy of 100 KeV.
Ans. 1.089 × 10^{-30} kg, 1.64 × 10^8 m/s

12.19 Calculate the amount of energy released when a neutron decays into a proton and an electron. Rest mass of neutron = 1.6747 × 10^{-24} gm, rest mass of proton = 1.6724 × 10^{-24} gm, and rest mass of electron = 9.11 × 10^{-28} gm. *Ans.* 0.79 MeV

12.20 Calculate the binding energy of the deuteron from the following data: rest mass of proton = 1.67265 × 10^{-24} g, rest mass of neutron =1.67496 × 10^{-24} g, and the rest mass of deuteron = 3.34365 × 10^{-24} g. *Ans.* 2.23 MeV

13

Charged Particle Dynamics

Application of the laws of classical mechanics to the motion of charged particles in electric and magnetic fields is the basis of so many devices in electronics, accelerator technology, electron and proton microscopy, mass spectrography, plasma physics, astrophysics, physics of cosmic rays, and so on. It would suffice to say that classical mechanics is adequate for describing the motion of charged particles in regions whose dimensions are much greater than the atomic radius ($\sim 10^{-14}$ m). Furthermore, we assume that the velocities of the charged particles are far lesser than the velocity of light so that the motion can be treated non-relativistically, ignoring the relativistic corrections.

Atomic particles behave like extended rigid bodies, and not as a mass point, in their behaviour. These particles can have spin angular momentum and the force on them may depend not only on their positions or velocities but on their internal motions as well. In our treatment we continue to treat the atomic particles as geometrical points, ignoring their spin angular momentum. In addition, it is assumed that a charged particle acts as a test particle.

13.1 KINETIC ENERGY OF A CHARGED PARTICLE IN AN ELECTRIC FIELD

The force expressed by a particle of charge q in an electric field $\mathbf{E}$ is

$$\mathbf{F} = q\,\mathbf{E}$$

The amount of work done on the particle by the field in an infinitesimal displacement dl is

$$dW = \mathbf{F}.d\mathbf{l} = q\mathbf{E}.d\mathbf{l}$$

Thus, the total work done on the particle by the field in displacing the particle from point A to B is

$$W = q\int_A^B \mathbf{E}.d\mathbf{l} \tag{13.1}$$

As the electric field is a conservative force field, the work done by the field must result in the equal decrease of its potential energy.

$$qV_A - qV_B = q\int_A^B \mathbf{E}.d\mathbf{l} \tag{13.2}$$

V_A and V_B are the potentials at the points A and B. This decrease in potential energy must be equal to the increase in the kinetic energy of the particle. Therefore,

$$\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 = q\,(V_A - V_B) \tag{13.3}$$

where m is the mass of the particle; v_A amd v_B are the speeds of the particle at points A and B, respectively. However, if the particle starts from rest, then $v_A = 0$ and calling the velocity acquired in moving to point B through a potential difference $V_A - V_B = V$, then

$$\frac{1}{2}\,mv^2 = qV \tag{13.4}$$

Thus, a particle carrying a positive charge gains or loses energy accordingly as it moves from a higher to lower potential or from a lower to a higher potential. Reverse will be the case if the charge carried by the particle is negative.

In esu system, the kinetic energy is measured in ergs and in rationalized MKS (or SI) system, it is measured in Joules. More often, a convenient unit of energy used in atomic and nuclear physics is the electron volt (ev). It is the energy acquired by an electron (charge e = 4.8×10^{-10} esu) on moving through a potential difference of 1 volt (equal to $\frac{1}{300}$ = esu). Thus,

In esu system $\quad 1 \text{ ev} = 4.8 \times 10^{-10} \times \frac{1}{300} = 1.6 \times 10^{-12}$ erg

In SI system $\quad 1 \text{ ev} = 1.6 \times 10^{-19}$ coulomb $\times$ 1 volt $= 1.6 \times 10^{-19}$ joules

13.2 MOTION OF A CHARGED PARTICLE IN A CONSTANT ELECTRIC FIELD

If the intensity of the electric field is the same at all the field points, it is said to be uniform. If the field intensity at a field point does not vary with time, the field is said to be constant. Consider a particle of mass m, charge q, placed in a uniform, constant field **E**. The force acting on it is q **E** and the equation of motion becomes

$$m\frac{d^2\mathbf{r}}{dt^2} = q\mathbf{E} \tag{13.5}$$

Integrating wrt time, we get the velocity of the particle

$$\frac{d\mathbf{r}}{dt} = \frac{q\mathbf{E}}{m}\,t + C_1$$

where C_1 is the constant of integration. If at $t = 0$, $\mathbf{v} = \mathbf{u}$ we get $C_1 = \mathbf{u}$, and therefore,

$$\frac{d\mathbf{r}}{dt} = \frac{q\mathbf{E}}{m}\,t + \mathbf{u} \tag{13.6}$$

Integrating it once again wrt time, we get the displacement of the particle

$$\mathbf{r} = \frac{1}{2}\frac{q\mathbf{E}}{m}\,t^2 + \mathbf{u}t + C_2$$

where C_2 is the constant of integration. If at $t = 0$, $\mathbf{r} = \mathbf{r}_0$, we get

$$\mathbf{r} = \frac{1}{2}\frac{q\mathbf{E}}{m}\,t^2 + \mathbf{u}t + \mathbf{r}_o \tag{13.7}$$

When $\mathbf{u} = 0$ and $\mathbf{r}_o = 0$, the displacement is given by

$$\mathbf{r} = \frac{1}{2}\frac{q\mathbf{E}}{m}t^2 \tag{13.8}$$

which is the equation of a parabola. The trajectory of the charged particle in a uniform constant field is parabolic. Below we will consider two special cases of the motion.

Case I: *Longitudinal Electric Field*

It is the case when the applied electric field is along the direction of motion of the charged particle. If the direction of motion of the particle is along x-axis, the resulting displacement vector will be along **E**, and thus, the problem becomes a simple, one-dimensional case. The acceleration, velocity, and position of the particle become

$$\frac{d^2x}{dt^2} = a_x = \frac{q}{m}E_x$$

$$\frac{dx}{dt} = v_x = \frac{q}{m}E_x t + v_{ox}$$

$$x = \frac{q}{2m}E_x t^2 + v_{ox}t + x_o$$

If the particle starts from origin, that is, $x_o = 0$ and initial velocity $v_{ox} = 0$, then the corresponding expressions become

$$v_x = \frac{q}{m}E_x t$$

$$x = \frac{q}{2m}E_x t^2$$

Case II: *Transverse Electric Field*

The particle is moving along x-axis and the applied electric field is along y-axis that is,

$$\mathbf{E} = \mathbf{E}_y\,\mathbf{j}$$

The resulting acceleration will be along the y-axis and is given by

$$a_y = \frac{qE_y}{m}; \quad a_x = a_z = 0$$

A particle with velocity v_x enters the field plates along x-direction.

The transit time of the particle in the field,

$$t = \frac{a}{v_x} \tag{13.9}$$

and the transverse velocity v_y is

$$v_y = \frac{qE_y}{m}\frac{a}{v_x} \tag{13.10}$$

where a is the length of the plates.

The displacement of the particle in the y-direction is given by

$$y = \frac{1}{2}a_y t^2$$

$$= \frac{1}{2}\frac{qE_y}{m}\frac{a^2}{v_x^2} \tag{13.11}$$

The particle moves in a parabolic path as shown by Eq. (13.11). The electric field E_y is effective only between the plates over a distance a, the length of the two plates, beyond which it abruptly drops to zero and the beam will proceed, straight along the direction it emerges from the field.

Let there be placed a fluorescent screen at a distance L from the ends of the plates and the beam strikes it at this point P, which is at a distance y from Q, the point where the undeflected beam would have struck the screen (Fig. 13.1).

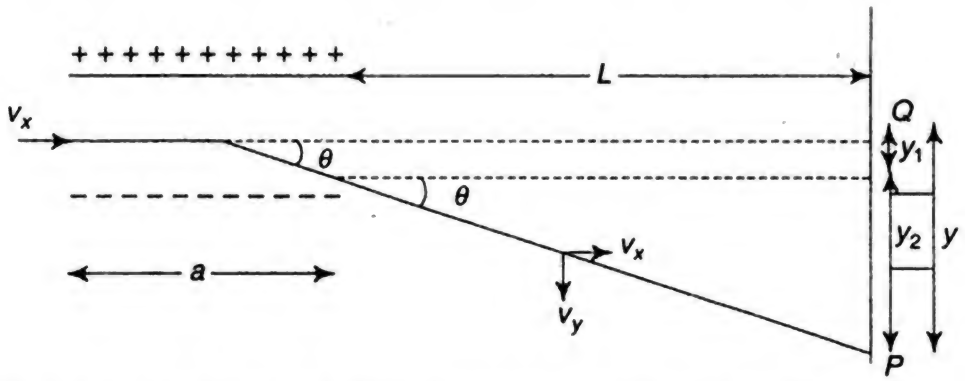


Fig. 13.1 Motion of a charged particle in a longitudinal electric field

Now $$y = y_1 + y_2 \tag{13.12}$$
where y_1 is the deflection within the plates and y_2 is the deflection in the field-free space.

Now, $$y_1 = \frac{qE_y}{2m}\frac{a^2}{v_x^2}$$

and $$y_2 = L \tan\theta$$

where $$\tan\theta = \frac{v_y}{v_x} = \frac{qE_y a}{mv_x^2} \tag{13.13}$$

therefore, $$y_2 = \frac{LqE_y a}{mv_x^2} \tag{!3.14}$$

The net displacement of the beam
$$y = y_1 + y_2$$
$$= \frac{qE_y a}{mv_x^2}\left(L + \frac{a}{2}\right) \tag{13.15}$$

The distance of the screen from the centre of the plates is $\left[L + \frac{a}{2}\right]$, a constant of the arrangement as it depends on its geometry, and calling it D, we get
$$y = D\tan\theta \tag{13.16}$$
where use has been made of Eq. (13.13).

The value of v_x is obtained from the accelerating potential applied to the beam before entering the deflecting plates. Thus,
$$\frac{1}{2}mv_x^2 = qV$$

so
$$v_x = \sqrt{\frac{2qV}{m}} \tag{13.17}$$

EXAMPLE 13.1

Find the trajectory of a particle of mass m, charge e in a uniform electric field, assuming zero velocity parallel to **E** at $t = 0$. Sketch the trajectory in the plane of motion.

Solution

The plane of motion of a particle will be defined by its initial velocity **v** and the direction of electric field **E**. Let the initial velocity coincide with the x-axis and **E** with the y-axis. The equation of motion of a charge in an electric field is

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} \tag{1}$$

where **p** is the momentum of the particle. Obviously, since there is no force in the direction perpendicular to the x-y plane, the particle will move in this plane at all later times. Rewriting Eq. (1) as

$$\frac{dp_x}{dt} = 0 \tag{2}$$

$$\frac{dp_y}{dt} = eE \tag{3}$$

Integrating Eqs (2) and (3) yields

$$p_x = p_{xo} = p_o \tag{4}$$

$$p_y = eEt \tag{5}$$

The energy E of the particle (without the potential energy due to the field) is given by

$$E = \sqrt{m^2c^4 + p^2c^2}$$

$$= \sqrt{m^2c^4 + p_o^2c^2 + c^2e^2\,E^2t^2}$$

$$= \sqrt{E_o^2 + (ecEt)^2} \tag{6}$$

where $E_o = \sqrt{m^2c^4 + p_o^2c^2}$ is the initial energy of the particle. The work done by the electric field changes the energy of the particle

$$\frac{dE}{dt} = eE\cdot\mathbf{v} = eEv_y = eE\frac{dy}{dt} \tag{7}$$

or
$$E = E_o + eEy \tag{8}$$

The Eqs (6) and(7) lead to

$$E_o + eEy = \sqrt{E_o^2 + (eEt)^2} \tag{9}$$

Therefore,
$$y = \frac{E_o}{ce}\left[\sqrt{1 + (ceEt)^2 / E_o^2} - 1\right] \tag{10}$$

and
$$t = \frac{\sqrt{(E_o + eEy)^2 - E_o^2}}{ceE} \tag{11}$$

Furthermore, $$\frac{p_y}{p_x} = \frac{\gamma m v_y}{\gamma m v_x} = \frac{v_y}{v_x} = \frac{dy/dt}{dx/dt} = \frac{dy}{dx} \tag{12}$$

Putting $p_x = p_o$ and $p_y = eEt$ into (12) and using t from (11), we get

$$\frac{dy}{dx} = \frac{eEt}{p_o} = \frac{\sqrt{(E_o + eEy)^2 - E_o^2}}{p_o c} \tag{13}$$

Integrating Eq. (13), we get

$$\frac{x}{p_o c} = \int \frac{dy}{\sqrt{(E_o + eEy)^2 - E_o^2}} = \frac{1}{eE} \cosh^{-1} \frac{eEy}{E_o} + \text{const}$$

The initial conditions $x_o = y_o = 0$ yield

$$y = \frac{E_o}{eE}\left(\cosh \frac{eEx}{p_0 c} - 1\right) \tag{14}$$

Thus, the particle in a constant electric field moves in a catenary (Fig. E13.1.) However for $v << c$, that is, a non-relativistic motion, $p_o = mv_o$, $E_o = mc^2$, and expanding $\cosh\left(\frac{eEx}{p_o c}\right)$, one gets

$$y \approx \frac{eE}{2m_o v^2} x^2 \tag{15}$$

which shows that the trajectory reduces to a parabola.

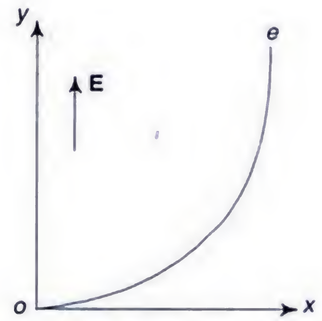


Fig. E13.1 The trajectory of the particle in non-relativistic motion

3.2.1 Cathode Ray Oscilloscope

In a cathode ray oscilloscope (Fig. 13.2), a beam of electrons accelerated to a high velocity by an accelerating electric field on the pair of anodes, is made to enter the space between two pairs of plates—the vertical deflecting plates and the horizontal deflecting plates. On applying a potential difference to these plates, a vertical and a horizontal electric field is set up between the vertical and horizontal deflecting

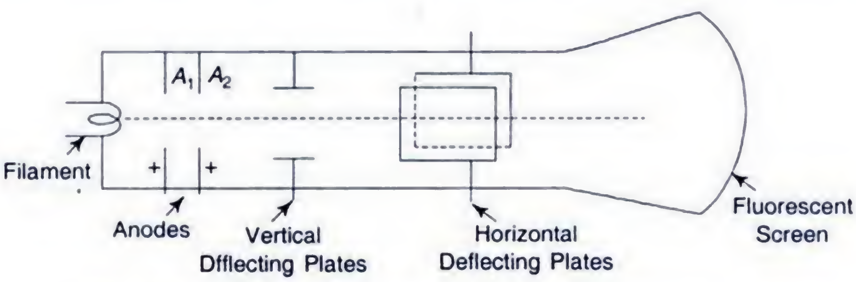


Fig. 13.2 A schematic diagram of cathode ray oscilloscope

However, if $\mathbf{v}_o = 0$ and $\mathbf{r}_o = 0$ at $t = 0$, Eq. (13.24) becomes

$$\mathbf{r}_o = \frac{q\mathbf{E}_o}{m\omega}\left(t - \frac{\sin \omega t}{\omega}\right) \tag{13.25}$$

Figure 13.3. gives the plots of acceleration, velocity and displacement as a function of ωt.

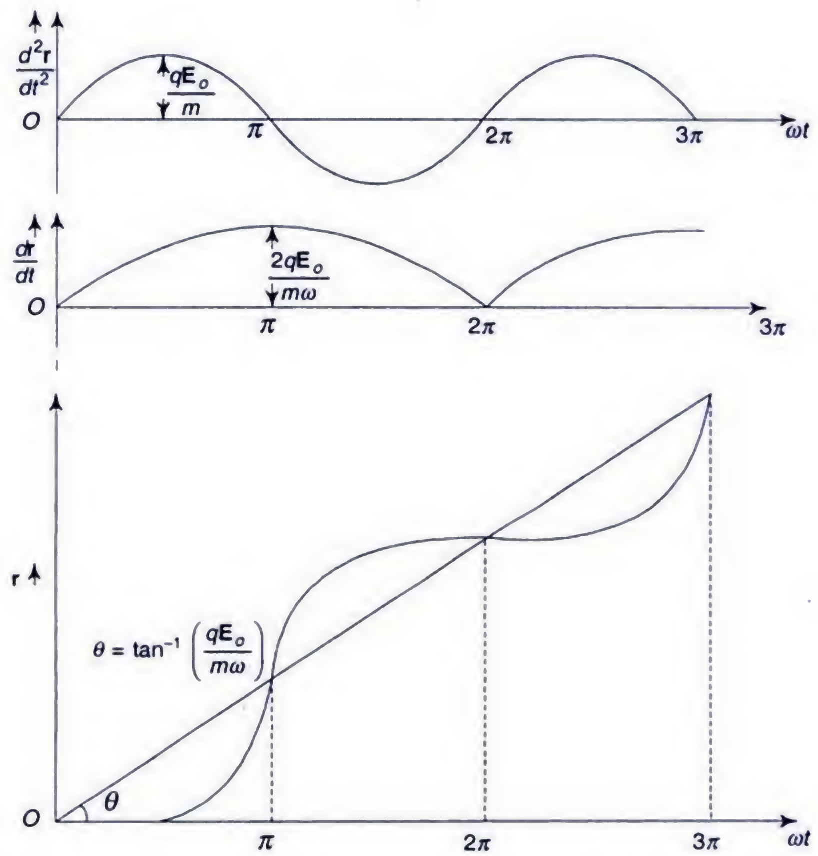


Fig. 13.3 Acceleration, velocity and displacement of a charged particle in a sinusoidal electric field

As is obvious, the velocity of the particle never reveres its sign.

The displacement is superposition of two motions Eq. (13.25), one varying linearly with time and the other varying sinusoidally with time. The slope of the linear part of displacement is $\theta = \tan^{-1}\left(\frac{qE_o}{m\omega}\right)$

EXAMPLE 13.2

A sinusoidal potential difference of peak value 300 volts and frequency 2×10^2 MHz is applied between the plane parallel cathode and plate 0.1 cm apart in a diode valve. The electrons are emitted from the cathode with almost zero velocity when

As a moving charge constitutes a current, it must experience a similar force in a magnetic field. If a charge q moves with velocity **v** making an angle θ with **B**, the distance travelled by the charge in time $\delta t = v\delta t$ and this is equal to a conductor of length $\delta l = v\delta t$. It makes an angle θ with the direction of **B** and carrying a current $I = \frac{q}{\delta t}$.

Thus, the force acting on the charge

$$Fm = \frac{q}{\delta t}\, v\delta t\, B \sin\theta$$

or $$\mathbf{F}m = q\mathbf{v} \times \mathbf{B} \qquad (13.28)$$

This force is called the Lorentz force and is directed at right angles to both **v** and **B**, Fig. 13.4. Eq. (13.28) holds if all the quantities are measured in the same system of units. If q, v, and B are measured in emu, $\mathbf{F}_m$ will be in dynes, but if measured in RMKS (or SI) system, F_m will be in Newtons.

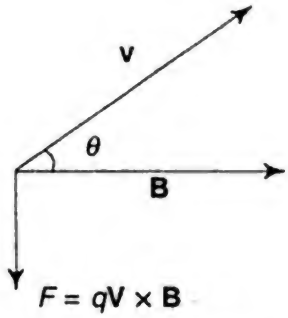


Fig. 13.4 Force on a charge in a magnetic field

Sometimes, q and B are measured in the Gaussian system of units, that is q in esu and B in emu (or gauss). Then, to convert the units of q into emu, it has to be divided by c. The Lorentz force equation in Gaussian system becomes

$$\mathbf{F}_m = \frac{q}{c}\, \mathbf{v} \times \mathbf{B} \qquad (13.29)$$

As the magnetic force $\mathbf{F}_m$ always acts in a direction perpendicular to **v**, there is no change in the magnitude of velocity but only a continual change in its direction. That is the reason why a magnetic field cannot feed energy into a system of charged particles.

In case, there is simultaneously present an electric field **E**, the total force acting on the charged particle is given by

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\mathbf{F} + \frac{q}{c}\, \mathbf{v} \times \mathbf{B} \qquad (13.30)$$

The force due to an electromagnetic field on a charged particle is also called Lorentz force.

13.5 CHARGED PARTICLE IN A UNIFORM AND CONSTANT MAGNETIC FIELD

Consider a charged particle of mass m and charge q esu, moving with velocity **v**, cm/s in a magnetic field of intensity **B**. It experiences a Lorentz force given by

$$\mathbf{F} = \mathbf{q}\, \frac{\mathbf{v} \times \mathbf{B}}{c} \qquad (13.31)$$

As no other force is acting on the particle, the equation of motion for the particle becomes

$$m\frac{d^2\mathbf{r}}{dt^2} = \frac{q}{c}\, \mathbf{v} \times \mathbf{B} \qquad (13.32)$$

Since the force exerted by the magnetic induction **B** is at right angles to the velocity **v** of the particle, the magnitude of velocity will remain unaltered and only its direction will keep on changing. Such a motion causes the charge to move in a circular path with constant speed and is always directed towards the centre of the circle (Fig. 13.5 (a)).

Therefore,
$$\left| q\frac{\mathbf{v}\times\mathbf{B}}{c} \right| = \frac{mv^2}{r}$$

or
$$r = \frac{cmv}{q\mathbf{B}} \tag{13.33}$$

If the initial velocity of the particles has an arbitrary direction it can be resolved into components along and perpendicular to the direction of **B**. The component of velocity, which is orthogonal to **B**, causes the particle to move in a circular path and the longitudinal component makes the particle move in the direction of **B** with constant speed. The superposition of these motions causes the particle to move in a helical path with its axis parallel to **B** (Fig. 13.5(b)). The radius r of the circular or the helical path described by a charged particle in a uniform magnetic field is called gyro-radius or cyclotron radius and is given by Eq. (13.33).

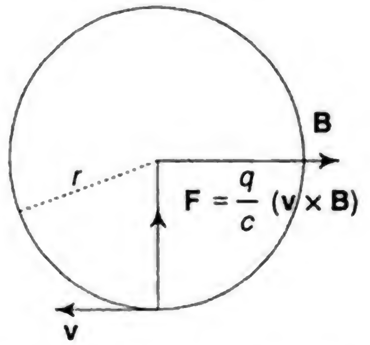


Fig. 13.5(a) A charged particle with velocity v moving in a magnetic field **B** and experiencing a centripetal force

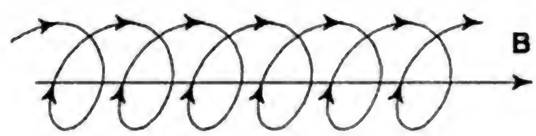


Fig. 13.5(b) Helical path of a charged particle moving in a magnetic field **B**

The frequency of motion, called gyro-frequency or cyclotron frequency, is given by

$$n = \frac{\omega}{2\pi} = \frac{v}{2\pi r} = \frac{qB}{2\pi mc} \tag{13.34}$$

As is obvious from Eq. (13.34), the gyro-frequency n is independent of the velocity (or energy) of the particle.

Alternatively, one can analyse the above problem analytically. Assuming that the magnetic field is along the x-axis, we have $\mathbf{B} = B\mathbf{i}$. The equation of motion is given by

$$m\frac{d^2\mathbf{r}}{dt^2} = \frac{q}{c}\,\mathbf{v}\times\mathbf{B}$$

or
$$m\left[\frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k}\right] = B\frac{q}{c}\,[v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}] \times B\mathbf{i}$$

Similarly, from Eq. (13.40)

$$\frac{dz}{dt} = v_2 \cos(\omega t + \phi) \tag{13.43}$$

Integrating Eqs (13.42) and (13.43), one gets

$$y = -\frac{v_2}{\omega} \cos(\omega t + \phi) \tag{13.44}$$

and
$$z = \frac{v_2}{\omega} \sin(\omega t + \phi) \tag{13.45}$$

Squaring and adding the above equations gives

$$y^2 + z^2 = \frac{v_2^2}{\omega^2} = r^2 \tag{13.46}$$

This is the equation of a circle of radius r in the y-z plane where

$$r = \frac{v_2}{\omega} = \frac{mcv_2}{qB} \tag{13.47}$$

Since the particle has constant velocity v_1 in the direction of **B**, the resultant motion of the particle will be helical (Fig. 13.5 (b)) the axis of the helix being in the direction of **B**. If the x-component of velocity is zero, the motion will be confined to a circle. The angular velocity ω and the gyro-radius r of the circular motion, being given by the equations

$$r = \frac{mcv_2}{qB} \tag{13.47}$$

$$\omega = \frac{qB}{mc} \tag{13.48}$$

the gyro-frequency is

$$n = \frac{\omega}{2\pi} = \frac{qB}{2\pi mc} \tag{13.49}$$

EXAMPLE 13.3

Find the gyro-radius and the gyrofrequency of a proton moving in a plane normal to a uniform magnetic field of 0.2 Wb/m^2 with a speed of 10^6 m/s. Mass of proton is = 1.67×10^{-27} kg; charge of proton = 1.6×10^{-19} coulomb.

Solution

The gyro-radius R is given by

$$\frac{mv^2}{R} = Bqv$$

or
$$R = \frac{mv}{Bq}$$

So, the gyro-radius

$$= \frac{1.67 \times 10^{-27} \times 10^6}{0.2 \times 1.6 \times 10^{-19}} \text{ m/s} = 52.19 \text{ mm/s}$$

The gyro-frequency is given by

$$f = \frac{1}{T} = \frac{qB}{2\pi m} = \frac{1.6 \times 10^{-19} \times 0.2}{2 \times 3.14 \times 1.67 \times 10^{-27}}$$
$$= 3.05 \times 10^6 \text{ Hz}$$

EXAMPLE 13.4

A deuteron of kinetic energy 40 keV is describing a circular orbit of radius 0.6 m in a plane perpendicular to a magnetic induction **B.** Calculate the kinetic energy of a proton that describes a circular trajectory of radius 0.8 m in the same plane with the same **B**.

Solution

Let m_d, v_d and q_d stand for the mass, velocity, and charge of the deuteron ; m_p, v_p and q_p for the mass, velocity, and charge of the proton.

The radius of circular orbit moving in a plane perpendicular to **B** is given by

$$r = \frac{mv}{qB} \tag{1}$$

Writing it for the deuteron and proton successively, we get

$$r_d = \frac{m_d v_d}{q_d B} \tag{2}$$

$$r_p = \frac{m_p v_p}{q_p B} \tag{3}$$

Dividing Eq. (2) by Eq. (3), we get

$$\frac{r_d}{r_p} = \left(\frac{m_d}{m_p}\right)\left(\frac{v_d}{v_p}\right)\left(\frac{q_p}{q_d}\right) = 2\,\frac{v_d}{v_p} \tag{4}$$

Therefore,
$$v_p = 2v_d \times \frac{r_p}{r_d} \tag{5}$$

Now, the kinetic energies (KB) of the proton and deuteron are

$$\text{KE (proton)} = \frac{1}{2} m_p v_p^2$$

$$\text{KE (deuteron)} = \frac{1}{2} m_d v_d^2$$

$$\therefore \quad \text{KE (proton)} = \frac{m_p}{m_d}\left(\frac{v_p}{v_d}\right)^2 \times \text{KE (deuteron)} = \frac{1}{2} \times 4 \left(\frac{r_p}{r_d}\right)^2 \times \text{KE (deuteron)}$$

$$= 2 \times \left(\frac{0.8}{0.6}\right)^2 \times 40 \text{ keV} = 142.22 \text{ keV}$$

13.4.1 The Cyclotron

The cyclotron is a circular accelerator where charged particles like protons, deuterons, or alpha particles are accelerated to high energies and these projectiles are used in the study of nuclear reaction mechanism. It is based on the principle of motion of a charged particle in a uniform magnetic field. When a charged particle is projected at right angles to a uniform magnetic field **B**, it describes a circular path with a time period that is independent of the particle speed or energy.

It consists of two halves of a hollow metallic cylindrical box and these halves, called dees are separated by a narrow gap (Fig. 13.6). A uniform magnetic field is

energy gain is equal to $2Nqv$. If R is the radius of the dee, one can obtain the velocity of the emergent ion from the relation

$$\frac{q}{c} Bv = \frac{mv^2}{R}$$

or
$$v = \frac{qBR}{mc} \tag{13.35}$$

The kinetic energy of the emergent ion is given by

$$\frac{1}{2} mv^2 = \frac{1}{2} \frac{q^2 B^2 R^2}{mc^2} \tag{13.36}$$

Thus, the kinetic energy gained by the particle varies as $\frac{q^2}{m}$ ratio and the strength of magnetic field B. So far the treatment has been non-relativistic under the assumption that the speed of the ion is much less than c. However, for very high energies, when the speed v is comparable to c, the relativistic corrections have to be applied as mass becomes a function of velocity and the particle starts radiating energy itself. This sets the practical limit for the acceleration of the particles.

EXAMPLE 13.5

The most energetic protons a cyclotron can produce, are N of energy 100 MeV for the maximum values of **B** and the path radius. What is the maximum energy that can be produced by the cyclotron for a beam of (a) α – particles and (b) deuterons?

Solution

If the radius of the final trajectory is R, then the magnetic rigidity of the charged particle is BR when it acquires maximum energy and its momentum is

$$|\mathbf{p}| = BRq$$

Treating the particle to be non-relativistic, its kinetic energy is given by

$$E_{kin} = \frac{p^2}{2m} = \frac{B^2 R^2 q^2}{2m}$$

Thus, the energy acquired depends on $\frac{q^2}{m}$ of the particle, the other parameters remaining the same. The maximum energy acquired by α–particles

$$= \left(\frac{q^2}{m}\right)_\alpha \Big/ \left(\frac{q^2}{m}\right)_p \times 100 \text{ MeV}$$

$$= 100 \text{ MeV}$$

Analogously, the maximum energy of the deuterons

$$= \left(\frac{q^2}{m}\right)_d \Big/ \left(\frac{q^2}{m}\right)_p \times 100 \text{ MeV}$$

$$= 50 \text{ MeV}$$

EXAMPLE 13.6

It is desired to obtain a beam of protons having a speed of $c/6$ where c is the velocity of light in a cyclotron under a magnetic field of 10^4 gauss. What is the radius of the dees of cyclotron? Treat the problem non-relativistically.

Solution

The radius of curvature of the path of the proton

$$r = \frac{mvc}{qB} \tag{1}$$

where q/m is the charge to mass ratio of the proton and **B** in gauss is the strength of magnetic induction. The velocity of the proton,

$$v = \frac{rqB}{mc} = \frac{r \times 4.8 \times 10^{-10} \times 10^4}{1.67 \times 10^{-24} \times 3 \times 10^{10}}$$

$$= 10^8 \text{ r cm/s}$$

Now, the value of, r for $v = c/6$, is

$$= \frac{3 \times 10^{10}}{6 \times 10^8} = 50 \text{ cm}$$

EXAMPLE 13.7

Calculate the frequencies of the radio-frequency oscillator in a cyclotron of 10,000 gauss field when accelerating (a) protons and (b) α–particles. Given that the radius of the dees is 50 cm, find the energy of these particles at the instant of emerging from the dees.

Solution

The frequency of radio oscillator

$$n = \frac{qB}{2\pi mc}$$

Thus, (a) n_p, the frequency for protons $= \dfrac{4.8 \times 10^{-8} \times 10^4}{2 \times 3.14 \times 1.67 \times 10^{-24} \times 3 \times 10^{10}}$

$$= 16 \text{ MHz}$$

(b) For α–particles, the q/m ratio is $\frac{1}{2}$ of that for protons, so

$$n_\alpha = \frac{n_p}{2} = 8 \text{ MHz}$$

The radius of curvature of the path of proton,

$$r = \frac{mcv}{qB}$$

so $$v = \frac{rqB}{mc}$$

Thus, the energy of the protons

$$= \frac{1}{2} mv^2$$

$$= \frac{1}{2} m \times \frac{r^2 q^2 B^2}{m^2 c^2} = \frac{1}{2} \frac{r^2 q^2 B^2}{mc^2}$$

$$= \frac{1}{2} \times \frac{(50)^2 \times (4.8 \times 10^{-10})^2 \times (10^4)^2}{1.67 \times 10^{-24} \times (3 \times 10^{10})^2}$$

$$= 19.16 \times 10^{-6} \text{ erg}$$

13.5.2 180° Magnetic Focusing and Momentum Selector

The charged particles of equal momentum, even if moving in different directions, can be brought to focus at very nearly the same point on a screen by a suitably applied magnetic field. This is called magnetic focusing. Since the particles come to a common focus after describing an angle of 180° from the point of entry into the magnetic field, it is given the name of 180° magnetic focusing.

A beam of charged ions enters through a slit S into a region where perpendicular magnetic field is acting. An ion of charge q and mass m will follow a circular path of radius r, given by

$$r = \frac{cmv}{qB} \tag{13.37}$$

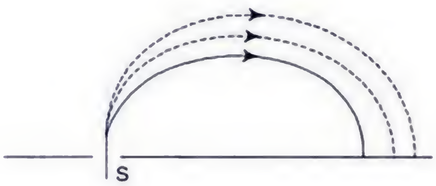
It is obvious that ions with the same charge but different momenta will have different radii of curvature and will come to focus at different points on a screen 180° apart from S. Such an arrangement is called momentum selector and finds application for separating different types of ions in a mass spectrograph (Fig. 13.7(a)).

In Fig. 13.7 (b), a conical beam of ions makes an angle θ to the normal AS to the screen. An ion moving along ASE strikes the screen at G, so that SG = $2r$. However, an ion moving along the path BSD will strike the screen at G_θ. The radius of curvature of this trajectory will also be $2r$ since all the particles have the same charge and momentum. Thus,

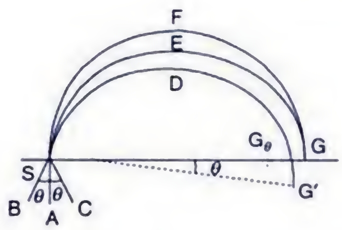
$$SG_\theta = 2r \cos\theta$$

Therefore,

$$GG_\theta = 2r - 2r \cos\theta$$
$$= 2r(1 - \cos\theta) \tag{13.38}$$



(a)



(b)

Fig. 13.7(a) A beam of ions with different momenta

Fig. 13.7(b) A conical beam of ions of same charge and momenta

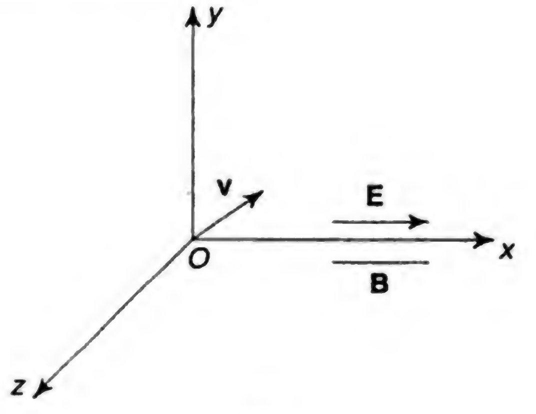
The particle along the path CSF will also strike the screen at G_θ. GG_θ represents the width of 180° magnetic focusing.

Making use of the series expansion of cos θ, we get

$$GG_\theta = 2r\left[1 - \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\cdots\right)\right]$$

$$= 2r\frac{\theta^2}{2} = r\theta^2 \tag{13.39}$$

For a small value of θ, θ^2 is very small with the consequence that the focusing width is indeed very small.



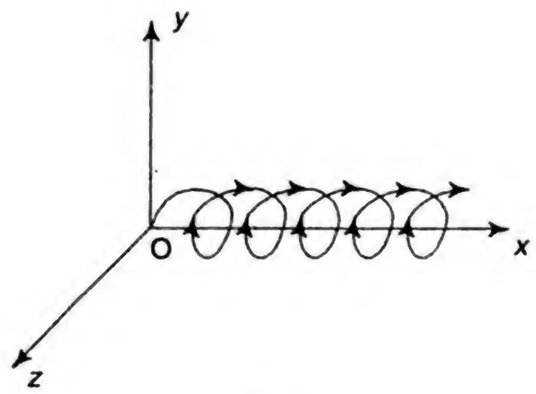


Fig. 13.8(a) Electric field $\mathbf{E} = E\mathbf{i}$ and magnetic field $\mathbf{B} = B\mathbf{i}$ applied on a particle moving with velocity $\mathbf{v}$

Fig. 13.8(b) The helical motion of a charged particle under parallel $\mathbf{E}$ and $\mathbf{B}$

The equation of motion becomes

$$m\frac{d^2\mathbf{r}}{dt^2} = q\left[\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right] \tag{13.40}$$

Rewriting its Cartesian components, one gets

$$m\frac{d^2}{dt^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = q\left[E\mathbf{i} + \frac{(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})\times(B\mathbf{i})}{c}\right] \tag{13.41}$$

Equating the coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ on both sides of Eq. (13.41), we get

$$\frac{d^2x}{dt^2} = \frac{qE}{m}$$

$$\frac{d^2y}{dt^2} = \frac{qB}{m}v_z \tag{13.42}$$

$$\frac{d^2z}{dt^2} = -\frac{qB}{m}v_y$$

These equations are identical to the Eqs (13.36) for the case of a charged particle moving in a constant and uniform magnetic field with the only difference that the particle moves with constant acceleration along the x-axis that is, the direction of applied $\mathbf{E}$ and $\mathbf{B}$. Thus the charged particle will move along a helix with its axis along the x-axis.

The projection for this motion on the y-z plane will be a circle of gyro-radius.

$$r = \frac{mc}{qB}\sqrt{v_y^2 + v_z^2} \tag{13.43}$$

and gyro-frequency $$n = \frac{qB}{2\pi mc} \tag{13.44}$$

Thomson parabolas and positive ray analysis

Let a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E}$ both be applied along the y-axis

$$t' = \frac{l'}{v}$$

and the corresponding displacement along the x-axis, calling it x_2, is

$$x_2 = a_x t' = \frac{qBl}{cm}\frac{(l')}{v} \tag{13.49}$$

The total displacement along the x-axis becomes

$$x = x_1 + x_2$$

$$= \frac{qBl^2}{2cm\,v} + \frac{qBll'}{cm\,v}$$

$$= \frac{qBl}{\text{cm}\,v}\left(\frac{l}{2} + l'\right) \tag{13.50}$$

Let us now analyse the effect of the electric field. The acceleration along the y-axis is given by

$$a_y = \frac{d^2 y}{dt^2} = \frac{qE}{m}$$

Let us call the length of the region of the electric field d and d', the distance of the screen from the end of the electric field. Analogously, the net displacement of the particle along the y-axis is given by

$$y = y_1 + y_2$$

$$= \frac{1}{2}\frac{qE}{m}\left(\frac{d}{v}\right)^2 + \frac{qEd}{mv}\left(\frac{d'}{v}\right)$$

$$= \frac{qE}{mv^2}\,d\left(\frac{d}{2} + d'\right) \tag{13.51}$$

Dividing Eq. (13.50) by Eq. (13.51), we get

$$\frac{x}{y} = \frac{vB}{c}\frac{\left(\frac{l^2}{2} + ll'\right)}{\left(\frac{d^2}{2} + dd'\right)} \tag{13.52}$$

The LHS of the above equation is a constant, provided v is constant. In that case, Eq. (13.52) represents a straight line and all the particles, irrespective of their q/m values, fall on it, which is shown by a solid line in Fig. 13.9 (a).
Eliminating v from Eqs (13.50) and (13.51), we get

$$\frac{x^2}{y} = \frac{q}{m}\frac{B^2}{c^2 E}\frac{\left(\frac{l^2}{2} + ll'\right)}{\left(\frac{d^2}{2} + dd'\right)} \tag{13.52}$$

The last factor on the LHS of the above is a constant depending upon the geometry. Denoting it by k, we write the above equation as

$$\frac{x^2}{y} = \frac{q}{m}\frac{B^2}{c^2 E}k \tag{13.53}$$

This equation represents a parabola about the y-axis. Thus all the particles of the same q/m value, irrespective of their velocities, will fall on the same parabola. However, those particles that have a different value of q/m will fall on a different parabola (Fig. 13.9 (a)) shows one branch of the parabola and the other branch (Fig. 13.9 (b)) is traced by reversing the direction of **B** so that **E** and **B** an antiparallel.

These parabolas are called Thomson parabolas as these were used by him to evaluate q/m ratio for positive rays. It is given by the equation

$$\frac{q}{m} = \frac{c^2 E}{B^2} \frac{x^2}{y} \frac{1}{k} \tag{13.54}$$

One can field the value of q/m by noting the values of x and y of any point on the parabolic trace and the factor k which depends on the geometry of the apparatus.

EXAMPLE 13.9

A beam of protons having a velocity 10^5 m/s along the x-axis enters a region of space where a combination of parallel electric and magnetic fields of 5×10^3 volts/m and 3×10^{-2}weber/m^2, respectively, act along the z-axis. A photographic plate is placed perpendicular to the beam at a distance of 0.1 m from the origin. Find the coordinates of the point where the beam will impinge the plate. Given $m_p = 1.67 \times 10^{-27}$ kg; charge on the proton = 1.6×10^{-19} coulomb.

Solution

The initial velocity of the proton that is, 10^5 m/s remains unchanged as there is no force acting in the x-direction. The time taken by the proton to reach the photographic plate, t

$$= \frac{0.1}{10^5}\text{s} = 10^{-6}\text{s}$$

The magnetic field exerts a force $q|\mathbf{v} \times \mathbf{B}| = qvB$ along the y-axis.
The displacement along y-axis

$$= \frac{1}{2} \frac{qvB}{m} t^2$$

$$= \frac{1}{2} \frac{1.6 \times 10^{-19} \times 10^5 \times 3 \times 10^{-2}}{1.67 \times 10^{-27}} \times (10^{-6})^2 \text{ m}$$

$$= 0.14\text{m}$$

The electric field exerts a force qE along the z-axis. The displacement along the z-axis

$$= \frac{1}{2} \frac{qE}{m} t^2$$

$$= \frac{1}{2} \frac{1.6 \times 10^{-19} \times 5 \times 10^3}{1.67 \times 10^{-27}} \times (10^{-6})^2 \text{ m}$$

$$= \frac{1.6 \times 5 \times 10^{-1}}{2 \times 1.67} \text{ m} = 0.24 \text{ m}$$

The coordinates of the point of impact of the proton on the photograph plate are (0.10,0.14,0.24) m.

EXAMPLE 13.10

In an experiment to determine the specific charge of positive ions by JJ Thomson method, the electric field produces deflection along the z-axis and the magnetic field produces deflection along the y-axis, the initial path of the ion being along the x-axis. Two parabolic traces are obtained on the photographic plate, for which the z-coordinates are found to be in the ratio 1:1.002 for the same value of the y-coordinates. If the charges on the ions are the same, what is the ratio of the masses of the two isotopes.

Solution

Let us take the initial direction of motion of the positive ion along the x-axis, the electric field **E** and the magnetic field **B** being applied along the z-axis.
The resulting parabola is represented by

$$\frac{y^2}{z} = \frac{q}{m}\frac{B^2}{c^2 E}k \tag{1}$$

where k is the geometry factor. All the particles having the same q/m ratio will fall on the parabola irrespective of their velocities.

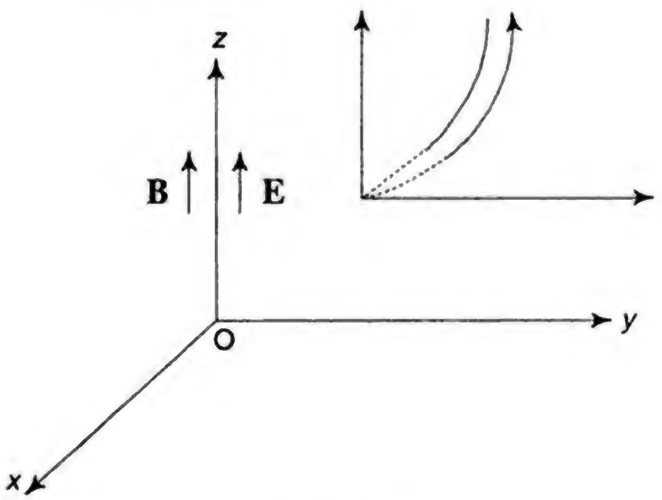


Fig. E13.2

The two parabolic traces are from Eq. (1)

$$\frac{y_1^2}{z_1} = \frac{q}{m_1}\frac{B^2}{c^2 E}k \tag{2}$$

$$\frac{y_2^2}{z_2} = \frac{q}{m_2}\frac{B^2}{c^2 E}k \tag{3}$$

If $y_1 = y_2$, we get from Eqs (2) and (3)

$$\frac{z_1}{z_2} = \frac{m_1}{m_2} \tag{4}$$

Thus the ratio of the masses of the positive ions is 1:1.002

Case II: *Electric and Magnetic fields are crossed (or mutually perpendicular)*

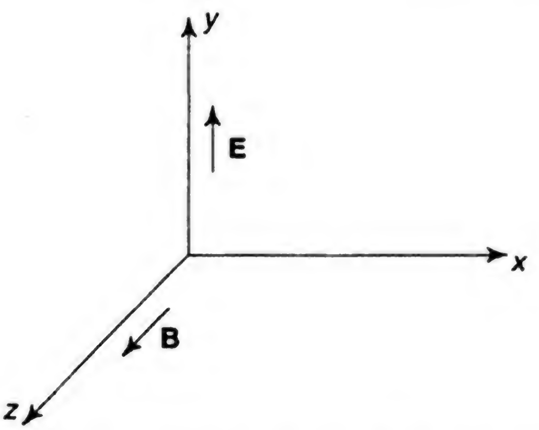


Fig. 13.10 A charged particle in combined electric field and magnetic field which are mutually perpendicular to each other

Consider a charged particle of mass m and charge e in combined electric and magnetic fields perpendicular to each other (Fig. 13.10). The electric field **E** and magnetic field **B** are

$$\mathbf{E} = E\mathbf{j}$$
$$\mathbf{B} = B\mathbf{k}$$

and let the velocity of the particle at time t be

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

The equation of motion is given by

$$m\,\frac{d^2\mathbf{r}}{dt^2} = q\left[\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right] \tag{13.55}$$

Rewriting the equation in its Cartesian components, we get

$$\frac{dv_x}{dt} = \frac{qB}{mc}v_y \qquad (1)$$
$$\frac{dv_y}{dt} = \frac{qE}{m} - \frac{qB}{mc}v_x \qquad (2) \qquad (13.56)$$
$$\frac{dv_z}{dt} = 0 \qquad (3)$$

Equation 13.56(1) shows that

$$v_z = \text{constant} \tag{13.57}$$

showing that the velocity component of the particle along the direction of the magnetic field remains constant in crossed electric and magnetic fields.

Let us deduce the values of the remaining components of velocity that is, v_x and v_y. Differentiating Eq. 13.56(2) wrt time and substituting the value of $\frac{dv_x}{dt}$ from Eq. 13.56 (1), one gets,

$$\frac{d^2v_y}{dt^2} = -\left(\frac{qB}{mc}\right)^2 v_y$$

Putting $\frac{qB}{mc} = \omega$, the above equation reduces to

$$\frac{d^2 v_y}{dt^2} + \omega^2 v_y = 0 \tag{13.58}$$

This is a simple harmonic equation for v_y and has the general solution

$$v_y = A \sin (\omega t + \phi) \tag{13.59}$$

where A and ϕ are undetermined constants

Putting the value of v_y from Eq. (13.59) into Eq. (13.56(2)), we get

$$A\omega \cos (\omega t + \phi) = \frac{qE}{m} - \frac{qB}{mc} v_x$$

or
$$v_x = \frac{cE}{B} - A \cos (\omega t + \phi) \tag{13.60}$$

Assuming that initially the particle is at rest that is, $v_x = v_y = v_z = 0$ at at $t = 0$, we evaluate the constants A and ϕ. From Eq. (13.59), we get at $t = 0$

$$0 = \sin \phi,$$

Therefore, $\phi = 0$

Again at $t = 0$, from Eq. (13.60), we get

$$0 = \frac{cE}{B} - A$$

Therefore, $A = \frac{cE}{B}$

Rewriting the equations for v_x, v_y, and v_z in view of the values of constants $A = \frac{cE}{B}$, $\phi = 0$, we have

$$\begin{aligned} v_x &= \frac{cE}{B} (1 - \cos \omega t) && \text{(i)} \\ v_y &= \frac{cE}{B} \sin \omega t && \text{(ii)} \\ v_z &= 0 && \text{(iii)} \end{aligned} \tag{13.61}$$

We obtain the values of displacement along the x-, y,- and z-axis by integrating the above equation wrt t. Thus,

$$x = \frac{cE}{B} \int (1 - \cos \omega t)\, dt$$

$$= \frac{cE}{B}\left(t - \frac{\sin \omega t}{\omega}\right) + C_1$$

where C_1 is a constant of integration.

At $t = 0$, $x = 0$, therefore $C_1 = 0$

Thus,
$$x = \frac{cE}{B}\left(t - \frac{\sin \omega t}{\omega}\right)$$

$$= \frac{cE}{B\omega} (\omega t - \sin \omega t) \tag{13.62}$$

Similarly,
$$y = \frac{cE}{B} \int \sin \omega t \, dt$$

$$= -\frac{cE}{B}\frac{\cos \omega t}{\omega} + C_2$$

At $t = 0$; $y = 0$ $$0 = -\frac{cE}{B\omega} + C_2$$

Therefore, $$C_2 = \frac{cE}{B\omega}$$

Thus, $$y = \frac{cE}{\omega B}(1 - \cos \omega t) \tag{13.63}$$

Lastly, $$z = 0 \tag{13.64}$$

Putting $\frac{cE}{\omega B} = R$ and $\omega t = \theta$, the equations for x, y and z become

$$\begin{aligned} x &= R(\theta - \sin \theta) \\ y &= R(1 - \cos \theta) \\ z &= 0 \end{aligned} \tag{13.65}$$

Equation (13.65) represents a cycloidal motion in the x-y plane. This is represented as the path traced out by a point on the path of a circular coin of radius R $\left(= \frac{cE}{\omega B}\right)$ rolling along a straight line, which is the x-axis (Fig. 13.11).

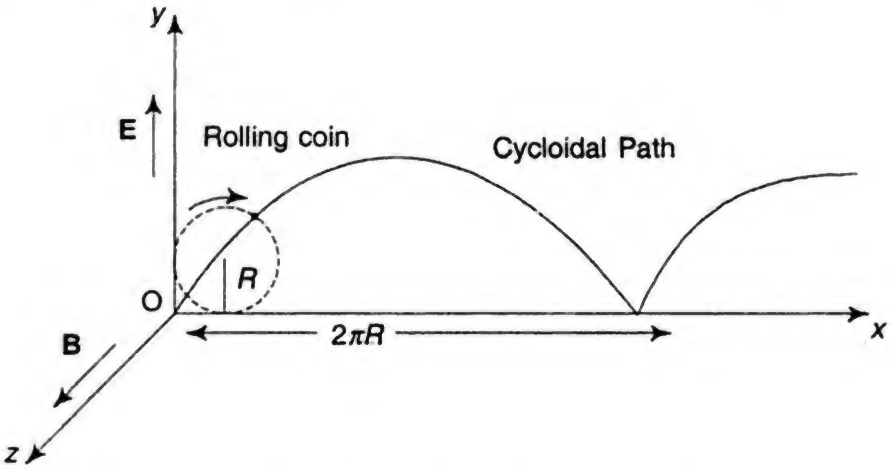


Fig. 13.11 Cycloidal path of a charged particle in crossed electric and magnetic fields

13.6.1 Velocity Selector

Consider a charged particle of charge q and mass m, moving along the x-axis so that $v_x \neq 0$, $v_y = v_z = 0$. On passing through crossed electric and magnetic fields along the y- and z-axes, respectively, the particle will experience a force along the y-axis equal to $q\mathbf{E}$ due to the electric field and magnetic force equal to $\frac{qv_x \mathbf{B}}{c}$ due to the magnetic field along the negative y-axis direction. If the initial velocity v_x of the particle is such that both these forces cancel each other, then

$$qE = \frac{qv_x B}{c}$$

or $$v_x = \frac{Ec}{B} \tag{13.66}$$

and the particle will move along the x-axis with constant velocity v_x. Such an arrangement of crossed electric and magnetic fields constitutes a velocity selector or velocity filter since only particles with velocity $v_x = \frac{cE}{B}$ will pass through and all others will be deflected sideways.

The schematic diagram of a velocity selector is shown in Fig. 13.12 below:

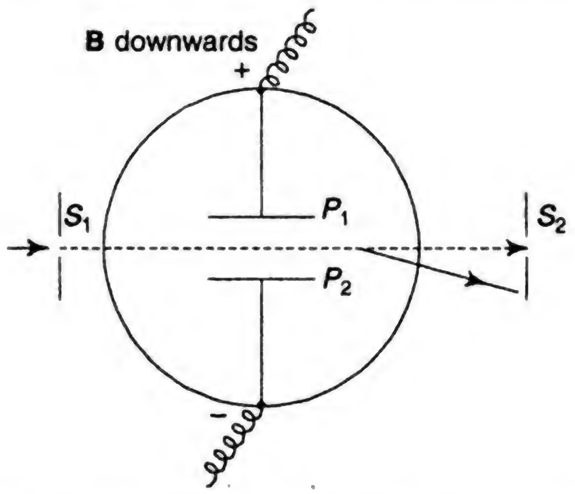


Fig. 13.12 A velocity selector for charged particles constituted by crossed **E** and **B**

A beam of ions enters through a slit S_1 and passes through a velocity selector constituted by crossed **E** and **B**. The electric field **E** is produced by applying a potential difference across the plates P_1 and P_2, and the magnetic field by an electromagnet in such a way that it is directed downwards and perpendicular to **E**. Only those ions that satisfy the relation (13.66) will pass undeflected through the slit S_2, while others will be deflected sideways.

EXAMPLE 13.11

A positive ion beam moving in the x-direction enters a region in which there is an electric field E_y = 6000 volts/cm and magnetic field B_z = 300 gauss. Deduce the speed of those ions that may pass undeflected through the region. What will happen to the ions that are(a) faster and (b) slower than these?

Solution

The crossed electric field along, say, the y-axis and the magnetic field along the z-axis constitute a velocity selector and an ion passing at right angles to the plane spanning the crossed fields will pass undeflected, provided its velocity is given by the relation

$$v_x = \frac{E_y}{B_z}$$

However, if E_y is expressed in statvolts/cm and B_z in gauss, that is, Gaussian units, then the above result becomes

$$v = \frac{cE}{B}$$

where c is the velocity of light in vacuum.

Now $\quad E_y = \dfrac{6000}{300} = 20$ statvolts/s

And $\quad B_z = 300$ gauss

so $\quad v_x = \dfrac{3 \times 10^{10} \times 20}{300} = 2 \times 10^9$ cm/s

The force exerted by E_y along the y-axis counterbalances the force exerted by E_z along the y-axis.

(i) If the velocity of the positive ion is greater than 2×10^9 cm/s, the velocity dependent magnetic force in the y- direction will increase and exceed that due to the electric field along the + y-direction, with the consequence that the positive ion beam will experience a resultant deflection in the – y-direction.

(ii) If the velocity of the positive ion is smaller than 2×10^9 cm/s, the deflection caused by the electric field will remain unchanged but the deflection caused by the magnetic field will decrease with the result that the beam will be deflected in the + y-direction.

EXAMPLE 13.12

Calculate the velocity of a stream of (a) protons and (b) α–particles in order that it may remain undeflected when passing through crossed electric and magnetic fields, $E = 600$ volts/cm and $B = 600$ gauss. Will these velocities be different?

Solution

When a beam of charged particles passes through crossed electric and magnetic fields constituting a velocity selector, it will remain undeflected if the force qE acting on the particle due to the electric field E along + y-axis and qv_xB along the – y-axis due to the magnetic field balance each other, so that

$$v_x = \frac{E}{B}$$

As it does not involve either the charge or mass, the velocity is the same for protons or the α-particles. In Gaussian units

$$v_x = \frac{cE_y}{B_z}$$

Now, $\quad E_y = \dfrac{600}{300} = 2$ statvolt/cm

$\quad B_z = 600$ gauss

Thus, $\quad v_x = \dfrac{3 \times 10^{10} \times 2}{600} = 10^8$ cm/s

QUESTIONS

13.1 What is the shape of trajectory of a positively charged particle that is projected along a uniform electric field and a parallel uniform magnetic field?

13.2 A charged particle is moving in a circular path in a uniform magnetic field. Show that its period is independent of the radius of the path. Further, show that its speed is proportional to the radius of the path.

13.3 Two particles of equal mass and charge are projected normally into a uniform magnetic field with the same velocity. If the charge of the particles is of opposite sign, how do their motions differ?

13.4 Why does the speed of a charged particle moving in a uniform magnetic field remain constant?

13.5 Show that the path of a charged particle in a uniform magnetic field is, in general, a helix. Under what conditions, is it reduced to a circle.

13.6 Show that the cyclotron frequency of a given kind of ion is independent of the energy. Is there any limiting condition?

13.7 Discuss the principle of 180° magnetic focusing.

13.8 Explain the principle of a velocity selector for charged particles, using crossed electric and magnetic fields.

13.9 Show that gain in kinetic energy of a charged particle in an electric field is equal to qV, where V is the potential difference between the initial and final positions.

13.10 What are Thomson's parabolas? How are they used to determine the q/m ratio for positive rays?

13.11 Calculate in terms of m, B, e, and ρ the kinetic energy of a particle of charge e and mass m, moving in a circle of radius ρ inside a cyclotron dee, subjected to a magnetic induction $\mathbf{B}$.

13.12 Show that the path of a charged particle moving with a uniform initial velocity in a constant transverse electric field is a parabola. Find out an expression for the direction of emergence of the particle from the field with the initial direction of motion.

PROBLEMS

13.1 A uniform electric field of magnitude 8×10^4 volts/m is directed along the x-axis and uniform magnetic induction of magnitude 0.06T is along the y-axis. What must be the speed of the electron that can be projected along the z-axis and pass through these crossed fields without getting deviated. *Ans.* (1.33×10^6 m/s)

13.2 Show that $\dfrac{E}{B}$ has dimensions of velocity.

13.3 Show that no charged particle can pass undeviated through crossed electric and magnetic fields if the intensity of the electric field in statvolts/cm is numerically greater than the strength of magnetic field measured in gauss.

13.4 Two isotopes of potassium(K^+) have masses 39 and 40 atomic mass units respectively and same kinetic energy in a mass spectrograph. The isotope with mass 39 moves along a circular path of radius 50 cm. Find out the separation of the two in the focal plane at 180° phase. What is the maximum permissible spread θ on each side of the initial ion beam so that two focal lines do not intersect? *Ans.* (1. 28 cm; 9.1°)

13.5 Two beams of Uranium isotopes U^{235} and U^{238} are focused by 180° deflection. Beams of U^{238} has a radius of 150 cm in a field of 10,000 gauss. Find the separation of the beams at the focus if the energies are equal. *Ans.* (1.26 cm)

13.6 A beam of protons with velocity 2.5×10^7 cm/s is allowed to pass through the space between two parallel plates 2 mm apart and a potential difference of 500 volts is applied across the plates. Find $\mathbf{B}$ applied at right angles both to the direction of the beam and $\mathbf{E}$, which is required to be applied so that the beam passes through undeflected. *Ans.* (10^4 gauss)

13.7 Calculate the value of the electric field which will give an alpha particle acceleration equal to the acceleration due to gravity. Mass of alpha particle is four times that of a proton and its charge twice to that of a proton. *Ans.* (2.05×10^{-9} volts/cm)

13.8 A cathode ray oscilloscope has deflecting plates of length 2.0 cm and separation 0.50 cm. Calculate the potential difference in volts between the plates which will cause angular deviation of 0.04 radians in an electron beam of speed 8.0×10^8 cm/s.

$\frac{e}{m} = 5 \times 10^7 \frac{\text{esu}}{\text{gm}}$ *Ans.* (3.84 volts)

13.9 A cyclotron dee has a diameter of 0.7 m. Calculate the maximum energy of protons that can be confined to the dees with a magnetic induction of 0.3 T

Ans. (0.527 MeV)

13.10 An electron of velocity $v = (3\mathbf{i} + 4\mathbf{j})10^8$ cm/s enters a region of uniform magnetic field $\mathbf{B} = 600i$ gauss so that its path becomes helical, Then

(i) In what direction does the helix axis lie?

(ii) Calculate the radius of the helix.

(iii) Calculate the number of revolutions the electron performs as it advances 20 cm along the axis of the helix.

Mass of electron = 9.1×10^{-28} gm; charge on the electron = 4.8×10^{-10} esu

Ans. [(i) x -axis; (ii) .038 cm; (iii) 93.45 revolutions/s]

13.11 A 100 eV electron is circulating in a plane at right angles to a uniform magnetic field of 10.00 gauss. Calculate

(i) radius of orbit

(ii) time period of revolution, and

(iii) direction of rotation as viewed along the magnetic field.

Ans. [(i) 33.72 cm; (ii) 3.57×10^{-8} s; (iii) clockwise]

14

Lagrangian and Hamiltonian Formalism

14.1 INTRODUCTION

Till now we have used the Newtonian mechanics from which the development of the motion can be followed. The solution of Keplers' equations is a very good example of mechanics based on second law of Newton, that is,

$$\mathbf{F} = m \frac{d^2\mathbf{r}}{dt^2} \tag{14.1}$$

It involves vectors and also requires the solution of second order differential equation. In the case of many particles, one has to write

$$\mathbf{F}_i = m_i \frac{d^2\mathbf{r}_i}{dt^2} = -\nabla_j \sum_j V_{ij} \tag{14.2}$$

where V_{ij} is the potential between its ith and jth particle. This leads to coupled equations. Further, many problems, say particles (atoms or molecules) in a solid, have constraints on them, which keep them in place. Such constraints lead to equation of restraint, say

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - C_{ij}^2 = 0 \tag{14.3}$$

where C_{ij}^2 may be constant; this relates $\mathbf{r}_i$ and $\mathbf{r}_j$ and they are not independent. This leads to more equations, which have to be simultaneously solved. Of course, there may be problems of constraint that are even, in principle, unsolvable. In other words, if we use Newton's equation of motion, it leads to these difficult situations.

This has been tackled by using generalised coordinates in such a way that the forces of constraint disappear.

14.2 VARIOUS COORDINATE SYSTEMS

Before the concept of generalised coordinate system was introduced and used for D'Alembert's principle and Lagrangian equations, the coordinate system used in

Newtonian mechanics were:

(i) rectangular cartesian coordinate system
(ii) spherical coordinates
(iii) cylindrical coordinates

Examples are

(i) Cartesian coordinates

As shown in Fig. (14.1), Cartesian coordinates are determined by rectangular coordinate system x, y, and z, perpendicular to each other. For a given point, at distance from the centre of coordinates in say x_1–y_1 plane,

$$x_1 = x_o + x_2$$
$$y_1 = y_o + y_2 \tag{14.4a}$$

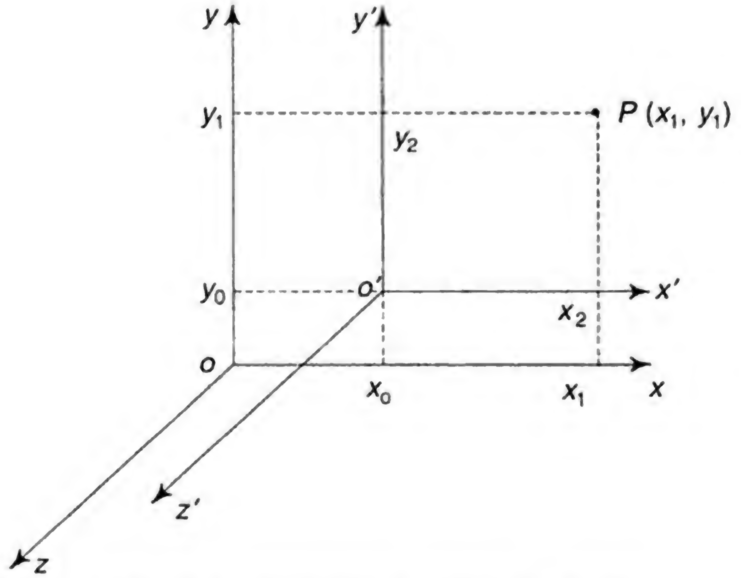


Fig. 14.1 Rectangular coordinate system

where x_o, y_o are coordinates of the centre of coordinate system of x_2, y_2, if the two frames of reference are parallel to each other. In general, if two coordinate systems are inclined to each other at an angle θ, then it can be shown that

$$x_1 = x_\theta + x_2 \cos\theta - y_2 \sin\theta$$
$$y_1 = y_\theta - x_2 \sin\theta + y_2 \cos\theta \tag{14.4b}$$

that is,

$$x_1 = x_1 (x_o, x_2, y_2)$$
$$y_1 = y_1 (x_o, x_2, y_2) \tag{14.5}$$

for fixed two-dimensional coordinate system. In a three-dimensional case, it can be similarly shown that for a moving three-dimensional system, one can write

$$x_1 = x_1 (x_2, y_2, z_2, t) \tag{14.6}$$
$$y_1 = y_1 (x_2, y_2, z_2, t) \tag{14.7}$$
$$z_1 = z_1 (x_2, y_2, z_2, t) \tag{14.8}$$

where $\theta = \omega t$ and ω is angular velocity.

(ii) Spherical coordinates

From Fig. 14.2(a), the relationship between (x,y,z) coordinate system and the spherical coordinate system (r, θ, φ) is given by

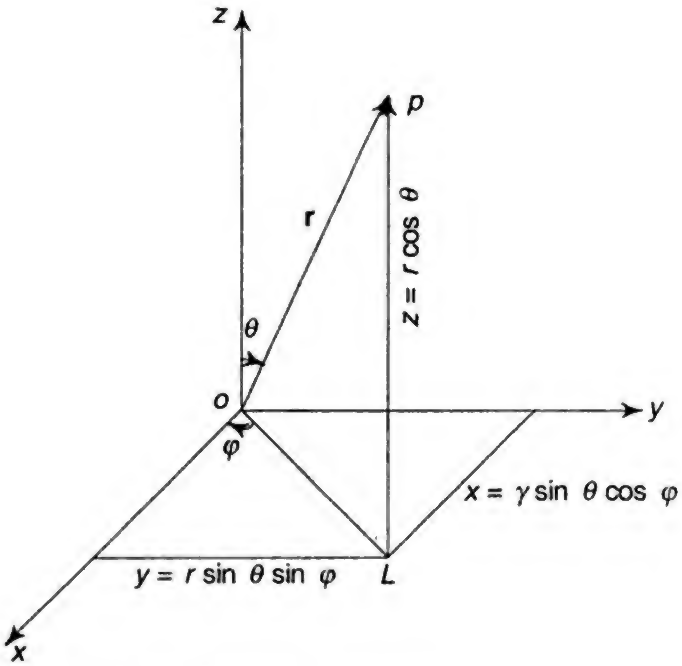


Fig. 14.2(a) Spherical coordinate system $\mathbf{r} = \mathbf{r}\ (r,\ \theta,\ \varphi,\ t)$

$$\begin{aligned} x &= r \sin \theta \cos \varphi \\ y &= r \sin \theta \sin \varphi \\ z &= r \cos \theta \end{aligned} \tag{14.9}$$

These equations again can be expressed for a moving system

$$\begin{aligned} \mathbf{r} &= \mathbf{r}\ (x,\ y,\ z,\ t) \\ &= \mathbf{r}\ (r,\ \theta,\ \varphi,\ t) \end{aligned} \tag{14.10}$$

(iii) Cylindrical coordinates

Here, as shown in Fig. 14.2 (b),

$$x = \rho \cos\phi;\ y = \rho \sin\phi;\ z = z$$

so that $$\mathbf{r} = \mathbf{r}\ (z,\ \rho,\ \phi,\ t) \tag{14.11}$$

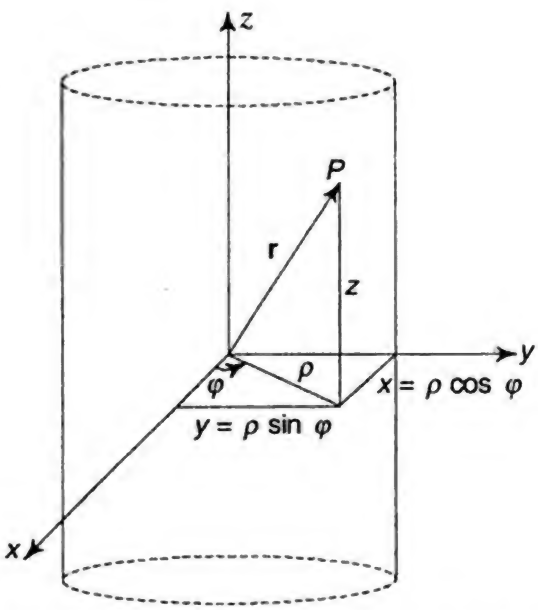


Fig. 14.2(b) Cylindrical coordinate system $\mathbf{r} = \mathbf{r}\ (z,\ \rho,\ \varphi,\ t)$

for a cylindrical moving system. In all the three equations, that is, (14.6) to (14.11), it can be seen that, in general, one can write for a moving system

$$\mathbf{r} = \mathbf{r}\,(q_1, q_2, q_3, t) \tag{14.12}$$

This is a general form of the dependence of r on general coordinates $q_1, q_2, q_3, \ldots$, which vary from one coordinate system to another.

14.3 CONSTRAINTS: HOLONOMIC AND NON-HOLONOMIC

An advantage of generalised coordinates is that they can be used where there are constraints, so that they absorb the constraints in their own expressions. It is important to understand the nature of constraints. There are two types of constraints:
(i) Holonomic
(ii) Non-Holonomic

Holonomic constraints are those, which can be absorbed in generalised coordinates, so that they can be expressed in the form of the equations

$$f\,(r_1, r_2, ---, t) = 0 \tag{14.13}$$

For example, for particles in a rigid body, one can write, like Eq. (14.3).

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - C_{ij}^2 = 0$$

It is an exact equation and in principle, can be solved.

On the other hand, non-holonomic constraints are those, which can be written in the form of inequality. As for example, a particle placed on the surface of a sphere. The particle rolls down the sphere, and therefore, satisfies the inequality

$$r^2 - R^2 \geq 0 \tag{14.14}$$

where R is the radius of the sphere and r is the distance of the particle from the centre of the sphere. After the particle leaves the surface of the sphere, its distance is greater than R, and hence, the inequality.

Another example of a non-holonomic constraint is the rolling of a disc on a rough surface. As a matter of fact, where there is friction, the constraints are non-holonomic and cannot be absorbed in an equation of the type Eq. (14.13). Generalised coordinates are useful only for holonomic constraints.

14.4 GENERALISED COORDINATES

Problems involving holonomic constraints can always find, at least formally, a solution in principle. So, it is almost invariably assumed, that any constraints, if present, are holonomic. Such is specially the case, for microscopic problems, involving molecules, atoms or smaller particles. Then constraints are used only as a mathematical idealisation to the actual physical case. Such constraints are always holonomic and fit smoothly into the framework of theory, involving generalised coordinates.

The use of generalised coordinates as expressed in Eqs (14.12) and (14.13) is such that, say, in a system of N particles free from constraints, and hence, having $3N$ independent coordinates or degrees of freedom, suppose there are holonomic restraints in k equations in the form of Eq. (14.13). Then, we may use these equations to eliminate k of $3N$ coordinates so that we are left with $3N$-k independent degrees

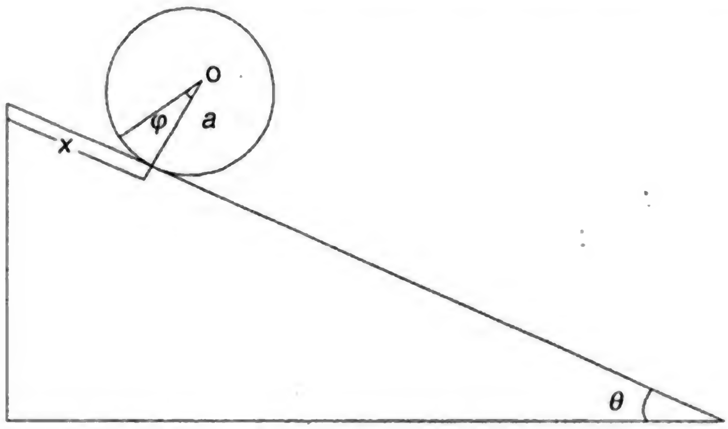


Fig. E14.1 A cylinder rolling down an inclined plane

$$T = \frac{m\dot{x}^2}{2} + \frac{ma^2}{4}\dot{\varphi}^2 = \frac{3m\dot{x}^2}{4} \tag{2}$$

as $a\dot{\phi} = \dot{x}$.

If l is the length of the inclined plane, then

$$V = mga + mg\,(\,l - x)\sin\theta \tag{3}$$

Then, $$L = T - V = \frac{3m\dot{x}^2}{4} - mg\sin\theta\,(l - x) - mga \tag{4}$$

Then, Lagrangian equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

or $$\frac{3m\ddot{x}}{2} - mg\sin\theta = 0 \tag{5}$$

This is the equation of motion.

EXAMPLE 14.2

A bead slides on a smooth rod, which is rotating about an end in a vertical plane with uniform angular velocity ω. Show that $\ddot{r} = \omega^2 r - g\sin\omega t$

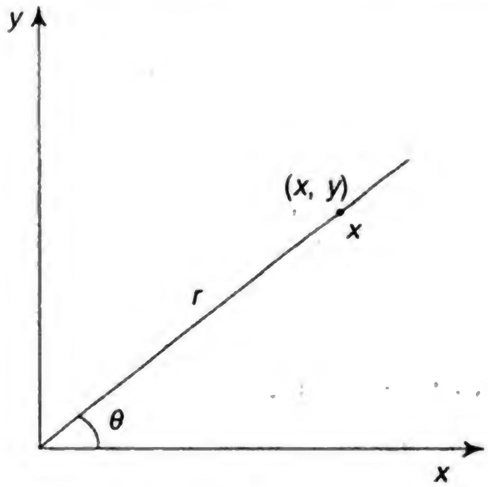


Fig. E14.2 A bead sliding on a rotating rod

Solution

We can write $\omega = \dot{\theta}$

or $\omega t = \theta$

and $x = r\cos\theta$, $y = r\sin\theta$

Then, $$T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = mgr\sin\theta$$

Therefore, Lagrangian is given by

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\sin\theta$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\sin\theta$$

and $\partial L/\partial \dot{r} = m\dot{r}$

Hence, with r as the generalized coordinate, Lagrangian equation becomes.

$$m\ddot{r} = mr\dot{\theta}^2 - mg\sin\theta$$

or $$\ddot{r} = r\dot{\theta}^2 - g\sin\theta$$

EXAMPLE 14.3

Construct a Lagrangian, and hence, equation of motion of a simple pendulum placed in a uniform gravitational field.

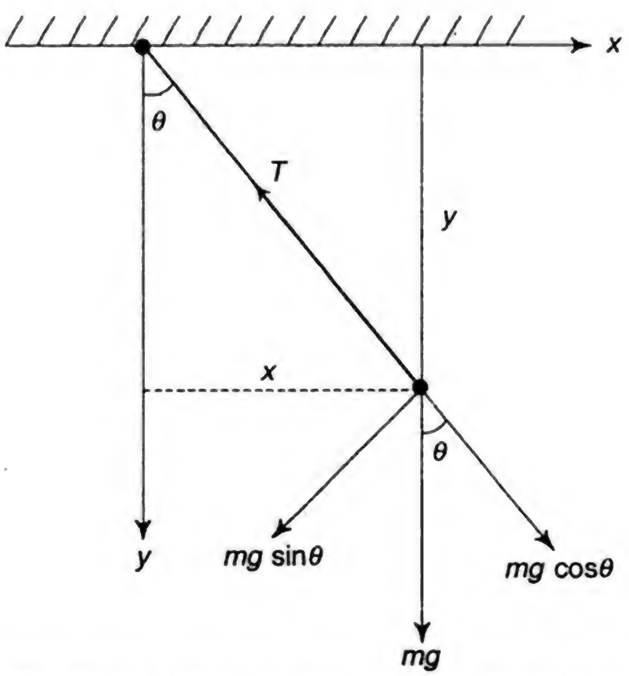


Fig. E14.3 Forces acting on a simple pendulum

Solution

A simple pendulum with a weightless rod of length l, has a mass m at its end. In swinging, the simple pendulum traverses an arc in a vertical plan. Because the pendulum has only one degree of freedom, that is, the angular displacement θ, this is the generalised coordinate that we select. We do not have any forces of constraint

in the equation of motion. Then, we can write x and y coordinates, in terms of the generalised coordinate θ as

$$x = l\sin\theta;\ y = l\cos\theta$$

The kinetic energy T of the mass is

$$T = \frac{1}{2}\,m\,(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}\,ml\dot{\theta}^2$$

and potential energy V is given by

$$V = mgy = -\,mgl\cos\theta$$

Then Lagrangian L of the simple pendulum is

$$L = \frac{1}{2}\,ml^2\dot{\theta}^2 + mgl\cos\theta$$

The Lagrangian equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

where $$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta},\ \frac{\partial L}{\partial \theta} = -\,mgl\sin\theta$$

So that the equation of motion is

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

For small oscillation ($\theta \ll 1$), we use $\sin\theta \simeq \theta$, and the equation of motion becomes

$$\ddot{\theta} + \omega^2\theta = 0$$

where $\omega = \sqrt{g/l} = 2\pi\nu$ is the angular frequency as expected in a simple harmonic motion.

EXAMPLE 14.4

In an Atwood's machine, the pully is frictionless, so the system is holonomic. Write down the equation of motion using Lagrangian formalism.

Solution

The problem has one independent coordinate x. So, one can write the expressions for kinetic energy T and potential energy V as

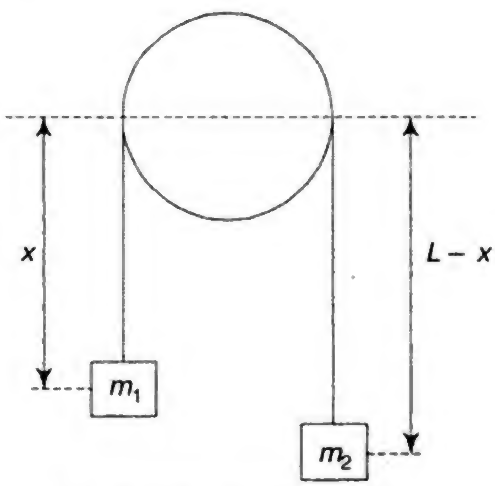


Fig. E14.4 Atwood machine

$$T = \frac{1}{2}(m_1 + m_2)\,\dot{x}^2$$

$$V = -m_1 g x - m_2 g\,(l - x)$$

Hence, the Lagrangian L is given by

$$L = (m_1 + m_2)\,\dot{x}^2/2 + m_1 g x + m_2 g\,(l - x)$$

The Lagrangian equation is then given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\,\dot{x}$$

and $$\frac{\partial L}{\partial x} = m_1 g - m_2 g$$

Hence, $$(m_1 + m_2)\,\ddot{x} - m_1 g + m_2 g = 0$$

or $$\ddot{x} = \frac{(m_2 - m_1)g}{m_1 + m_2}$$

14.7 HAMILTON'S CANONICAL EQUATIONS

The development of Lagrangian formalism was followed, both historically and logically, by Hamiltonian formalism. It is only an alternate statement of the mechanics of the motions of the particle and no new physics is added, which is basically Newtonian. But as is now well-known. Hamilton's equation gives another and more powerful tool for working the physical principles already established. Though we will not discuss Hamilton-Jacobi equation here; that this formalism laid the foundation for the formalism in quantum mechanics through Hamilton-Jacobi equation. Here, we only give the elementary discussion of Hamilton's canonical equations.

Lagrangian equation for conservative system

A conservative Lagrangian system is such, wherein Lagrangian does not contain time explicitly. Then,

$$L \equiv L\,(q, \dot{q}) \tag{14.41}$$

Hence, $$\frac{dL}{dt} = \sum_j \left(\frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial t} + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{dt}\right) \tag{14.42}$$

For a conservative system, the Lagrangian equation has the form

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \tag{14.43}$$

Using Eqs (14.42) and (14.43), we obtain

$$\frac{dL}{dt} = \sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \left(\frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt}\right)\right] = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) \tag{14.44}$$

or

$$\frac{d}{dt}\left(L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = 0 \tag{14.45}$$

This mean $L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}$ is constant in time and we denote it as H, H being called

the Hamiltonian, that is,

$$H \equiv \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \tag{14.46}$$

It can be proved that

$$\frac{\partial L}{\partial \dot{q}_j} = p_j \tag{14.47}$$

This can be seen easily; if we take a Cartesian coordinate system with x, y, z as general coordinates. Then, for x-component we can write $\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i}$ for conservative cases. Then,

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i}\sum_i \frac{1}{2} m_i\,[\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2] = m_i\dot{x}_i = p_{xi} \tag{14.48}$$

Similarly, one can write for any generalised coordinate. We can then write for a general case

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{14.49}$$

From Eqs (14.46) and (14.49) we can write

$$H = \sum_j \dot{q}_j p_j - L \tag{14.50}$$

Then $\frac{dH}{dt} = 0$; hence, H is a constant of motion.

As for a conservative case, V is independent of velocity. Therefore,

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} \tag{14.51}$$

Hence, the first term in Eq. (14.46) can be written as

$$\sum_j \dot{q}_j p_j = \sum_j \dot{q}_j \left(\frac{\partial T}{\partial \dot{q}_j}\right) \tag{14.52}$$

Now, one can write T in generalised coordinates if $\mathbf{r}_i$ does not explicitly depend on time.

$$T = \sum_{jk} a_k a_j \dot{q}_k \dot{q}_j = \left(\sum_m a_m \dot{q}_m\right)^2 \tag{14.53}$$

or
$$\frac{\partial T}{\partial \dot{q}_n} = 2a_n\dot{q}_n \text{ (for } m = n) \tag{14.54}$$

Therefore,
$$\sum_j \dot{q}_j \left(\frac{\partial T}{\partial \dot{q}_j}\right) = 2\left(\sum_j a_j \dot{q}_j\right)^2 = 2T \tag{14.55}$$

Hence,
$$H = 2T - L = 2T - T + V = T + V \tag{14.56}$$

Therefore, H is the total energy of the system and is called Hamiltonian. Because $\frac{dH}{dt} = 0$, the total energy of the system is conserved. However, for a general case, when H and L may explicitly depend upon time, we can write

$$H = (p, q, t) = \sum_i \dot{q}_i p_i - L(q, \dot{q}, t) \tag{14.57}$$

As H is a function of (p, q, t), we can write,

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i}\right) dq_i + \sum_i \left(\frac{\partial H}{\partial p_i}\right) dp_i + \left(\frac{\partial H}{\partial t}\right) dt \tag{14.58}$$

Then, using Eq. (14.57), we can write

$$dH = \sum_i \dot{q}_i dp_i + \sum_i p_i d\dot{q}_i - \sum_i \left(\frac{\partial L}{\partial q_i}\right) dq_i - \sum_i \left(\frac{\partial L}{\partial \dot{q}_i}\right) d\dot{q}_i - \left(\frac{\partial L}{\partial t}\right) dt \tag{14.59}$$

Noting that $p_i = \frac{\partial L}{\partial \dot{q}_i}$ we can write Eq. (14.59) as

$$dH = \sum \dot{q}_i dp_i - \sum_i \left(\frac{\partial L}{\partial q_i}\right) dq_i - \left(\frac{\partial L}{\partial t}\right) dt \tag{14.60}$$

Also,
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_i}\right) = \frac{\partial L}{\partial q_i} = \dot{p}_i \tag{14.61}$$

Thus, Eq. (14.60) becomes

$$dH = \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i - \left(\frac{\partial L}{\partial t}\right) dt \tag{14.62}$$

Comparing Eqs (14.58) and Eq. (14.62) we can write

$$\frac{\partial H}{\partial q_i} = -q_i; \frac{\partial H}{\partial p_i} = \dot{q}_i$$

and
$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{14.63}$$

These are called canonical equations of Hamilton.

EXAMPLE 14.5

Let a projectile of mass m be projected upward. Find its equation of motion using Hamilton's canonical equations.

Solution

Taking the vertical as z-axis, we can write

$$L = \frac{1}{2}\, m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \tag{1}$$

Hence,
$$H = (p_x^2 + p_y^2 + p_z^2)/2m + mgz \tag{2}$$

Using Hamilton's Canonical equations (14.63)

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x/m;$$

$$\dot{y} = dH/\partial p_y = p_y/m;\ \dot{z} = \frac{\partial H}{\partial p_z}\ p_z/m \tag{3}$$

$$-\dot{p}_x = \frac{\partial H}{\partial x} = 0; -\dot{p}_y = \frac{\partial H}{\partial y} = 0$$

$$-\dot{p}_z = \frac{\partial H}{\partial z} = mg \tag{4}$$

Differentiating Eq. (3) with respect to time, and eliminating, p_x, p_y and p_z using Eqs (3) and (4)we can write

$$m\ddot{x} = 0,\ m\ddot{y} = 0,\ m\ddot{z} = -\ mg \tag{5}$$

This is the equation of motion.

EXAMPLE 14.6

Find the equation of motion of a particle in a central potential, that is, $V = V(r)$, using Hamilton's canonical equations.

Solution

We write
$$T = \frac{mv^2}{2} = m(\dot{r}^2 + r^2\dot{\theta}^2)/2 \tag{1}$$

(See Eq. (3.34) for the expression in terms of r and θ)

Then,
$$H = T + V$$

$$= \frac{m\dot{r}^2}{2} + \frac{mr^2}{2}\dot{\theta}^2 + V(r) \tag{2}$$

Then,
$$p_r = \frac{\partial H}{\partial \dot{r}} = m\dot{r} = mv_r$$

$$p_\theta = \frac{\partial H}{\partial \dot{\theta}} = mr^2\dot{\theta} = mrv_\theta \tag{3}$$

$$\dot{r} = p_r/m;\ \dot{\theta} = p_\theta/mr^2 \tag{4}$$

Therefore,
$$H = T + V = \frac{p_r^2}{2m} + p_\theta^2/2mr^2 + V(r)$$

$$\frac{dH}{\partial p_r} = \dot{r} \text{ and } -\frac{\partial H}{\partial r} = p_\theta^2/m_r^3 - \frac{\partial V}{\partial r} \text{ and } p_\theta = \frac{\partial H}{\partial \dot{\theta}} = 0 \tag{5}$$

Hence, using $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{6}$$

or $$\dot{p}_r + \frac{\partial H}{\partial r} = 0$$

or $$\frac{d}{dt}(m\dot{r}) - \frac{p_\theta^2}{mr^3} + \frac{\partial V}{\partial r} = 0$$

or $$\frac{d}{dt}(m\dot{r}) - \frac{(mr^2\dot{\theta})^2}{mr^3} + \frac{\partial V}{\partial r} = 0$$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \tag{7}$$

This is the equation in r-coordinates.

QUESTIONS

14.1 How many coordinate systems are generally used in physics? Name these and draw figures to give the coordinates of a point in each system.

14.2 Give an example of holonomic restraint and describe the constraint using cartesian coordinates.

14.3 What is a non-holonomic constraint? Give a few examples.

14.4 Why do we require generalised coordinates for absorbing holonomic constraints? Give some examples.

14.5 What is virtual work? What is the role of time in it?

14.6 How is virtual displacement different from real displacement?

14.7 Is D' Alembert's equation true for real displacement or virtual displacement? If the latter, why?

14.8 How in D'Alembert's principle a dynamic case has been reduced to static case?

14.9 Give the example of Q_j, the component of generalised force, where q's are not lengths, but some other quantities. Here,

$$Q_j = \sum_i \mathbf{F}_i\,(\partial \mathbf{r}_i/\partial q_j)$$

and $$\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_j Q_j \cdot \delta q_j$$

If we change q_j, what happens to $\mathbf{F}_i$?

14.10 Derive the relationship

$$H = \sum_j q_j \frac{\partial L}{\partial \dot{q}_j} - L$$

from the properties of Lagrangian.

14.11 Prove that $H = T + V$

14.12 When is $\frac{dH}{dt} = 0$ and when is it not? Describe the two physical conditions.

14.13 Derive the canonical equations of Hamilton.

14.14 What are the advantages of Lagrangian equation over Newton's equation in solving the problems of the motion of a particle?

14.15 Give an example of the motion of the particle where Hamilton canonical equations yield results easily.

PROBLEMS

14.1 A bead is sliding on a uniformly rotating rod in a horizontal plane in a force-free space. Find its equation of motion. *Ans.* [$\ddot{r} = r\omega^2$]

14.2 Construct (a) Lagrangian and (b) equation of motion of a coplanar double pendulum placed in a uniform gravitational field.

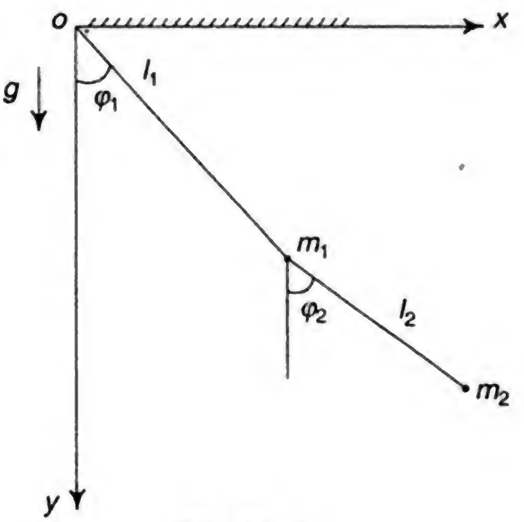


Fig. P14.2

Ans. [(a) $L = T_1 + T_2 - U_1 - U_2 = \frac{1}{2}(m_1 + m_2)\, l_1^2\dot{\varphi}_1^2 + (m_1 + m_2)$

$g l_1 \cos\varphi_1 + m_2 g l_2 \cos\varphi + \frac{1}{2} m_2\, l_2^2\, \dot{\varphi}_2^2 + m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)$

(b) $\omega_1^2 = (2 + \sqrt{2})g/l; w_2^2 = (2 - \sqrt{2})\; g/l$]

14.3 Consider a ladder sliding down a wall. Assume that the floor and the wall are smooth. Find the equations of motion for the ladder, assuming that the motion takes place in a plane. k is the radius of $\dot{\theta}^2/2$ gyration.

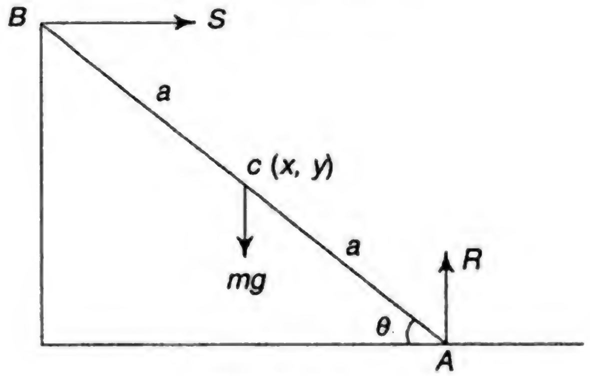


Fig. P14.3

Ans. [$L = M(a^2 + k^2)\,\theta^2/2 - Mg\,a\sin\theta$. $d^2\theta/dt^2 = -\,ag\cos\theta/(a^2 + k^2)$]

14.4 Write the Hamilton's equations in
(a) Cartesian coordinates
(b) Spherical coordinates

Ans. [(a) $\dot{x} = p_x/m$; $\dot{y} = p_y/m$; $\dot{z} = p_z/m$
(b) $\dot{r} = p_r/m$; $\dot{\theta} = p_o/mr^2$
$\dot{\phi} = p_\phi/mr^2 \sin^2\theta$]

14.5 Find the equation of motion of a pendulum bob, suspended by a spring and allowed to swing in a vertical plane. *Ans.* [$mr_2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr \sin\theta = 0$
$m\ddot{r} - m\ddot{r}\dot{\theta}^2 - mg \cos + k(r - r_j) = 0$]

14.6 Consider a wire bent in the form of a parabola $z = ar^2$ and bead sliding on the wire without friction. This wire is rotated by means of a shaft with a constant acceleration α. Show that $p_r = m(1 + 4a^2r^2)\dot{r}$.

14.7 A bead of mass m is free to slide on a circular wire of radius a, as shown in the Fig. P14.7. The wire itself rotates in a horizontal plane about a point O with a constant angular velocity. Determine the motion of the bead. Show that

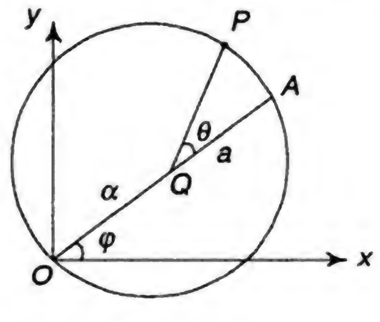


Fig. P14.7

$$T = \frac{ma^2}{2}(\dot{\theta} + \omega)^2 + 2(\dot{\theta} + \omega)\cos\theta + \omega^2$$

where $\varphi = \omega t$

Ans. [$\ddot{\theta} = -\omega^2 \sin\theta$]

14.8 Determine the Hamiltonian of a harmonic oscillator if its Lagrangian is given by

$$L = \frac{1}{2}\dot{x}^2 + \beta x\dot{x}^2 - \frac{1}{2}\omega^2x^2 - \alpha x^2$$

14.9 Construct the Lagrangian and equation of motion of a spherical pendulum placed in a uniform gravitational field.
(A spherical pendulum is a case of the bob of the pendulum suspended in such a fashion that it is able to move on the surface of a sphere of radius l, l being the length of the pendulum).

Ans. [$\ddot{\theta} - \frac{1}{2}\sin 2\theta\dot{\varphi}^2 + \frac{g}{l}\sin\theta$]

14.10 A triple pendulum consists of masses αm, m and m attached to a single light string at distances a, $2a$ and $3a$ respectively from its point of suspension, Fig. P14.10. Determine the value of α such that one of the normal frequencies of the system will equal the frequency of a simple pendulum of length $a/2$ and mass m. The displacement of the masses from the equilibrium position is assumed to be small.

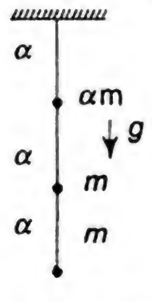


Fig. P14.10

15

Mechanics of Continuous Media

Matter, generally, has three forms: solid, liquid, or gas. A solid may be defined as that form of matter in which the external shape and the internal arrangement of its constituents remain unchanged under constant external conditions like temperature, pressure, and so on. Generally, in a solid, the molecules (or atoms) vibrate around their mean positions. All the known solids can be made to change their internal arrangements, and hence, external shape under high pressure or temperature. Some solids like diamond require very large pressure to change their shape. Others, say, a piece of rubber, require very little pressure to change their shape.

Liquids, on the other hand, are defined as substances, which can flow on the application of a very negligible external force. The molecules or atoms in a liquid are relatively free to change their position with respect to each other but are restricted by cohesive forces so as to maintain a fixed volume. They can slide by each other but the average distance between them remains the same.

The gases have not only the property of flow like liquids but are also compressible that is, it is easy to change their volume as well as their shape on the application of a comparatively small force. In a gas there are nearly no forces between molecules. A gas has neither a definite volume nor a definite shape.

In this manner solids, liquids, and gases are distinguishable from each other. Solids require very large external forces to deform and compress them. Liquids require very little force to deform them but a large force to compress them. Gases require no force to deform them and a much smaller force to change their volume, compared to the case of liquids.

These three forms are found at ordinary temperature. At very high temperatures and very high pressures, two other forms of matter are of importance: plasma and nuclear matter.

Plasma is a mixture of negative and positive charges moving around in a somewhat random manner. Such plasma exists in discharge tubes or in stars like sun etc. In metals where electrons move freely, combination of electrons and positive ions can be treated as a plasma. Nuclear matter is the matter where only nucleons exist without any electrons. Nuclear matter exists in neutron stars where under tremendous pressure all the electrons have been driven into nuclei so that protons in nuclei

have been neutralised leaving behind only neutrons. We will, however, not discuss these two forms of matter here as they are very special forms and are beyond the scope of this book.

SECTION A
ELASTICITY

15.1A FORCES BETWEEN ATOMS OR MOLECULES IN A SUBSTANCE

To understand the difference between solids, liquids, and gases at atomic level, we note the following facts:

(i) Density of solids varies from 1.0 gm/cm^3 to about 22 gm/cm^3 except for some fibrous materials like woods for which it varies form 0.1 to 1.0 gm/cm^3

(ii) Density of liquids varies from 0.97 gm/cm^3 for petroleum to 1.60 gm/cm^3 for carbon disulphide and carbon tetra chloride except for mercury for which it is 13.6 gm/cm^3.

(iii) At normal atmospheric pressure and temperature, the density of air is 1.29×10^{-3} gm/cm^3 and for hydrogen it is 0.09×10^{-3} gm/cm^3. Obviously, the density of gases is smaller than that of liquids by a factor of thousand or more

What do we learn form this about the internal structure of these three forms of matter? Knowing the density of a substance and the atomic weight of the atoms or molecules, one can estimate the number or atoms/molecules per cc of the substance. From this, one can estimate, on the average, the distance between the atoms or molecules if the structure of the material is known. Higher the density, less will be the distance between atoms or molecules.

It has been found that the average distance between two lead atoms is 3.2×10^{-8} cm, between two H_2O molecules is 3.4×10^{-8} cm, and the molecular distance in oxygen gas at 0 °C it is about 10^{-5} cm. What is the significance of this? We see that the atoms/molecules of gases are 1000 times further apart compared to those in liquids or solids. Therefore, one can assume that the atoms or molecules in a gas, which are, in general, neutral have no potential energy between them. Therefore, they move freely. But the inter-atomic distances in solids and liquids are of the same order as the size of atoms or molecules. Therefore, in liquids and solids, the outermost electrons in atoms and molecules will overlap. It is, therefore, a case of electrical binding of a somewhat complex nature in which the electrons of one atom overlap the electrons of adjoining atoms (Fig. 15.1A). Electrical force between them is obtained through a detailed application of Coulomb law. These bonds are ionic, covalent, or Van der Waal type. Of course at these distances and sizes, laws of quantum physics will be applicable.

The results of such an interaction have been calculated for various cases by many research workers. It is agreed now that if we plot a curve of the potential energy between the two neighbouring atoms of molecules in any solid or a liquid, it will have the form shown in Fig. 15.2A. Positive potential energy means repulsion and negative potential energy means attraction. The distance at which the atoms actually

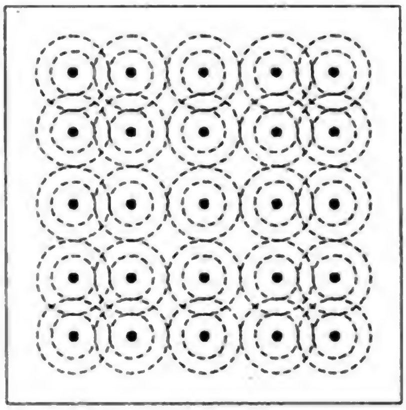


Fig. 15.1A(a) Atoms in a solid or liquid

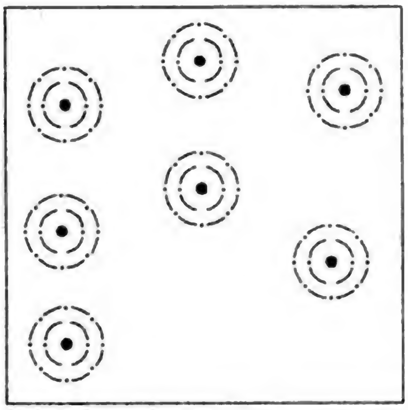


Fig. 15.1A(b) Gaseous atoms

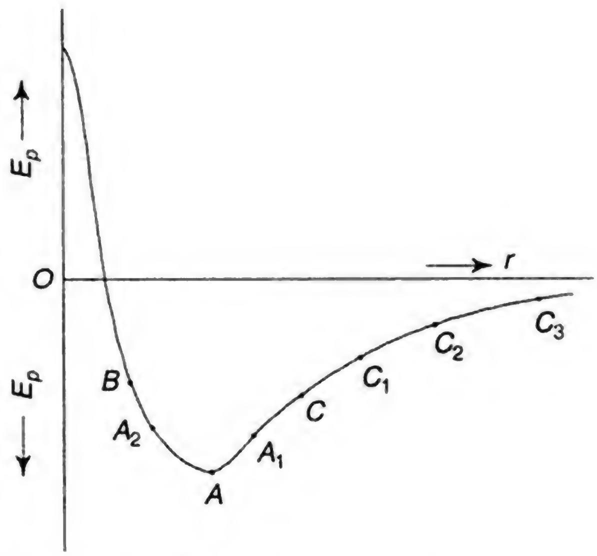


Fig. 15.2A Potential energy curve

exist will be that at which the potential energy is negative at its minimum value as shown at point A in Fig. 15.2A.

Any system always tends to exist at the minimum possible potential energy as the minimum potential energy position is the most stable one. Apart from the potential energy due to electrical interaction, we should take into account the kinetic energy due to the fact that at a given temperature, each atom/molecule possesses some kinetic energy. On the basis of kinetic theory of matter, this kinetic energy in a gas is given by 3/2 kT, where k is Boltzman's constant and T is the absolute temperature. In solids and liquids also, there will be kinetic energy but different from that in gases as in solids and liquids we have different degrees of freedom. Each atom will oscillate around its mean position shown at A (Fig. 15.2A), the limits of its motion being say A_1 and A_2.

The shape of the potential energy curve around mean position A in solids has been found by many research workers to be represented by

$$E_p = Cx^2 \tag{15.1A}$$

where x is the displacement of the atom from its equilibrium position and C is constant. Therefore, the force acting on each atom, displaced by a distance x from its mean position is given by

$$F = -\frac{dE_p}{dx}$$
$$= -2Cx$$
$$= -kx \tag{15.2A}$$

where k = $2C$. Thus, the atom/molecule will perform simple harmonic motion around its position of equilibrium.

If we compress such material, this will tend to bring the atoms nearer to each other. In terms of the potential energy curve in Fig. 15.2A, this means the position of the atom has been displaced form A to, say, B, which corresponds to less inter-atomic distance than A. But point B has a higher potential energy. The atoms will, therefore, like to come to point A corresponding to minimum potential energy because that is more stable. Hence, a restoring force will act on each atom to oppose the compression.

Similarly, if the material is extended, the atoms will be further apart from each other. In terms of the potential energy curve in Fig. 15.2A, this means that the distance between the atoms now corresponds to, say, point C, at a larger inter-atomic distance than point A. But this is also at a higher potential energy than point A. Hence, restoring force is created to bring the atom to its minimum potential energy corresponding to point A. This restoring force opposes the force responsible for extension. In this manner we see that due to the shape of the potential energy curve between atoms in a solid, whenever one tries to compress or extend a solid substance, a restoring force comes into play, which opposes the applied force. In equilibrium position, the restoring force is equal and opposite to the applied force. This constitutes the basis of elasticity of the various solids and liquids.

A substance that requires more force to shift the position of its atoms, say, from position A to B or A to C is more elastic because the atoms have more tendency to come back to their original positions. So, according to Eq. 15.2A, larger the value of C, more elastic is the substance, which is the same thing as saying that deeper the minima, more elastic the substance.

15.2A ELASTICITY, STRESS AND STRAIN

A body can be deformed in many ways.

(i) It can be compressed or stretched in one dimension so that only length changes

(ii) It can be compressed or stretched in all the three dimensions so that the whole volume changes

(iii) One can apply force in such a manner that one portion of the body is displaced with respect to the other

In all the three cases the external forces applied will be opposed by the internal forces. To understand these three phenomena quantitatively we have to define the amount of change and the forces required to bring about the change.

The internal force per unit area, which is called into play to oppose the external force applied for deforming a body, is called stress. It is assumed in this definition that internal force is opposite and equal to the applied force.

The fractional deformation, that is, the ratio between the change (in length, volume, or displacement) and the original state (i.e. length, volume or distance between layers) is called the strain. Because of the three types of deformation mentioned we have the following three kinds of strains.

(i) **Tensile strain**: This corresponds to a change in the length of a substance either through compression or extension or stretching (Fig. 15.3A(a)).

(ii) **Volumetric strain or bulk strain**: This corresponds to a change in the volume of a substance (Fig. 15.3A(b)).

(iii) **Shear or shearing strain**: This corresponds to the displacement of one layer of the substance with respect to the other (Fig. 15.3A(c)).
Their detailed discussion follows.

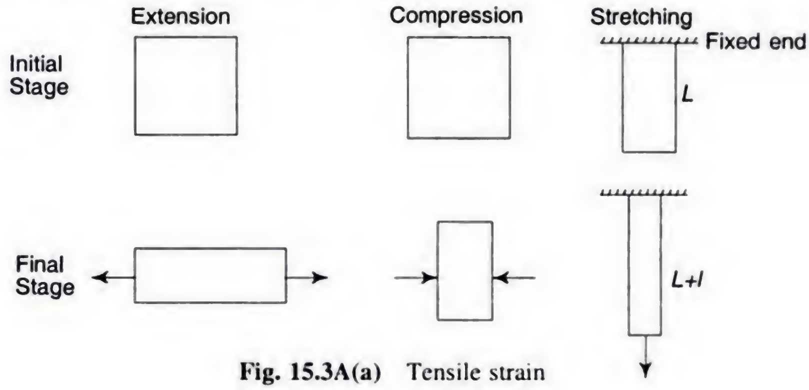


Fig. 15.3A(a) Tensile strain

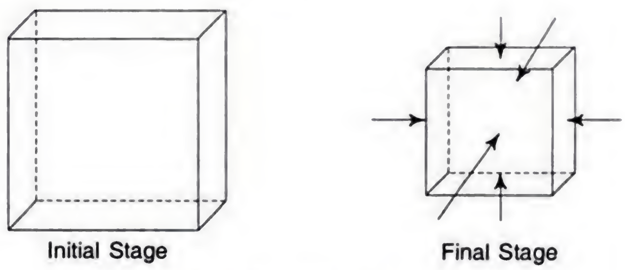


Fig. 15.3A(b) Volumetric strain

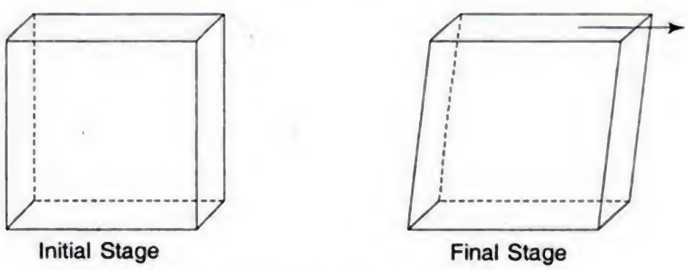


Fig. 15.3A(c) Shear strain

wire so that its elastic limit is not exceeded? (Elastic limit of cooper = 1.5×10^9 dynes /cm^2)

Solution:

$$Y = \frac{FL}{al}$$

$$a = \pi r^2$$

$$= \frac{22}{7} \times (5 \times 10^{-2})^2 = 7.85 \times 10^{-3}\ cm^2$$

Therefore, $$l = \frac{FL}{AY}$$

$$= \frac{5 \times 10^3 \times 980 \times 200}{7.85 \times 10^{-3} \times 1.1 \times 10^{12}}$$

$$= 0.11\ cm$$

The elastic limit of copper is 1.5×10^9 dynes/cm^2.

Hence, $$\frac{F}{a} = 1.5 \times 10^9\ dynes/cm^2$$

or $$a = \frac{5 \times 980 \times 1000}{1.5 \times 10^9} = 3.27 \times 10^{-3}\ cm^2$$

Now, $$r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{3.27 \times 10^{-3}}{\pi}}$$

$$= \sqrt{1.04 \times 10^{-3}}$$

$$= 3.2 \times 10^{-2}\ cm$$

Hence, the diameter should be 6.4×10^{-2} cm. In other words, when the diameter becomes less than this value the wire will not obey Hook's law.

15.2.2A Bulk Modulus

Bulk modulus applies to the change in volume in the same manner as the Young's Modulus applies to the change in length. Bulk modulus is defined as the ratio of the force per unit area (applied normal to the whole surface of the body) to the change in volume per unit volume, without any change in shape. As in Young's modulus, this definition of bulk modulus, holds good within the elastic limits. Thus bulk-modules B is given by

$$B = \frac{\text{Volumetric stress}}{\text{Volumetric strain}}$$

$$= \frac{F/a}{v/V}$$

15.3A EQUIVALENCE OF SHEAR STRAIN TO COMPRESSION AND EXTENSION STRAINS

The shear strain (Fig. 15.6A) is given by

$$\theta = \frac{DD'}{AD} = \frac{CC'}{BC} \text{ (for small value of } \theta\text{)}$$

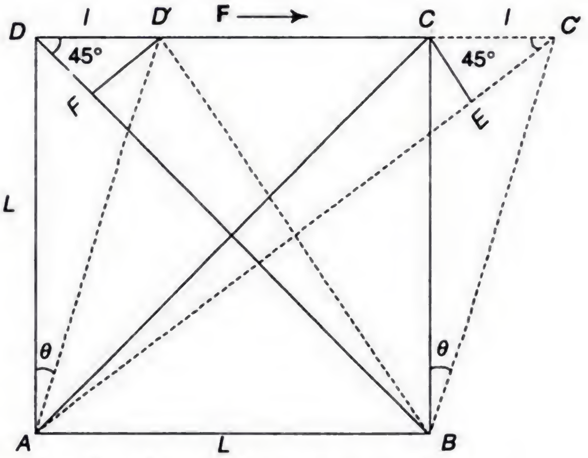


Fig. 15.6A A force F applied along DC

We also see that diagonal AC has been extended to AC' and diagonal BD has been compressed to BD' if the force F is applied along DC as shown in the Fig. 15.6A. Drawing perpendicular CE on AC', it is obvious EC' gives extension of $AC = AE$. Similarly, DF gives the compression of BD.

Now, extension strain along AC can be written as

$$\frac{\text{Increase in length}}{\text{Original length}} = \frac{EC'}{AE} = \frac{EC'}{AC}$$

Further, if angle θ is very small, angle $CC'B$ is nearly 90° and angle $CC'E$ is nearly 45°. Thus, from triangle CEC'

$$EC' = CC' \cos 45° = \frac{CC'}{\sqrt{2}}$$

Also, in triangle ABC, $$\frac{BC}{AC} = \sin 45° = \sqrt{\frac{1}{2}}$$

Therefore, $$AC = BC\sqrt{2} = L\sqrt{2}$$

Hence, $$\frac{EC'}{AC} = \frac{CC'}{\sqrt{2}} \Big/ BC \times \sqrt{2}$$

$$= \frac{1}{2}\frac{CC'}{BC}$$

But $$\frac{CC'}{BC} = \theta$$

Therefore, $$\frac{EC'}{AC} = \theta/2 \tag{15.13A}$$

Similarly, the compression strain along BD is given by

$$\frac{\text{Decrease in length}}{\text{Original length}} = \frac{DF}{BD}$$

Again, by the similar argument as before, it can be shown that

$$DB = AD \times \sqrt{2}$$

Therefore, compression strain

$$\frac{DF}{BD} = \frac{\theta}{2} \tag{15.14A}$$

Thus, total shear strain = extension strain + compression strain

This shows that total shear strain is a result of two tensile strains perpendicular to each other—one corresponding to extension and the other corresponding to compression.

15.4A POISSON'S RATIO

While discussing the Young's modulus, we had assumed that the area of cross-section perpendicular to the increase in the length of the wire, remains constant. This is only approximately true when increment in length is very small. In actual practice, it is the volume of the solid, which stays constant so that when length along the direction of weight increases, the cross-sectional area perpendicular to the increase in length shrinks. This is shown in Fig. 15.7A.

Let L be original length and N be the original breadth of, say, a long bar. If l is the increase in length and n is the decrease in breadth then

Longitudinal strain $= \dfrac{l}{L}$

Transverse strain $= \dfrac{n}{N}$

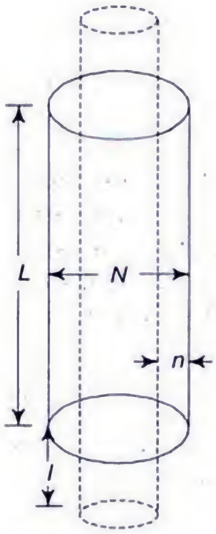


Fig. 15.7A Calculation of Poisson's ratio

We, then, define a quantity called Poisson's ratio σ that is given by the ratio of the transverse strain to the longitudinal strain. Thus,

$$\sigma = -\frac{n/N}{l/L} = -\frac{nL}{lN} \tag{!5.15A}$$

where the minus sign has been introduced because n (a decrease in breadth) is negative. Hence, σ is positive. The Poisson's ratio is a useful concept and is used in many relations.

15.5A RELATION BETWEEN ELASTIC CONSTANTS

The quantities Y, η, B and σ are not independent. They are related to each other as shown below. This can be seen physically. The quantity Y, the Young's modulus, is connected with longitudinal strain and so is Poisson's ratio σ. If α is the longitudinal strain per unit stress, then

$$\alpha = \frac{l/L}{F/a} = \frac{1}{Y}$$

or
$$Y = \frac{1}{\alpha} \tag{15.16A}$$

Similarly, if β is the transverse strain per unit stress, then

$$\beta = \frac{n/N}{F/a}$$

and
$$\sigma = \beta/\alpha = \beta Y \tag{15.17A}$$

Again, the bulk-modulus is an extension of the Young's modulus Y in three dimensions. Further, η, the modulus of rigidity, involves the ratio of the change in length in one direction with the length perpendicular to it. We state below these relations, which will be proved in the next section.

$$\beta = \frac{Y}{3(1-2\sigma)} \tag{15.18A}$$

$$\eta = \frac{Y}{2(1+\sigma)} \tag{15.19A}$$

Thus, we see that out of four quantities, Y, B, and σ, only two are independent. If we know the values of any two, other two can be calculated.

Let us derive the relation between B, Y, and σ, Eq. (15.18A) first. We know that

$$B = \frac{F/a}{v/V} = \frac{T}{v/V}$$

where v is the change in volume when acted upon by a stress, $F/a = T$, on all sides of the body. Let us consider a unit cube $ABCDEFGH$ so that $EH = ED = EF = 1$. Let stress T_x act outwards on the faces $ADEF$ and $BCHG$ respectively. Similarly, the stress T_z acts on faces $ABGF$ and $CHED$ and T_y acts, on $ABCD$ and $EFGH$ as shown in Fig. 15.8A.

Defining α and β as before, we see that increase Δl_x in the length EH is given by

$$\Delta l_x = \alpha T_x - \beta T_y - \beta T_z$$

If
$$T_x = T_y = T_z \equiv T,$$

then
$$\Delta l_x = (\alpha - 2\beta)T \tag{15.20A(a)}$$

Similarly, the increase along Y and Z axes are

$$\Delta l_y = \alpha T_x - \beta T_y - \beta T_z$$

or
$$\Delta l_y = (\alpha - 2\beta)T \tag{15.20A(b)}$$

and
$$\Delta l_z = \alpha T_z - \beta T_y - \beta T_x = (\alpha - 2\beta)T \tag{15.20A(c)}$$

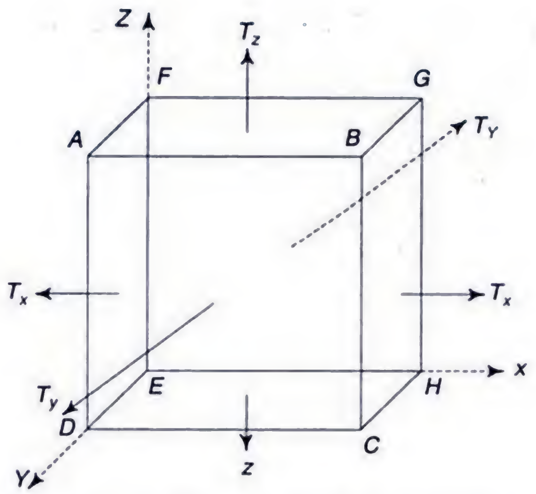


Fig. 15.8A Relations between elastic constants

The deformed volume V of the cube is given by

$$V = [l + (\alpha - 2\beta)\ T]^3$$

Neglecting the terms containing squares and cubes in α and β (because α and β are very small quantities), we get

$$V = l^3 + 3T\ (\alpha - 2\beta) \tag{15.21A}$$

But the original volume was l. Hence, the change in volume v is given by

$$v = V - l^3 = 3T\ (\alpha - 2\beta)$$

Remembering that $F/a = T$, we get

$$B = \frac{F/a}{v/V} = \frac{F/a}{3T(\alpha - 2\beta)} = \frac{T}{3T(\alpha - 2\beta)}$$

$$= \frac{1}{3(\alpha - 2\beta)} \tag{15.22A}$$

Putting $Y = 1/\alpha$ and $\dfrac{\beta}{\alpha} = \sigma$, the above equation may be written as

$$B = \frac{Y}{3(1 - 2\sigma)} \tag{15.23A}$$

Next, we proceed to establish the relation of Eq. (15.9A).

Let us consider the Fig. 15.9A where a square $ABCD$ with each side equal to L is deformed to a parallelogram $ABC'D'$ by a force acting along DC so that the angle $DAD' = CBC' = \theta$, say. We assume that θ is very small. The new diagonal AC' has been extended compared to AC but the diagonal BD' has been compressed as compared to BD as is evident from the diagram. If we draw a perpendicular CE on AC', then $AE = AC$, if the angle CAE is very small. The extension of AC' from AC is given by EC'. The next step is to write EC' in terms of α and β.

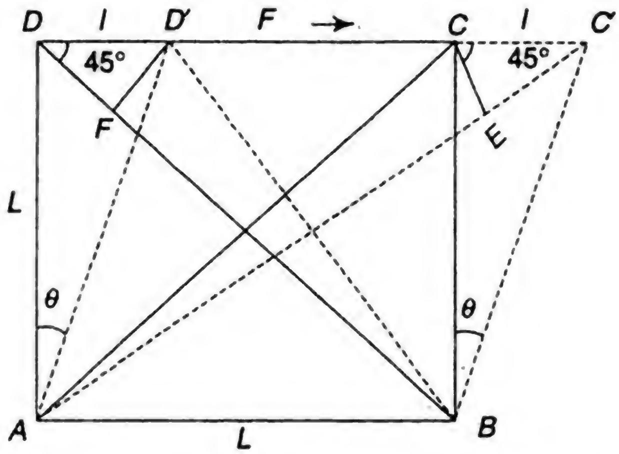


Fig. 15.9A Relation between elastic constants

It should be remembered that α is the extension per unit length per unit tension and β is the compression per unit length per unit tension. Increase in AC is the result of tension and contraction in BD is due to extension in AC.
The extension along AC due to extension stress along $AC = AC\ \alpha\ T$
The extension along AC due to compression stress along $BD = AC\ \beta\ T$
The total extension along $AC = AC\ (\alpha \times \beta) \times T$ (15.24A)
We can also write EC^+ from the geometry, in term of CC'. Considering the triangle CEC', because θ is very small, the angle $CC'B$ is very nearly equal to 90° and hence, $\angle CC'A$ is nearly 45°. Thus,

$$EC' = \frac{CC'}{\sqrt{2}} \tag{15.25A}$$

which gives $$\frac{CC'}{\sqrt{2}} = AC\ (\alpha + \beta)T \tag{15.26A}$$

But $CC' = l =$ increment along the force

and AC is related to the side of the cube as

$$(AC)^2 = (AB)^2 + (BC)^2 = 2L^2$$

or $$AC = \sqrt{2}L \tag{15.27A}$$

Therefore, $$\frac{CC'}{AC} = \frac{l}{\sqrt{2}L} = \frac{\theta}{\sqrt{2}} \tag{15.28A}$$

From Eqs (15.26A) and (15.28A)

$$\frac{\theta}{\sqrt{2}} = \sqrt{2}\ (\alpha + \beta)T$$

or $$\theta = 2\ (\alpha + \beta)\ T$$

$$\frac{\theta}{T} = 2\ (\alpha + \beta)$$

But η, the modulus of rigidity has been defined as

$$\eta = \frac{F/a}{\theta} = \frac{T}{\theta}$$

Now, volume of the wire = area of cross-section × length = aL

Therefore, work done per unit volume of the wire

$$= \frac{1}{2} \frac{F}{a} \times \frac{l}{L}$$

$$= \frac{1}{2} \text{ stress} \times \text{strain} \tag{15.36A}$$

Case II: *Volume strain*

In the case of volume strain, we subject the whole body to a pressure P perpendicular to the surface. If the pressure is considered only in one direction, and dx, is the change in length in the direction of P, then work done is

$$dW = F\,dx = (Pa)\,dx = PdV \tag{15.37A}$$

where a is area perpendicular to P and $dV = adx$ is the change in volume. Equation (15.37A) applies, similarly, in other directions.

Therefore, the total work done is

$$W = \int_0^V Pdv$$

where v is the total change in volume.

Now, $$B = \frac{P}{v/V}$$

or $$P = B\,v/V$$

Therefore, $$W = \int_0^v \frac{Bv}{V}\,dv = \frac{B}{V}\int_0^v vdv$$

$$= \frac{B}{V}\cdot\frac{v^2}{2} = \frac{1}{2}\frac{Bv}{V}\cdot v$$

$$= \frac{1}{2} Pv = \frac{1}{2} \text{ stress} \times \text{change in volume} \tag{15.38A}$$

Work done per unit volume

$$W = \frac{1}{2}\frac{Pv}{V} = \frac{1}{2} P \frac{v}{V} = \frac{1}{2} \text{ stress} \times \text{strain} \tag{15.39A}$$

Case III: *Shearing strain*

Let us take a cube of side L and subject it to a tangential force F on its upper face, the lower face remaining fixed. Let the shear angle be θ (Fig. 15.12A).Then, according to Eq. (15.12A)), the modules of rigidity is given by

$$\eta = \frac{F/L^2}{\theta} = \frac{F}{L^2\theta}$$

or $$F = \eta L^2 \theta \tag{15.40A}$$

Therefore, work done by a small angle of shear $d\theta$ is given by F multiplied by the displacement in that direction. Let displacement be dx so that

$$d\theta = \frac{dx}{L} \tag{15.41A}$$

Therefore, work done for shear $d\theta$ is

$$dW = F\,dx = FLd\theta \tag{15.42A}$$

or

$$dW = \eta L^3 \theta d\theta$$

or

$$W = \int_0^\theta \eta L^3 \,\theta d\theta = \frac{1}{2}\,\eta L^3 \theta^2 = \frac{1}{2}\,(\eta L^2 \theta^2)L$$

$$= \frac{1}{2}\,LF\theta$$

$$= \frac{1}{2}\text{ tangential force} \times \text{displacement} \tag{15.43A}$$

We can further write it as

$$W = \frac{1}{2}\,(\eta L^2 \theta^2)\,L = (FL)\theta$$

$$= \frac{1}{2}\text{ couple} \times \text{shear strain} \tag{15.44A}$$

Work done per unit volume

$$= \frac{1}{2}\,\eta\theta^2$$

$$= \frac{1}{2}\,(\eta\theta)\theta$$

$$= \frac{1}{2}\left(\frac{F}{L^2}\right)\theta = \frac{1}{2}\text{ stress} \times \text{strain} \tag{15.45A}$$

EXAMPLE 15.3A

From a 100 cm long copper wire of 1mm radius is hung a block weighing 20 kg. The wire breaks suddenly. Does its temperature decrease or increase? Calculate the potential energy gained [$Y_{cu} = 1.2 \times 10^{12}$ dynes/cm^2].

Solution

When the wire is elongated, its potential energy increases equal to the work done on it. When the wire suddenly breaks down, the molecules return to their original position, and hence, the potential energy is released. This energy is converted into heat, and hence, the temperature of the wire will increase.

Gain in potential energy of the molecules = work done on the wire

$$= \frac{1}{2}\,F\,l$$

We know that

$$Y = \frac{FL}{al}$$

Therefore,

$$l = \frac{FL}{aY} = \frac{20 \times 1000 \times 980 \times 100}{\pi \times (0.1)^2 \times 12 \times 10^{11}}$$

$$= 5.2 \times 10^{-2}\text{ cm}$$

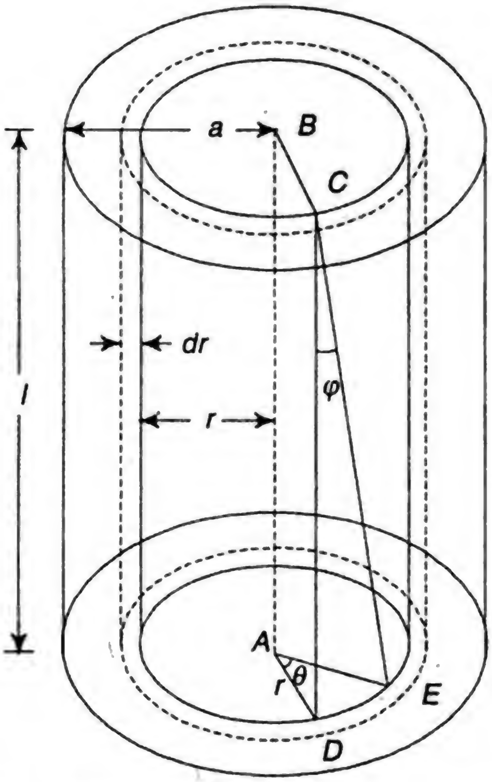


Fig. 15.10A Couple for twisting a cylinder

$$= T/\varphi = \frac{F}{2\pi r\,dr} \times \frac{l}{r\theta}$$

or
$$\eta = \frac{Fl}{2\pi r^2\,dr\theta} \tag{15.48A}$$

The moment of this force about the axis AB is given by

$$d\tau = \text{force} \times \text{perpendicular distance}$$
$$= F \times r$$
$$= \frac{2\pi\eta\theta}{l}\,.\,r^3 dr \tag{15.49A}$$

This is the couple required to twist the shell, between r and $r + dr$. The total moment of this force to twist the whole wire is obtained by integrating the expression (15.49a) for $r = 0$ to $r = a$. Thus,

$$\tau = \int_0^a d\tau$$

$$= \frac{2\pi\eta\theta}{l}\int_0^a r^3\,dr$$

$$= \frac{2\pi\eta\theta}{l}\,\frac{a^4}{4} = \frac{\eta\pi a^4}{2l}\,\theta \tag{15.50A}$$

If the cylinder is hollow with inner radius a_l and outer radius a_2, then, integrating between the limits a_l to a_2, we obtain

$$\tau = \frac{2\pi\eta\theta}{l}\int_{a_1}^{a_2} r^3\,dr = \frac{\eta\pi\theta}{2l}\left(a_2^4 - a_1^4\right) \tag{15.51A}$$

In Eq. (15.51A) the quantity, $C = \frac{\tau}{\theta} = \frac{\eta\pi a^4}{2l}$, is the couple to produce a unit twist and is called the torsional rigidity.

It is because of the constancy of this quantity that the lower end of the wire will execute the simple harmonic motion if it is twisted and then released.

EXAMPLE 15.5A

You are provided with two shafts of the same material, mass, and length. One of them is solid while the other one is hollow. Which will you prefer and why?

Solution

The couple required to twist a solid cylindrical rod of length l and radius r through an angle θ radians is given by

$$\tau = \frac{\pi\eta r^4\theta}{2l} \tag{i}$$

The couple required to twist a hollow cylindrical rod of length l and inner and outer radii r_1 and r_2, respectively, through an angle θ is

$$\tau' = \frac{\pi\eta\left(r_2^4 - r_1^4\right)\theta}{2l} \tag{ii}$$

Dividing Eq. (ii) by (i) we have,

$$\frac{\tau}{\tau'} = \frac{r^4}{r_2^4 - r_1^4} = \frac{r^4}{\left(r_2^2 + r_1^2\right)\left(r_2^2 - r_1^2\right)} \tag{iii}$$

Mass of solid shaft = $\pi r^2 ld$

where d is the density of the material.

Similarly, mass of hollow shaft = $\pi(r_2^2 - r_1^2)ld$

But mass of solid shaft = mass of hollow shaft therefore

$$\pi r^2 ld = (r_2^2 - r_1^2)ld$$

or
$$r^2 = r_2^2 - r_1^2$$

Hence, Eq. (iii) can be written as

$$\frac{\tau}{\tau'} = \frac{r^2}{r_2^2 + r_1^2} = \frac{r_2^2 - r_1^2}{\left(r_2^2 + r_1^2\right)}$$

or $\frac{\tau}{\tau'} < 1$ because $(r_2^2 - r_1^2) < (r_2^2 + r_1^2)$

or
$$\tau' > \tau$$

This shows that greater couple will be needed to twist a hollow shaft than a solid one of the same material, length and mass, and hence, it is stronger than a solid one. This is why hollow shafts are used in motor cars.

EXAMPLE 15.6A

A wire of 2 mm diameter and length 2 m is twisted through 90°. Calculate the angle of shear at the surface, at the axis of wire, and at a point midway between the axis and the surface. If the modulus of rigidity is 5 × 10^{11} dynes/cm^2, what is the torsional couple?

Solution

$$\text{Angle of shear } \phi = \frac{x\theta}{l}$$

(i) At the surface $\phi = \dfrac{1 \times 90°}{2 \times 10^3} = 45 \times 10^{-3}$ degree

(ii) At the axis $x = 0$, $\phi = 0$

(iii) At the point midway between the axis and the surface

$$\phi = \frac{0.5 \times 90°}{2 \times 10^3} = 22.5 \times 10^{-3} \text{ degree}$$

$$\text{Torsional couple} = \frac{\pi\eta\theta r^4}{2l} = \frac{3.14 \times 5 \times 10^{11} \times 3.14 \times (0.1)^4}{2 \times 200 \times 2}$$

$$= 0.616 \times 10^6 \text{ Joule}$$

15.7.1A Determination of η by Maxwell's needle

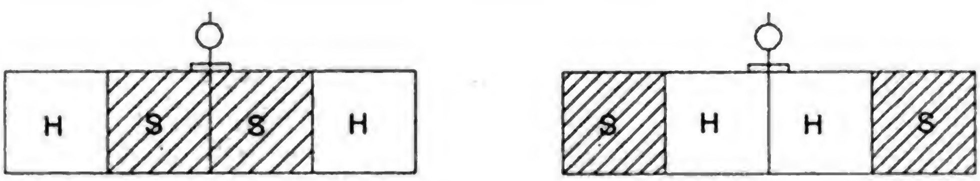
The time period of a torsion pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{\tau}} \tag{9.31}$$

where I is the moment of inertia of torsion pendulum about the suspension wire and τ is the restoring couple per unit twist.

A torsion pendulum can be used for finding η, the coefficient of rigidity, but the only snag lies in finding the value of I accurately. This difficulty was circumvented by Maxwell by using a hollow rod fitted with four cylinders; two hollow (H,H) and two solid (S,S) of equal length and radii (Fig. 15.11A). The experiment is performed first with the two solid cylinders in inner position and the two hollow cylinders in the outer position (Fig. 15.11(a) A); and then repeated with the positions of the solid and hollow cylinders interchanged (Fig. 15.11(b)A).

Let I_l and I_2 be the moments of inertia in the first and second cases, respectively, and the corresponding time periods of T_1 and T_2. Then



(a) (b)

Fig. 15.11A Maxwell needle

$$T_1 = 2\pi \sqrt{\frac{I_1}{\tau}} \tag{15.52A}$$

$$T_2 = 2\pi \sqrt{\frac{I_2}{\tau}} \tag{15.53A}$$

Squaring and subtracting Eq. (1552A) from Eq. (15.53A), we get

$$T_2^2 - T_1^2 = \frac{4\pi^2 (I_2 - I_1)}{\tau} \tag{15.54A}$$

Next, we proceed to calculate the change in moment of inertia ($I_2 - I_l$) on changing from the first configuration to the second. Let m_1 be the mass of each hollow cylinder, m_2 the mass of the solid cylinder, $2a$, the length of hollow tube, and therefore, $a/2$ the length of each cylinder. The distances of the centre of mass of the inner and outer cylinders from the axis of oscillation are $a/4$ and $3a/4$, respectively. Each solid cylinder has mass ($m_2 - m_1$) more than a hollow cylinder. Thus, in changing the first configuration to the second, we are essentially transferring ($m_2 - m_1$) mass from the core of each solid cylinder to that of each outer hollow cylinder. The moment of inertia is increased, as on each side the centre of gravity of mass ($m_2 - m_1$) is shifted from a distance $a/4$ to $3a/4$ from the axis. According to the principle of parallel axes,

$$\begin{aligned} I_2 - I_1 &= 2\,(m_2 - m_1)\left[\left(\frac{3a}{4}\right)^2 - \left(\frac{a}{4}\right)^2\right] \\ &= (m_2 - m_1)a^2 \end{aligned} \tag{15.55A}$$

Putting Eq. 15.55A into Eq. 15.54A, one gets

$$T_2^2 - T_1^2 = \frac{4\pi^2}{\tau}(m_2 - m_1)a^2$$

Putting

$$\tau = \frac{\eta\pi a^4}{2l} \tag{15.50A}$$

$$\eta = \frac{8\pi l(m_2 - m_1)}{a^2 (T_2^2 - T_1^2)} \tag{15.56A}$$

15.8A STATICS OF SOLID BEAMS AND COLUMNS

A solid beam is a bulk solid body, which has a rectangular or circular cross-section whose dimensions are much smaller than the length, which is anchored horizontally. A solid column is similar to a solid beam, except that its length is anchored vertically. Both these solid body configurations are used extensively in all phases of mechanical engineering. Hence, the importance of understanding the statics, that is, to know the balance of forces operating on them in a given condition. Therefore, it becomes an important problem in physics in general, and particularly in mechanics.

There are two primary reasons for wanting to know these forces (i) one wants to know if the materials and the configuration of anchoring of the beams and columns will withstand the forces without breaking or permanent deformation and (ii) no solid materials used in beams and columns is permanently and inflexibly rigid, and undergoes deformation, both elastic [which after removal of forces allows the material to come to its original shape] and plastic [for which the material becomes permanently deformed and does not retain its original shape after the removal of forces]. One wants to know the amount of this deformation and relate it to the intrinsic properties of the solid.

This requires us to develop theoretical concepts connected with the interaction of forces with solid materials in general. We have discussed already the concept of (i) stress, (ii) strain, (iii) the Young's modulus, Y (iv) bulk modulus, B (v) shear modulus, η and (vi) Poisson's ratio, σ. All these quantities are connected with elastic deformation of the solid material. We will assume in this discussion that solids are only deformed up to their elastic limits and non-recoverable deformation (plasticity) does not set in.

A normal solid material, under no forces acting from outside is generally stable and in equilibrium. There are a large number of molecules in a solid, arranged in a particular manner. For the equilibrium of such a system, there must be microscopic internal forces on these molecules acting in such a way that their resultant is zero. Not only that, the moments of these forces should also vanish, otherwise there will be couples acting on different portions of the solid, which will not be in equilibrium. So one can write, for any solid in equilibrium,

$$\mathbf{F} = \sum_i \mathbf{F}_i = 0 \tag{15.57A}$$

and

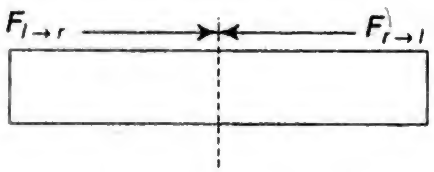
$$\mathbf{N} = \sum_i \mathbf{N}_i = 0 \tag{15.58A}$$

where F is the total external force and F_i are the microscopic internal forces mentioned above. Similarly, N is the total moment on the solid microscopic couples, created due to these internal forces.

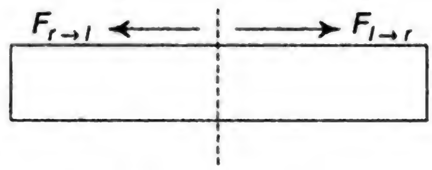
With this background, we will now discuss the problem of equilibrium of solid beams and the bending of beams in terms of shearing forces and bending moments. If one imagines a surface, which cuts across through any part of a solid structure (a rod or a beam), then the material on one side (say A) of the surface will exert a force on the other side (say B) and an equal force will be exerted by the surface on the side of B on the surface that is on the side A. These forces and the moments that they create will obey Eqs (15.57A) and (15.58A). Figure 15.12A shows three configurations of such a balance of forces. As shown in the figure, these correspond to (a) compression (b) tension and (c) shear.

Equilibrium of Solid Beams

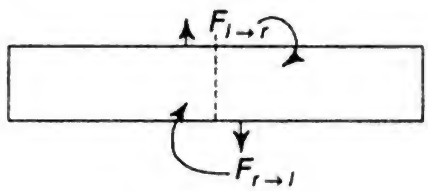
As mentioned earlier, a solid beam is generally anchored horizontally under the following conditions: (i) one end of the beam is fixed in a wall and the other end is free, where a load may be applied as shown in Fig. 15.13A(a). As we will see later, this configuration is useful for the analysis of the forces acting on the beam. This



(a) Forces of compression



(b) Forces of tension



(c) Shear forces

Fig. 15.12A Stresses in a beam (a) compression (b) tension (c) shear

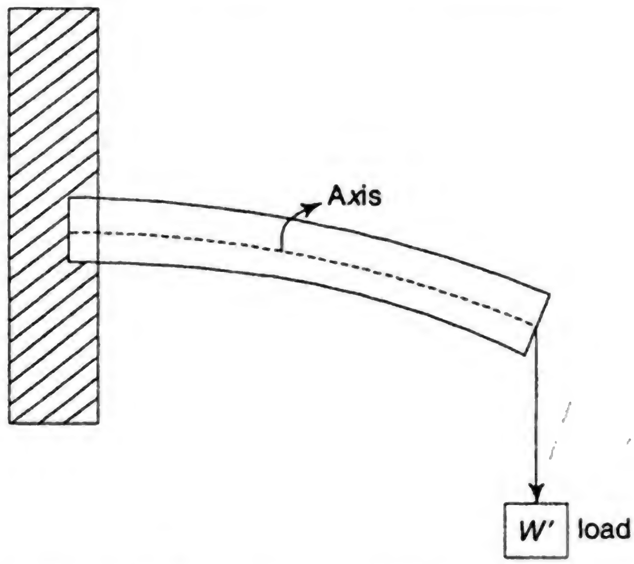
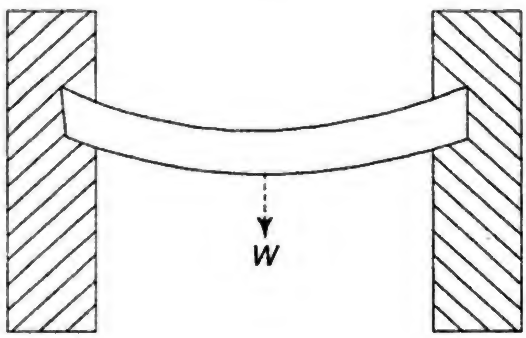
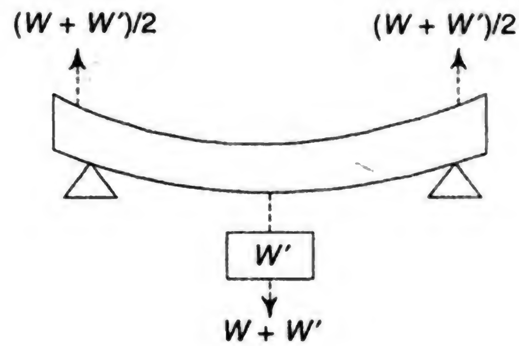


Fig. 15.13A(a) A cantilever with load W' on one end and beam fixed on the other hand

configuration is called cantilever and (ii) when beam is anchored on both ends and a load, including the weight of the beam, is applied in the middle as shown in Fig. 15.13 A(b). This is the general configuration found in buildings or mechanical structures.



(i)



(ii)

Fig. 15.13A(b) A cantilever (i) without or (ii) with load in the middle, both ends being fixed

We will analyse both these cases separately under both these two conditions and even under other more complicated conditions. If the beam is in any stable condition under equilibrium, Eqs (15.57A) and (15.58A) will hold good. As a matter of fact, we give below a few theorems, which will be applicable under these conditions. We will, however, formally not prove them as a reader can find their veracity on some reflection. Here, we give a proof of theorem 1 to illustrate the type of arguments that go into the proof.

Theorem 1: Every system of forces, is equivalent to a single force through an arbitrary point, plus a couple (either or both of which may be zero).

Proof of theorem 1

Let us select an arbitrary point P, on the beam, and let the sum of all the forces be $\mathbf{F} = \sum \mathbf{F}_i$ and let their total torque about a line passing through P be called $\mathbf{N}$. Since the couple can be composed of two forces, one of which may be allowed to act at an arbitrary point, which may be taken to act on P and added to F, so that a single force acts at P, plus a couple with other force of the couple. This proves the theorem.

Theorem 2: Any system of forces can be reduced to an equivalent system of forces, that contains at the most two forces.

Theorem 3: A single non-zero force and a couple in the same plane (such that the torque vector of the couple is perpendicular to the single force) have a resultant (a force); and conversely, a single force is equivalent to an equal force through any point, plus a couple.

Theorem 4: Every system of forces is equivalent to a single force plus a couple whose torque is parallel to the single force, or alternatively, every system of forces is equivalent to a couple plus a single force perpendicular to the plane of the couple.

Theorem 5: Any system of forces in a plane has a resultant (a force), unless it is couple.

Keeping the above theorems in mind we discuss the problem of a cantilever that is, a beam fixed at one end and with a load on the other end (Fig. 15.14A).

In terms of the theorems discussed above, this is a case that has a force F acting upward along DA, which represents the face fixed in the wall. Another force F acts downward at the face BC, representing the open end of the beam, which is bent due to this weight. A force $F_2 = W$ = weight of the beam acts downward through centre of gravity (C.G.). As the beam, though bent, is in equilibrium, theorem 1 will hold good; and these three forces will join together to give rise to one resultant force say S, which will create a couple around any arbitrary plane along, say, $A'D'$. We show in Fig. 15.15A (a) and (b) the configuration of these forces under conditions of undisturbed beam and beam bent and in shear due to its own weight $W = F_2$ along with the distances of the various forces from the wall.

Figure 15.15A(b) shows when extra load W' is applied and extra bending takes place due to the extra load, say, W' at the end. Before discussing the analytical method of calculating the bending, a few definitions are in order:

1. As theorem 1 states, a system of forces under which a rod is in equilibrium, though bent, gives rise to a resultant force at a point plus a couple around any arbitrary line in the beam. This couple is called the bending couple or bending moment.

2. The solid beam, especially with a rectangular cross-section, can be imagined to be divided into different planes, parallel to the plane passing though the axis OO', and parallel to the surface of the beam. The bent plane passing through OO' is called the neutral surface or the plane of bending.
3. The bent beam has its surfaces and the neutral surface all bent into arcs so that the spherical planes under the neutral surface have been compressed, while spherical planes outside the neutral surface are extended so that an arc of length l on the neutral surface contracts to $(l - \Delta l)$ on the inside concentric surface. These bent spherical surfaces have their centres of curvature on a straight line perpendicular to the plane of bending and is referred to as the axis of bending.

We discuss now three cases:

(a) When the beam is not bent, so that we neglect the weight of the beam, and also no extra weight is loaded at the end. In such a case according to Eq. (15.57A), we can write from Fig. 15.15A(a),

$$F = \sum_{x_i < x} F_i + S = 0 \tag{15.59A}$$

and

$$-N_0 - \sum_{x_i < x} (x - x_i)F_i + N = 0 \tag{15.60A}$$

where the sums are taken over all forces acting to the left of $A'D'$ (Fig. 15.15A(a)), and N_0 is the bending moment, if any, exerted by the left end of the beam against its support. The torque N_0 appears if the beam is fixed at the left hand. The force exerted by any clamp or other support at the end is to be included among the forces F_i.

(b) When the weight of the beam is taken into account. If the beam has the weight w per unit length, this should be included in equilibrium Eqs (15.59A) and (15.60A). Then

$$\sum_{x_i < x} F_i - \int_0^x w dx + S = 0 \tag{15.61A}$$

and

$$-N_0 - \sum_{x_i < x} (x - x_i) F_i + \int_0^x (x - x')w dx' + N = 0 \tag{15.62A}$$

The shearing force and bending moment at a distance x from the end are then given by:

$$S = -\sum_{x_i < x} F_i + \int_0^x w dx \tag{15.63A}$$

and

$$N = N_0 + \sum_{x_i < x} F_i(x - x_i) F_i - \int_0^x (x - x')w dx' \tag{15.64A}$$

If the beam is free at its two ends, the shearing force S and bending moment N must be zero at the ends. If we set $S = N = 0$ at the right end then Eq. (15.63A) has two components of force; but Eq. (15.64A) has three components of torque. On the other hand, if the beam is clamped on

both ends, or on either end, then shearing force S and bending moment N are determined through these two equations.

If there is an additional force due to, say, uniform distribution of extra weight this can be included in w.

One can have the following situation:

(i) A general case, so that shearing force S and bending moment N may be plotted as a function of x. Then

$$\frac{dS}{dx} = w(\text{except at } x_i) \tag{15.65A}$$

and

$$\frac{dN}{dx} = \sum_{x_i < x} F_i - \int_0^x w dx' = -S \tag{15.66A}$$

The shearing force S, however, increases by $-F_i$ from left to right across a point x_i, on which F_i is acting.

Expression for Bending Moment

Referring to Fig. 15.14A, the horizontal beam bends through an angle θ, which can be related to the shearing stress S, and shearing modulus η, through the definition of η as:

$$\eta = \frac{\text{shearing stress}}{\text{shearing strain}} = \frac{S}{A \tan \theta}$$

$$\cong \frac{S}{A\theta}$$

or

$$\theta \cong \frac{S}{A\eta} \tag{15.67A}$$

Again referring to Fig. 15.16A(b),when extra load W' is put on the beam, the relationship of Δl with $\Delta\phi$, the change in the shearing angle is

$$\frac{\Delta l}{z} = d\phi \tag{15.68A}$$

One can also write the relation of Y the Young modulus as:

$$Y = \frac{dF}{dA} \Big/ \frac{\Delta l}{l}$$

or

$$\frac{dF}{dA} = Y\frac{\Delta l}{l} = Yz\frac{d\phi}{l} \tag{15.69A}$$

If l is very small, that is, in the neighbouring segments of bent arcs in OO' in Figs. 15.16A(b) 15.16A(c), then one can write

$$l = ds \tag{15.70A}$$

Eq. (15.69A) then may be written as

$$\frac{dF}{dA} = Yz\frac{d\phi}{ds} \tag{15.71A}$$

Here F, the total compressive force to reduce l to $(l-\Delta l)$ can be determined from

$$F = \iint_A dF = Y\frac{d\phi}{ds}\iint_A z dA \tag{15.72A}$$

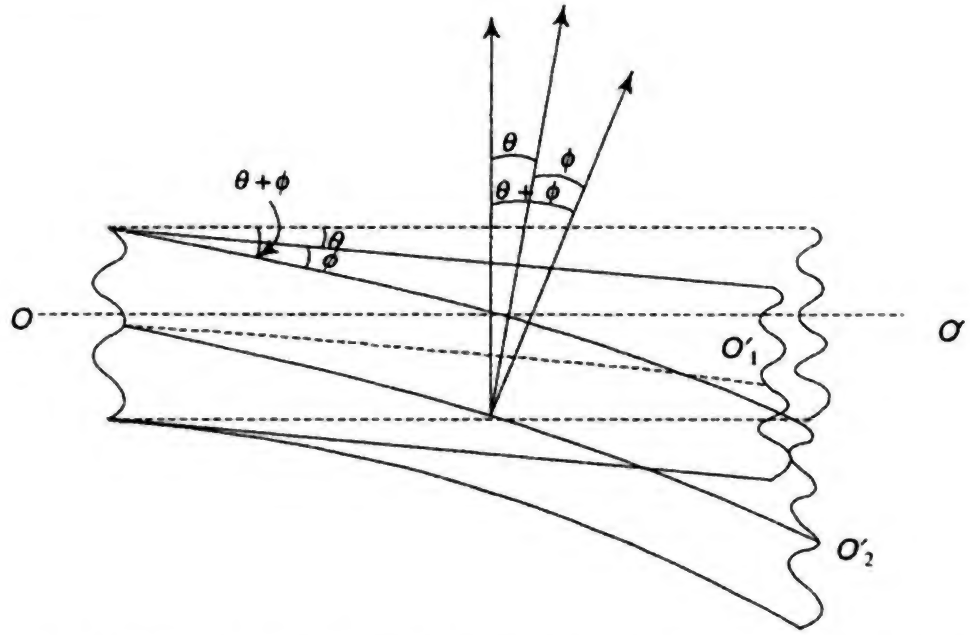


Fig. 15.16A(a) Bent beam due to its own weight and weight at the end

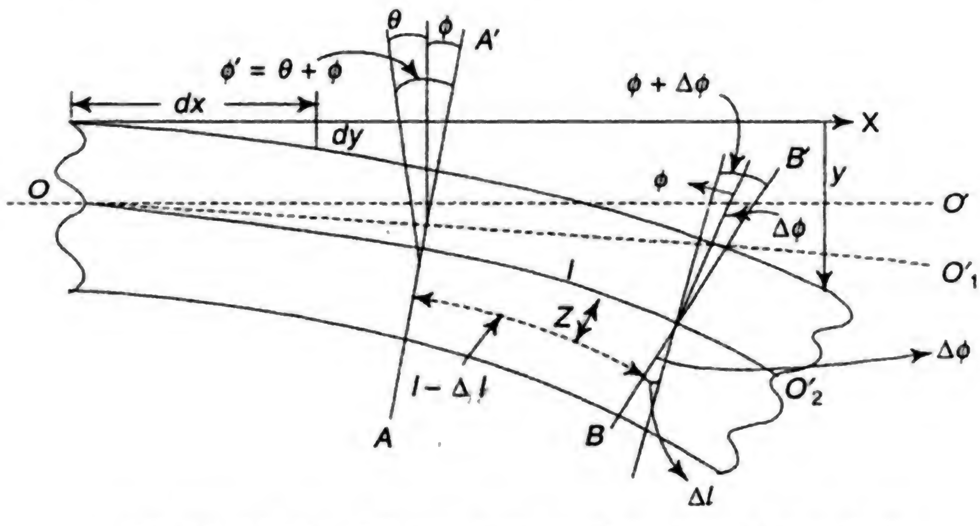


Fig. 15.16A(b) The configuration of angles of a bent beam

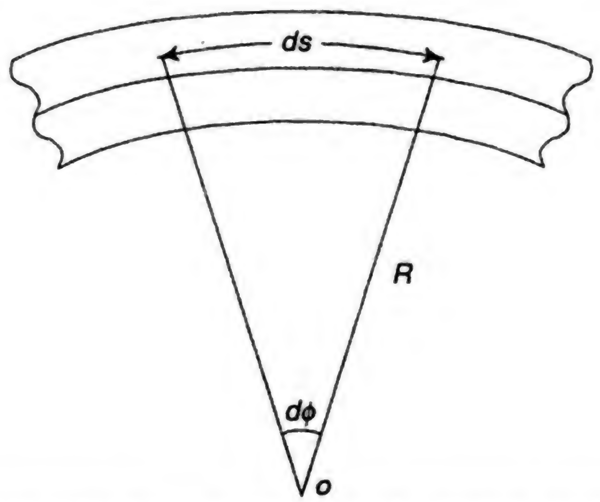


Fig. 15.16A(c) The relationship between the radius of curvature and length of arc (ds) and angle $d\phi$

The bending moment exerted by the force dF is, now, given by

$$N = \iint_A z dF = Y \frac{d\phi}{ds} \iint_A z^2 dA$$

$$= Yk^2 A \frac{d\phi}{ds} \qquad (15.73A)$$

where $$k^2 = \frac{1}{A} \iint_A z^2 dA \qquad (15.74(a))$$

k is called the radius of gyration of the cross-sectional area of the beam about a horizontal axis through its centroid and N in Eq. (15.73A) is called the bending moment of the beam. If we express

$$Ig \equiv \int z^2 dA \qquad (15.74(b))$$

Ig is called the geometrical moment of inertia. Then

$$N = YIg \frac{d\phi}{ds} \qquad (15.75A)$$

Writing $\frac{ds}{R} = d\phi$ or $\frac{d\phi}{ds} = \frac{1}{R}$, as shown in Fig. 15.16A(c), we can write Eq. (15.75A) as

$$N = Y\, Ig/R \qquad (15.76A)$$

YIg, which is obviously the external bending moment, required to produce a curvature of unit radius in the solid beam ($R = 1$), is called flexural rigidity of the beam.

For rectangular cross-section of the beam, $Ig = bd^3/12$, where b is the breadth and d is the depth of the beam. Then bending moment N is given by

$$N = \frac{Ybd^3}{12R} \qquad (15.77A)$$

Similarly for a circular cross-section

$$Ig = \pi R^4/4$$

Hence $$N = Y\, \pi R^4/4R = 1/4 Y\pi R^3 \qquad (15.78A)$$

It should be realized that it is being assumed that there is no net tension or compression of the beam when it is bent. Hence, $F = 0$, which leads to $\iint_A z dA = 0$. This implies that the neutral layer contains the centroid of the area A of the beam, and we may require OO' to be drawn through the centroid of the cross-section of the beam.

Depression (y) of the beam at the end cantilever

From Eq. (15.75A), it can be seen that

$$\frac{d\phi}{ds} = \frac{N}{Yk^2 A} \qquad (15.79A)$$

Let $y(x)$ be the downward deflection from a horizontal x-axis measured to the line OO'. Then $y(x)$ is obtained by solving the equation

$$\frac{dy}{dx} = \tan(\theta + \phi) = \tan \phi' \tag{15.80A}$$

where θ and ϕ are determined from Eqs (15.67A) and (15.68A). Assuming both θ and ϕ to be small, we can write Eq. (15.80A) as

$$\frac{d\phi}{dx} = \frac{N}{Yk^2 A} \tag{15.81A}$$

and
$$\frac{dy}{dx} = \theta + \phi = \phi' \tag{15.82A}$$

When there are no concentrated forces along the beam, we may differentiate Eq. (15.82A) and make use of Eqs (15.67A), (15.73A) and (15.79A) and obtain.

$$\frac{dy^2}{dx^2} = \frac{w}{nA} + \frac{N'}{Yk^2 A} \tag{15.83A}$$

and
$$dy^4/dx^4 = \frac{S}{nA}\frac{d^2 w}{dx^2} - \frac{w}{Yk^2 A} \tag{15.84A}$$

One can use Eqs (15.83A) and (15.84A) for two situations: (i) short, thick beam and (ii) long beam. In the former case, bending can be neglected, hence N can be taken to be zero. The application of these two equations, can give appropriate results, under various conditions of forces F_i. The solution of Eq. (15.83A) will give two constants and the solution of Eq. (15.84A) four constants, which will be determined by the conditions at the ends of the beam or segment of the beam.

Let us consider the case of a uniform beam of weight W, length L, clamped in a horizontal position at its left end ($x = 0$), and with only one force $F_i = -W$ that is, downward at the right end ($x = L$) that is, it becomes a cantilever. Then Eq. (15.84A) can be written as:

$$dy^4/dx^4 = -\frac{W}{Yk^2 AL} \tag{15.85A}$$

because, for horizontal beam, $\phi = 0$ at its left and hence plane AA′ is vertical; and beam is horizontal corresponding to no shearing strain, or the first term in Eq. (15.84A) becomes zero. Also $W/L = w$, in Eq. (15.84A).
The solution of Eq. (15.85A) can be written as:

$$y = -\frac{Wx^4}{24Yk^2 AL} + \frac{1}{6}C_3 x^3 + \frac{1}{2}C_2 x^2 + C_1 x + C_0 \tag{15.86A}$$

To obtain the values of C_0, C_1, C_2 and C_3, we obtain from Eq. (15.86A) the values of y, dy/dx, d^2y/dx^2 and d^3y/dx^3 for $x = 0$. The first term then vanishes and one gets

(i) $y = C_0 = 0$ at the left end of the beam.

(ii) $\dfrac{dy}{dx} = C_1 = \theta$ (as $\phi = 0$)

$$= \frac{S}{\eta A} = -\frac{W + W'}{\eta A} \tag{15.87A}$$

from Eqs (15.67A) and (15.82A)

(iii) Then, we derive from Eq. (15.86A)

$$\frac{d^2y}{dx^2} = C_2 = \frac{W}{\eta AL} - \frac{W'L + \frac{1}{2}WL}{Yk^2A} \tag{15.88A}$$

and
$$\frac{d^3y}{dx^3} = C_3 = \frac{\left(\frac{dN}{dx}\right)}{YA^2k} = \frac{-S}{Yk^2A} = \frac{W' + W}{Yk^2A} \tag{15.89A}$$

where we have used Eqs (15.75A) and (15.83A)

We finally write the expression for y, at any point (by collecting the expressions from various Eqs (15.86A), (15.87A), (15.88A) and (15.89A). Then we obtain

$$y = -\frac{L^3}{Yk^2A}\left[\frac{Wx^2}{4L^2}\left(1 - \frac{2x}{3L} + \frac{1}{6}\frac{x^2}{6L^2}\right) + \frac{W'x^2}{2L^2}\left(1 - \frac{x}{3L}\right)\right] -$$

$$\frac{L}{\eta A}\left[\frac{Wx}{L}\left(1 - \frac{x}{2L}\right) + \frac{W'x}{L}\right] \tag{15.90A}$$

The deflection at $x = L$ is then given by

$$y = -\frac{L^3}{Yk^2A}\left[\frac{1}{8}W + \frac{1}{3}W'\right] - \frac{L}{\eta A}\left(\frac{1}{2}W + W'\right) \tag{15.91A}$$

where the first term is deflection due to bending and second term is due to shear. Because the first term is more important for long beams; the bending is important for long beams.

(i) **When the weight of cantilever can be neglected:** If we neglect the weight of the rod that is, W may be neglected in comparison to W', then keeping in mind that the sign of W' is minus (downward), we get

$$y = \frac{W'L^3}{3Yk^2A} = \frac{W'L^3}{3YIg} \tag{15.92A}$$

where $Ig = k^2A$ from Eq. (15.74A(a)) and (b).

For a rectangular cross-section of breadth b and depth d, $Ig = \frac{bd^3}{12}$, then

$$y = \frac{W'L^3}{3Y(bd^3/12)} = \frac{4W'L^3}{Ybd^3} \tag{15.93A}$$

For a beam of circular cross-section, $Ig = \pi r^4/4$, then

$$y = -\frac{W'L^3}{3Y(\pi r^4/4)} = \frac{4W'L^3}{3Y\pi r^4} \tag{15.94A}$$

(ii) **When the weight of the cantilever is effective:** Then taking the whole of the first term, but neglecting the second term, we can write:

$$y = -\frac{L^3}{Yk^2A}(1/8W + 1/3W') \tag{15.95A}$$

or using $k^2A = Ig$, we get

$$y = -\frac{L^3}{3YI_g}(W' + 3/8W) \tag{15.96A}$$

The beam behaves as though the load W', at its free end is increased by 3/8 of its own weight.

(iii) **Cantilever uniformly loaded but no weight at the end:** This is a case when $W' = 0$ in Eq. (15.96A); and therefore,

$$y = -\frac{WL^3}{8YI_g} \tag{15.97A}$$

(iv) **Beam supported on two ends but centrally loaded; but neglecting beam's own weight:** Referring to Fig. (15.13A(b), where the load at the centre is zero, that is, $W = 0$, then we should realize that the distance of the point of depression—at the centre of the beam from the point where it is fixed is $L/2$; and the weight of the rod is now $W/2$. So from Eq. (15.92A), we can write the depression y as

$$y = \left(\frac{L}{2}\right)^3 \frac{W'/2}{3YI_g} = \frac{W'L^3}{48YI_g} \tag{15.98A}$$

Then for a circular cross-section of radius r, $Ig = \pi r^4/4$, and hence,

$$y = \frac{W'L^3 \times 4}{48Y(\pi r^4)} = \frac{W'L^3}{12Y(\pi r^4)} \tag{15.99A}$$

For rectangular cross-section of the beam:

$$y = \frac{W'L^3}{48Y} \times \frac{12}{(bd^3)} = \frac{W'L^3}{4Ybd^3} \tag{15.100A}$$

(v) **When the weight of the beam is effective and load W' loaded at the centre:** Then again referring to Fig. (15.13A(b)) and Eq. (15.91A), we put

$$L \to L/2$$
$$W \to W/2$$
$$W' \to W'/2$$

because it is divided between two sides from the centre to the support on both sides. Hence,

$$y = -\frac{L^3}{8Yk^2A}\left[\frac{W}{8} + \frac{W'}{6}\right]$$

$$= -\frac{L^3}{48Yk^2A}\left[W' + \frac{6W}{8}\right]$$

$$= -\frac{L^3}{48YIg}\left[W' + \frac{3W}{4}\right] \tag{15.101A}$$

(vi) **Supported beam uniformly loaded, but no central load:** Again referring to Eq. (15.101A) and (Fig. 15.17A), we have put an extra load W_0 uniformly. This will only mean that this should be added to W in Eq. (15.101A), so that y becomes (with $W' = 0$).

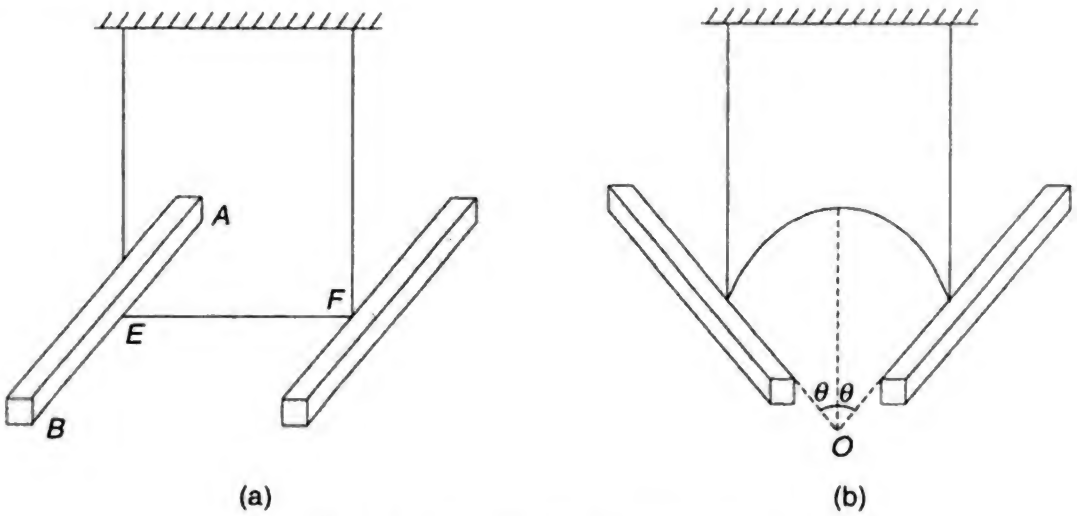


Fig. 15.18A Searle's method for Y

of the bars are brought near each other symmetrically by equal distances, so that the wire is bent in the form of a circular arc, Fig. 15.18A(b) and then released.

A torque is exerted by the wire on the bars and on release the bars vibrate in a horizontal plane, from circular arc on one side to a similar arc on the other. The mid points E and F remain almost stationary so that the action of the wire on the bars and their reaction constitute a couple only. If 2θ is the angle subtended by the wire of length l at the centre of curvature of the circular arc of radius R, then

$$R = \frac{l}{2\theta} \tag{15.103A}$$

The bending moment of the wire, $M = Y\dfrac{Ig}{R}$

where Ig is the geometrical moment of inertia of the cross-section of the wire equal to $\dfrac{\pi r^4}{4}$, where r is the radius of the wire. Thus, on substituting the values of R and Ig, we have

$$M = Y\frac{\pi r^4}{4} \times \frac{2\theta}{l}$$

This produces angular acceleration in each rod about its suspension and if I is the moment of inertia of a rod about its suspension or an axis passing through its middle and perpendicular to its length, we get

$$\frac{d^2\theta}{dt^2} = -\frac{Y\pi r^4\theta}{2lI}$$

the motion is simple harmonic with a time period

$$T_1 = 2\pi\sqrt{\frac{2lI}{Y\pi r^4}}$$

Therefore

$$Y = \frac{8\pi lI}{r^4 T_1^2} \tag{15.104A}$$

This method has the merit of requiring only a short length of the wire and yielding the value of σ in terms of two accurately measurable quantities T_1 and T_2 eliminating thereby the measurement of the radius of wire r, which is the main source of error.

QUESTIONS

15.1A Define elasticity and explain it from atomic point of view.

15.2A Define stress and strain and explain how these quantities are useful in studying the elastic behaviour of a material?

15.3A What do you mean by (i) Hook's law (ii) elastic limit (iii) yield strength and (iv) perfectly elastic body. Draw curves showing relationships between stress and strain, extension and load to illustrate your point.

15.4A Define Young's modulus of elasticity. When will Y be equal to stress?

15.5A Derive expression for Young's modulus, bulk modulus and modulus of rigidity.

15.6A Distinguish between Y, B and η. Why η. cannot be defined for liquids and gases?

15.7A Show that the units and dimensions of the three constants of elasticity (Y, B and η.) are the same.

15.8A A brittle wire such as cast iron is hung from a rigid support. Describe the changes that will take place when it is subjected steadily to increasing load. Illustrate your answer with a sketch graph.

15.9A What do you mean when you say that a substance is 'elastic'? Which is more elastic—a copper wire or a rubber tube? Explain.

15.10A Show that the shear strain is equal to the compression and extension strains.

15.11A Prove that the energy stored in a strained body in case of longitudinal strain is equal to $\frac{1}{2}$ stress $\times$ strain.

15.12A Show that a shearing stress is equivalent to an equal linear tensile stress and an equal compression stress at right angles to each other.

15.13A Derive an expression for the couple required to twisting one end of a cylinder when its other end is fixed.

15.14A What is a cantilever? A light beam of circular cross–section is clamped horizontally at one end and a heavy mass is attached at the other end. Determine the depression at the loaded end.

15.15A Why is a cantilever of uniform cross-section more likely to break near its fixed end?

15.16A Derive a relation between the Young's modulus, bulk modulus and Poisson's ratio of the substance.

15.17A What is Poisson's ratio? Show that the theoretical limiting values of Poisson's ratio are -1 and 0.5.

15.18A Define the terms: beam, neutral surface, neutral axis, and bending moment.

15.19A Derive an expression for the depression produced at the free end of a weightless cantilever of rectangular cross-section when a load is put at its free end.

15.20A Explain why a beam of square cross-section is stiffer than one of the circular cross-section of the same material, length and cross-sectional area?

15.21A Define bending moment of a beam and derive an expression for the same.

PROBLEMS

15.1A Calculate the maximum load that may be placed on a steel wire of radius 0.05 cm if the permitted strain must not exceed 1/1000 and Young's modulus for steel is 2.0×10^{12} dynes/cm^2 *Ans.* (157 N)

15.2A To a 7.0 m long steel wire of radius 1.0 mm is attached a load of 10 kg. Calculate the elongation produced. $Y_{steel} = 21.0 \times 10^{11}$ dynes/cm^2 *Ans.* (10.4×10^{-3} cm)

15.3A What load attached to a steel wire of diameter 0.6 mm and two metres long will produce an extension of 0.5 mm? ($Y_{steel} = 21.0 \times 10^{12}$ dynes/cm^2)
Ans. ($4.5\ \pi$ N)

15.4A Strain in a long vertical wire is 4×10^{-4} when it is stretched by a load of 2.0 kg. Calculate the Young's modulus of the wire and energy stored per unit volume if the diameter of the wire be 0.5 mm. *Ans.* ($Y = 2.5 \times 10^{12}$ dynes/cm^2
$E = 2.0 \times 10^5$ ergs/cm^2)

15.5A To one end of a 4 m long wire, a load of 20 kg is attached and it produces an elongation of 0.24 mm. Calculate the stress, strain and Young's modulus of the wire if its radius be 1 mm. *Ans.* (Stress = 6.2×10^7 dynes/cm^2
strain = 6.0×10^{-5}
$Y = 1.0 \times 10^{12}$ dynes/cm^2)

15.6A Calculate the load in kg needed to produce an extension of 1 mm on a wire of diameter 1.6 mm and 6 m in length. ($Y_{steel} = 2.0 \times 10^{12}$ dynes/cm^2)
Ans. (6.8 kg)

15.7A Calculate the work done in stretching steel wire 1m long and cross-sectional area 0.030 cm^2 when a load of 100 kg is slowly applied without the elastic limit being reached. ($Y_{steel} = 2.0 \times 10^{12}$ dynes/cm^2)

Ans. $\left(\frac{1}{20} \text{ Joules}\right)$

15.8A An extension of 0.01% is produced in a wire of radius 0.2 mm when it supports a load of 1 kg. Calculate the Young's modulus of the wire.
Ans. ($Y = 7.8 \times 10^{11}$ dynes/cm^2)

15.9A Calculate the new volume of a block of lead at sea bed when it is thrown into the sea. The pressure at seabed is 8.6×10^8 dynes/cm^2 greater than at the surface. The original volume of the block is 0.493 cm^3. Bulk modulus for lead is 4.8×10^{11} dynes/cm^2. *Ans.* (.492 cm^3)

15.10A Find the increase in pressure needed to decrease the volume of 1 m^3 of water by 10^{-4}m^3. The bulk modulus for water is 2.1×10^{10} dynes/cm^2.
Ans. (2.1×10^6 dynes/cm^2)

15.11A A steel strip is clamped horizontally at one end. On applying a load of 1 kg at the free end, the bending in equilibrium state is 10.0 cm. Calculate (i) the potential energy stored in the strip, (ii) the frequency of vibration if the load is disturbed from equilibrium. The mass of the strip may be neglected.
Ans. (5.0×10^6 ergs; 1.58/s)

SECTION B
FLUID DYNAMICS

In a stationary liquid, the molecules or atoms of the liquid have two motions (i) vibration around their mean positions and (ii) the diffusion due to Brownian motion.

The diffusion of the molecules in a liquid can take place because the forces on a molecule in a liquid are such that a molecule can easily slide by the side of another molecule. It is because of this reason that a liquid cannot bear any shearing strain.

If a pressure difference is applied to a liquid, the molecules of the liquid will start moving from higher pressure to the lower pressure following the process of diffusion. It is this motion of a large number of molecules moving together, by the side of other molecules, that constitute the process of flow in a liquid. Suppose we consider the flow of the liquid in an open channel, Fig. 15.1B.

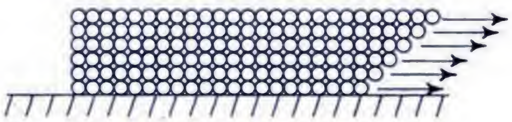


Fig. 15.1B Motion of molecules

The molecules of the liquid in contact with the molecules of the floor are nearly stationary due to the forces of adhesion. The molecules above them may slide past these molecules with a relative velocity, say v. The molecules in the second layer above the first layer may slide past the molecules in the first layer, again by the same velocity so that their velocity with respect to the molecules on the floor channel may be $2v$. In this manner, as we move away form the floor of the channel, the velocity of the molecules goes on increasing, so that the molecules on the uppermost layer have the maximum velocity. This is the molecular picture of the flow of the liquid. It may be mentioned that in actual practice, the molecules will not move in straight lines as shown in Fig. 15.1B but will move in a somewhat zig-zag manner. We have only shown the average motion of the molecules.

15.1B VISCOSITY

The molecular picture of the flow of a liquid as described in the previous section is due to the phenomenon of viscosity.

It is well-known that under similar pressure difference, different liquids flow with different rates. For example, if we take a simple apparatus as shown in Fig. 15.2B and fill the vessel A with a given liquid up to a certain height h, one can find out the time in which the whole liquid flows out into the vessel B.

If the experiment is performed for different liquids keeping h same for all liquids, we will find that some liquids, say alcohol or water, take much less time than other liquids say honey, glycerine, ghee and so on. Honey or glycerine, therefore, are said to have more resistance to flow than water or alcohol. This resistance to flow is called viscosity.

With a view to define the viscosity quantitatively, let us consider the case of a liquid flowing smoothly. We have seen already that when a liquid flows, the layer of the liquid nearest to the solid surface has the lowest velocity and the layer farther form it will have the largest velocity. In other words, in a smoothly flowing liquid, there will be a velocity gradient. It can be understood from the molecular picture of the flow of the liquid that this velocity gradient arises because of the resistance

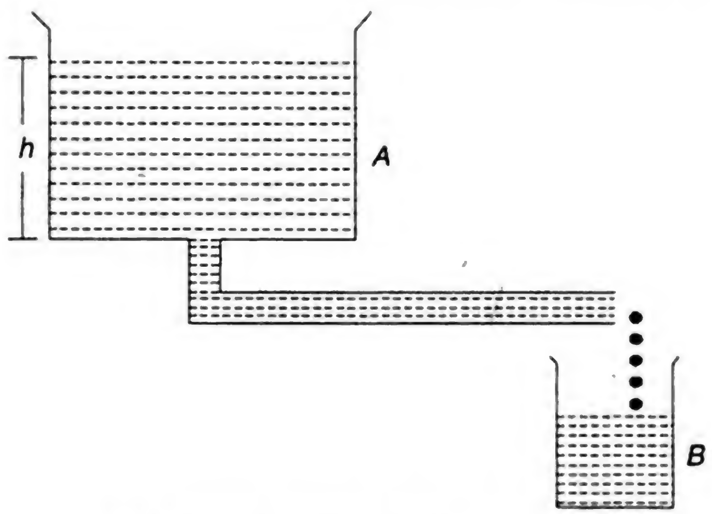


Fig. 15.2B Flow of a liquid

offered by the molecules in the layer, nearer to the floor of the channel, to the molecules away from it. Hence there is associated to a flowing liquid a velocity gradient. This force will act in the opposite direction to the direction of flow of the liquid. The viscosity is defined quantitatively in terms of this force acting on the layers of the flowing liquid and the velocity gradient.

Let us consider two nearby layers, Fig. 15.3B with a distance Δz between them. Let v be the velocity of one layer and $v + dv$ that of the other. Then the velocity gradient is given by $\frac{\Delta v}{\Delta z} = \frac{dv}{dz}$ for $\Delta z \to 0$. The coefficient of viscosity η is then defined as

$$F/A = -\eta \frac{dv}{dz} \tag{15.1B}$$

where F/A is the force per unit area acting on the liquid layer due to the viscosity. The area A is the area of the surface of the layer. The negative sign in Eq. (15.1B) shows that the force acting due to viscosity is in a direction opposite to that of the flow of the liquid. One has to apply an equal and opposite force from outside to make the liquid flow. An external pressure on the liquid may provide this.

From Eq. (15.1B), one can define the coefficient of viscosity as the tangential force per unit area required to maintain a unit velocity gradient or a unit relative velocity between two layers, which are a unit distance apart. If the tangential force

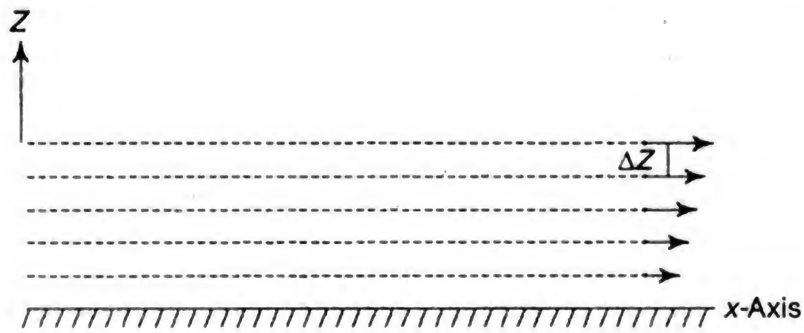


Fig. 15.3B Different layers move with different velocities

is 1 dyne/cm^2 for a velocity gradient of 1 cm/sec for 1 cm distance between layers, the coefficient of viscosity is unity in CGS system. This unit coefficient of viscosity in CGS is called *Poise*.

The following table gives the value of the coefficient of viscosity (η) for different fluids at 20^o C (viscosity is found to change with temperature)

S.No.	*Fluid*	$\eta \times 10^{-2}$ *Poise*	*S.No.*	*Fluid*	$\eta \times 10^{-2}$ *Poise*
1.	Water	1.0019	7.	Acetone	0.324
2.	Mercury	1.552	8.	Aniline	4.39
3.	Sulphuric acid	27.000	9.	Co_2 (liquid)	0.071
4.	Choloroform	0.569	10.	Caster oil	986.00
5.	Benzene	0.647	11.	Olive oil	84.00
6.	Acetic Acid	1.219	12.	Toluene	0.585

EXAMPLE 15.1B

A square plate of each side 10 cm rests on a layer of olive oil 2 mm thick whose coefficient of viscosity is 84 centipoise (1centipoise=10^{-2} poise). Calculate the horizontal force required to impart the plate a speed of 3 cm/sec.

Solution

The horizontal force required is given by

$$F = \eta A \frac{v}{x}$$

Here
$$A = 100 \text{ cm}^2$$
$$\eta = 84 \text{ cp} = 84 \times 10^{-2} \text{ Poise}$$
$$v = 3 \text{ cm/sec}$$
$$x = 0.2 \text{ cm}$$

Therefore,
$$F = \frac{84 \times 10^{-2} \times 100 \times 3}{0.2} = 1260 \text{ dynes}$$

15.2B EQUATION OF CONTINUITY

If a liquid is flowing, say through a pipe, it is easy to understand that the amount of fluid flowing through any cross-section of the pipe in a given time will be the same as flowing through any other cross-section in the pipe. If this were not so, then (i) either liquid in between the two sections is getting absorbed, which is not possible unless there is a leakage from the sides of the tube or (ii) the liquid is being created, which is not possible in a tube with no side openings. This physical fact is the basis of the equation of continuity.

Let us consider the flow of a liquid in a pipe as shown in Fig. 15.4B and consider the two cross–sections at A and B. At point A, let velocity be v_1, density ρ_1 and cross-sectional area A_1. At point B, let velocity be v_2, density ρ_2 and cross–sectional area A_2. Then, the mass of the liquid Δm_1 flowing through point A in time Δt is given by

$$\Delta m_1 = A_1 v_1 \rho_1 \Delta t \tag{15.2B}$$

Similarly, the extra energy gained (or lost) due to the change in velocity is given by

$$\Delta W_2 = \frac{1}{2}\, mv_2^2 - \frac{1}{2}\, mv_1^2 \tag{15.9B}$$

Therefore, the total change in the energy is given by

$$\Delta W_1 + \Delta W_2 = mg(h_2 - h_1) + \left(\frac{1}{2}\, mv_2^2 - \frac{1}{2}\, mv_1^2\right) \tag{15.10B}$$

This energy is provided by the work done by difference of pressure, as given by Eq. (15.7B). Thus

$$P_1A_1\Delta L_1 - P_1A_2\Delta L_2 = mg(h_2 - h_1) + \left(\frac{1}{2}\, mv_2^2 - \frac{1}{2}\, mv_1^2\right) \tag{15.11B}$$

Using Eq. (15.6B), we can write Eq. (15.11B) as

$$(P_1 - P_2)\,\frac{m}{\rho} = mg(h_2 - h_1) + \frac{1}{2}\, m(v_2^2 - v_1^2)$$

or

$$P_1 + \rho g h_1 + \frac{1}{2}\, v_1^2 \rho = P_2 + \rho g h_2 + \frac{1}{2} v_2^2 \rho$$

Hence

$$P + \rho g\, h + \frac{1}{2}\, \rho v^2 = \text{constant} \tag{15.12B}$$

at any point in the flow of a liquid in a pipe. It may be mentioned that by using equation (15.6B), we have assumed that the fluid obeys the equation of continuity or in other words there is no source or sink in the flow of the liquid. Equation (15.12B) is called the Bernoulli's equation. If the liquid is flowing in a horizontal pipe, the $\rho\, g\, h$ is constant. Hence, Eq. (15.12B) can be written as

$$P + \frac{1}{2}\, \rho v^2 = \text{constant} \tag{15.13B}$$

When the liquid is at rest, $v = 0$, then

$$P + \rho\, g\, h = \text{constant} \tag{15.14B}$$

The quantity $(P + \rho g h)$ is called static pressure and $\frac{1}{2}\rho v^2$ is called dynamic pressure, and $P + \rho g\, h + \frac{1}{2}\, \rho v^2$ is the total effective pressure. Dividing equation (15.12B) by ρg, we get.

$$h + \frac{v^2}{2g} + P/\rho g = H \text{ (constant)} \tag{15.15B}$$

H is called the total head.

EXAMPLE 15.3B

Figure 15.6B shows an instrument called Venturimeter, which is a device to measure the speed of a liquid in a pipe. It has a manometer M, which has mercury in it and is connected to the pipe at two points A and B. If the diameter of the pipe at point A is 5 cm, and at point B it is 2 cm, find out the velocity of the water in the pipe at point A if $P_1 - P_2 = 10$ cm of mercury. The pipe is placed horizontally.

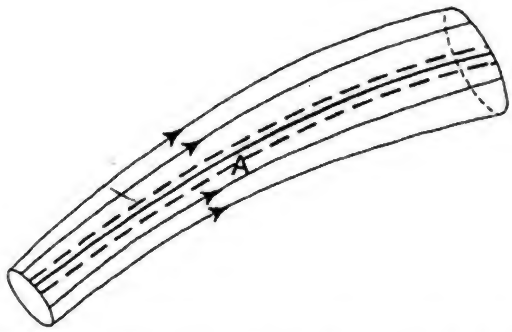


Fig. 15.7B Streamlines bounding a tube of flow

But any other element entering A will have the same velocity as the previous liquid element. Because of this basic property, no two streamlines will intersect because then at the point of intersection the liquid element will have two velocities. The streamlines, therefore, are parallel to the velocity of the particles, but the contour of a streamline will depend on the container. For a liquid flowing in a streamlined manner, no part of the liquid element should possess any acceleration. This happens when the opposing forces of viscosity and applied pressure at a given point are exactly balanced. The constant velocities at a given point for streamlined flow will vary for different liquids. More viscous the liquid, less will be the velocity under such conditions. The streamlined flow is also called *irrotational flow* because liquid does not have any whirlpools or rotations in it.

A turbulent flow, on the other hand, occurs when different liquid-elements follow no set paths and the velocity at a given point in a liquid changes with time. This occurs when the applied force for the flow of the liquid is much more than the force required to overcome resistance due to viscosity. Under these conditions, whirlpools may get formed, and the surface of the flowing liquid may not be smooth. Different liquid-elements will have accelerations and the paths of different liquid-elements will cross each other creating a disorderly and turbulent flow. Figure 15.8B(a) shows streamline flow and Figure 15.8B(b) turbulent flow.

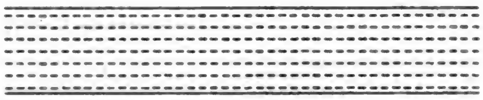


Fig. 15.8B(a) Streamline flow

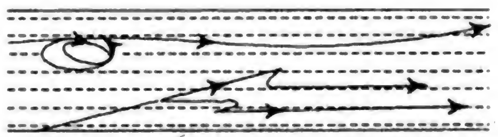


Fig. 15.8B(b) Turbulent flow

15.5B LINES OF FLOW IN AIRFOIL

Streamline flow of a fluid assumes different shapes, depending on the contour of the surfaces. An interesting case is that of the flow of air around aircraft wing or airfoil. The shapes of the streamlines in this case are shown in Fig. 15.9B. The following points emerge:

(i) If the airfoil surface is nearly parallel to the streamlines, the shape of the stream lines is least disturbed, and they resume their original paths very near behind the airfoil; Fig. 15.9B(a).

(ii) If the airfoil is tilted, the streamlines are disturbed more and resume their shapes at larger distance behind the airfoil; Fig. 15.9B (b).

(a) (b)

Fig. 15.9B Airfoil

(iii) The streamlines below the airfoil are much less disturbed, but above it, they are squeezed.

These are the experimental facts.

More density of air lines means more velocity above the foil. More the velocity, lesser will be the pressure, according to Bernouli's law. It is because of this low pressure on the upper side of an aircraft wing, that the aircraft has an upward thrust and can be made to be 'airborne'. It may be mentioned that it is the special shape of the airfoil that is responsible for this lift of the aircraft. The combination of the thick edge in front and thin edge at the back gives the streamlines with more density on the upper surface.

15.6B FLOW OF LIQUID THROUGH A NARROW TUBE: POISSEULLE'S LAW

If a liquid flows through a narrow tube or a capillary, the following consideration will determine the rate of flow of the liquid:

(i) If we assume the liquid flow to be streamlined, then the opposing force due to viscosity balances the force due to difference of pressure.

(ii) The molecules of the liquid touching the tube will be stationary, while those moving in the centre of the tube will be moving with the maximum velocity. There will, therefore, be a velocity gradient $\frac{dv}{dr}$ across the tube at any cross-section of the tube.

Let us consider the flow of the liquid in the cylinder between the radius r and $r + dr$, Fig. 15.10B. Let the velocity at r be v, and the velocity gradient $\frac{dv}{dr}$. Then according to Eq. (15.1B), the viscous drag per unit area is given by $\eta \frac{dv}{dr}$. The area on which this drag acts is the area of the surface of the cylinder of radius r and length l, which is $2\pi rl$. Hence, the total drag F on the cylinder of radius r is given by

$$F = -\eta \frac{dv}{d\tau} 2\pi rl \qquad (15.16B)$$

The negative sign shows that F is a retarding force. This force is balanced by the difference of pressure between the two ends of the tube. The total force F exerted by such a pressure is given by

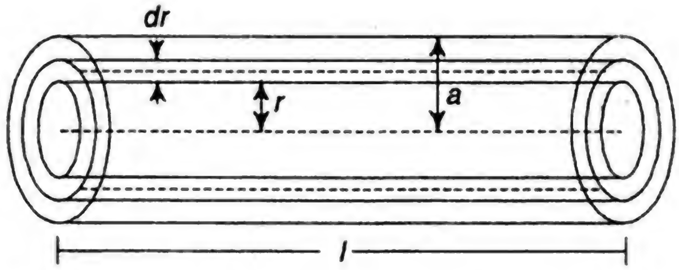


Fig. 15.10B Flow of liquid in a narrow tube

$$F = (P_1 - P_2)\,\pi r^2 \tag{15.17B}$$

Hence,
$$(P_1 - P_2)\,\pi r^2 = -\eta \frac{dv}{dr}\,2\pi r l$$

or
$$-dv = \frac{(P_1 - P_2)}{2l}\,r dr$$

Integrating both sides, we get

$$-\int_v^0 dv = \frac{(P_1 - P_2)}{2\eta l}\int_r^a r dr$$

Hence,
$$v = \frac{(P_1 - P_2)}{4\eta l}\,(a^2 - r^2) \tag{15.18B}$$

This is the velocity at radius r. To find the flow rate of the liquid, that is, the volume of the liquid flowing through any cross-section per unit time, we realize that volume of the liquid flowing any cross-section of the tube per unit time is given by the area of the cross-section multiplied by velocity; we get

$$dV = 2\pi r\, dr\, v$$

$$= \frac{2\pi(P_1 - P_2)}{4\eta l}\,(a^2 - r^2)\, r dr \tag{15.19B}$$

Hence, volume of the liquid flowing per unit time

$$V = \frac{2\pi(P_1 - P_2)}{4\eta l}\int_0^a (a^2 - r^2)\, r dr$$

$$= \frac{\pi(P_1 - P_2)}{8\eta l}\,a^4 \tag{15.20B}$$

This is called the *Poissuelle's law*. Eq. (15.20B) holds good for the amount of liquid flowing per unit time, through any section of the tube.

15.6.1 Flow of Blood in Human Body

The circulation of blood in human body illustrates how Bernouli's and Poisuelle's laws influence the velocities and pressures of the blood in different parts of the human body.

Second Edition

MECHANICS

This well-received and authoritative text has been revised with the objective to increase the coverage and to aid the students in their efforts at self-study by including solved examples at the end of the chapters. The presentation is lucid and direct, and the pedagogy followed in the book would help the students to have a firm grip on the basic concepts in mechanics.

Although the book is largely based on Newtonian formulation, a separate chapter on Lagarangian and Hamiltonian formulations has been added in this edition. In addition two new chapters on Mechanics of Continuous Media and Charged Particle Dynamics make the book more useful.

About 90 additional solved examples and another 90 unsolved problems give ample opportunity to the students for practice.

The book would meet the requirements of the B Sc Physics students.

H S Hans was formerly UGC Emeritus Fellow in the Department of Physics, Panjab University, Chandigarh, where he had been Professor and Head since 1967. He obtained his M Sc from Banaras Hindu University and his Ph D from Aligarh Muslim University in experimental nuclear physics, an area which he pursued abroad in Swarthmore, Texas, A and M, Agronne National Laboratory, Rochester University (USA) and Birmingam University (UK). A cyclotron laboratory at Panjab University is an outcome of his efforts. Professor Hans has published a large number of research papers in international and national journals of repute. He has been UGC National Lecturer and National Fellow. He has been, for many years, the coordinator of ULP (COSIP) of his department, which inspired a number of authors to develop a series of textbooks of which this book is the first attempt. He has also authored a book on Nuclear Physics (2001) for postgraduate students.

S P Puri, ex UGC Emeritus Fellow, was Professor and Chairman, Department of Physics, Panjab University, Chandigarh. He received his M Sc from Panjab University and Ph D from Aligarh Muslim University, Aligarh. He did his postdoctoral research at Bartol Research Foundation, Swarthmore PA (USA). He later joined University of Alabama (USA) as Visiting Assistant Professor of Physics and also visited University of Uppsala, Sweden. He was one of the earliest in the country to establish the studies of Nuclear Gamma-Ray Resonance (Mossbauer Effect) in Roorkee in 1963. His other special fields of interest include Crystal and Molecular Field Theories. He has authored a number of books including *Special Theory of Relativity* (1972); *Fundamentals of Vibrations and Waves* (1989); *Classical Electrodynamics* (1990) and compiled Collected Scientific Papers of Prof P S Gill (1971).

The McGraw-Hill Companies

ISBN 0-07-047360-9

9780070473607

Visit *Tata McGraw-Hill* at www.tatamcgrawhill.com